Summary Sec. 1.3:
The interacting Green's function
Feynman diagrams

From Sec. 1.2.1: Remember the most important properties of the one-particle Green's function for non-interacting electrons:

$$G^0_{\alpha\beta}(\mathbf{r}, t; \mathbf{r}'t') = \delta_{\alpha,\beta} G^0(\mathbf{r} - \mathbf{r}'; t - t') \,.$$

That means:

1. As long as the interaction between the electrons is spin-independent, G^0 is diagonal in spin space.
2. As long as no relativistic effects concerning the interaction are taken into account, G^0 is temporally homogenous.
3. As long as the electrons are not subjected to any local (external) potential, G^0 is spatially homogeneous.

If the above conditions are fulfilled, the corresponding properties are also valid for a system of interacting electrons:

$$G_{\alpha\beta}(\mathbf{r}, t; \mathbf{r}'t') = \delta_{\alpha,\beta} G(\mathbf{r} - \mathbf{r}'; t - t') \,.$$

Perturbation expansion of G, based on the non-interacting functions G^0 ? What does that mean?

$$G_{\alpha\beta} = \sum_{\nu=0}^{\infty} f_\nu(\cdots G^0_{\alpha\beta} \cdots \hat{V} \cdots)$$

ν means the order of the term with respect of the interaction operator $\hat{V}$ ("how often appears $\hat{V}$ in the term?").

The electron-electron interaction operator:

$$\hat{V} = V(\mathbf{r} - \mathbf{r}'; t - t') = \frac{e^2}{|\mathbf{r} - \mathbf{r}'|}\,\delta(t - t')$$

with

$$V(\mathbf{r} - \mathbf{r}') = \frac{1}{\Omega}\sum_{\mathbf{k}} V(\mathbf{k})\,\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \qquad \text{and} \quad V(\mathbf{k}) = \frac{4\pi e^2}{k^2}\,.$$

What's about the divergence of $V(\mathbf{k})$ for $\mathbf{k} = \mathbf{0}$?

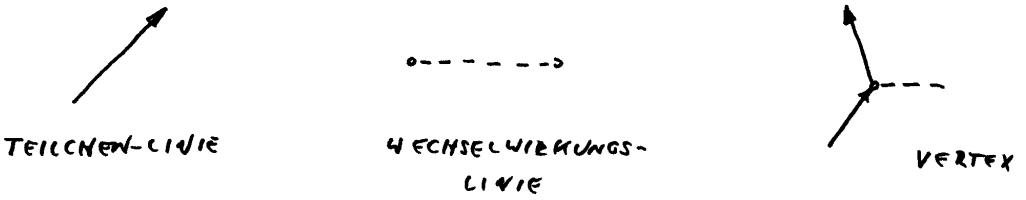
$V(\mathbf{0})$ refers to the spatial mean value of the potential energy of the electrons. Quantummechanically, such a constant energy value can always be shifted to zero without any change of the physical content of the Schrödinger equation.

Therefore, for all following formulas, one can write per definition:

$$V(\mathbf{k} = \mathbf{0}) = 0$$

Feynman diagrams in $(\mathbf{k}, \omega)$ space:

- R. Feynman (1918-1988), Nobel prize 1965, invented the diagram technique for his work in quantum electrodynamics; meanwhile, it has become extremely useful in many other fields in physics.
- Elements of Feynman diagrams:



- Rule 1: Draw all topographically different, connected diagrams which consist von ν interaction lines and $2\nu + 1$ particle lines (propagator lines) (ν means the *order* of the diagram).
  Each graph has to be entered and left by *one and ony one* particle line.

- Each particle line means a non-interacting Green's function which is named by a wave vector, a frequency, and two spin indices denoting the spin orientation of the electron *at the beginning and the end of the propagator*.
- At each *vertex*, momentum and energy conservation has to be fulfilled.
- In case of a spin-independent interaction (as, e.g., a Coulomb interaction), no *spin-flip* must happen at the vertices.

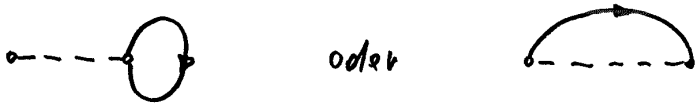
Mathematical evaluation:

- Each particle line is replaced by a non-interacting Green's function $G^0_{\alpha\beta}(\mathbf{k},\omega)$, each interaction line by a Fourier coefficient of the bare Coulomb potential $V(\mathbf{q})$.
- Summations over all *internal* spin indices and integrations over all *internal* wave vectors.
- Each diagram of order ν has to be multiplied by the factor

$$\left(\frac{i}{\hbar}\right)^{\nu} (2\pi)^{-4\nu} (-1)^F$$

with F as the number of closed particle loops in the diagram.

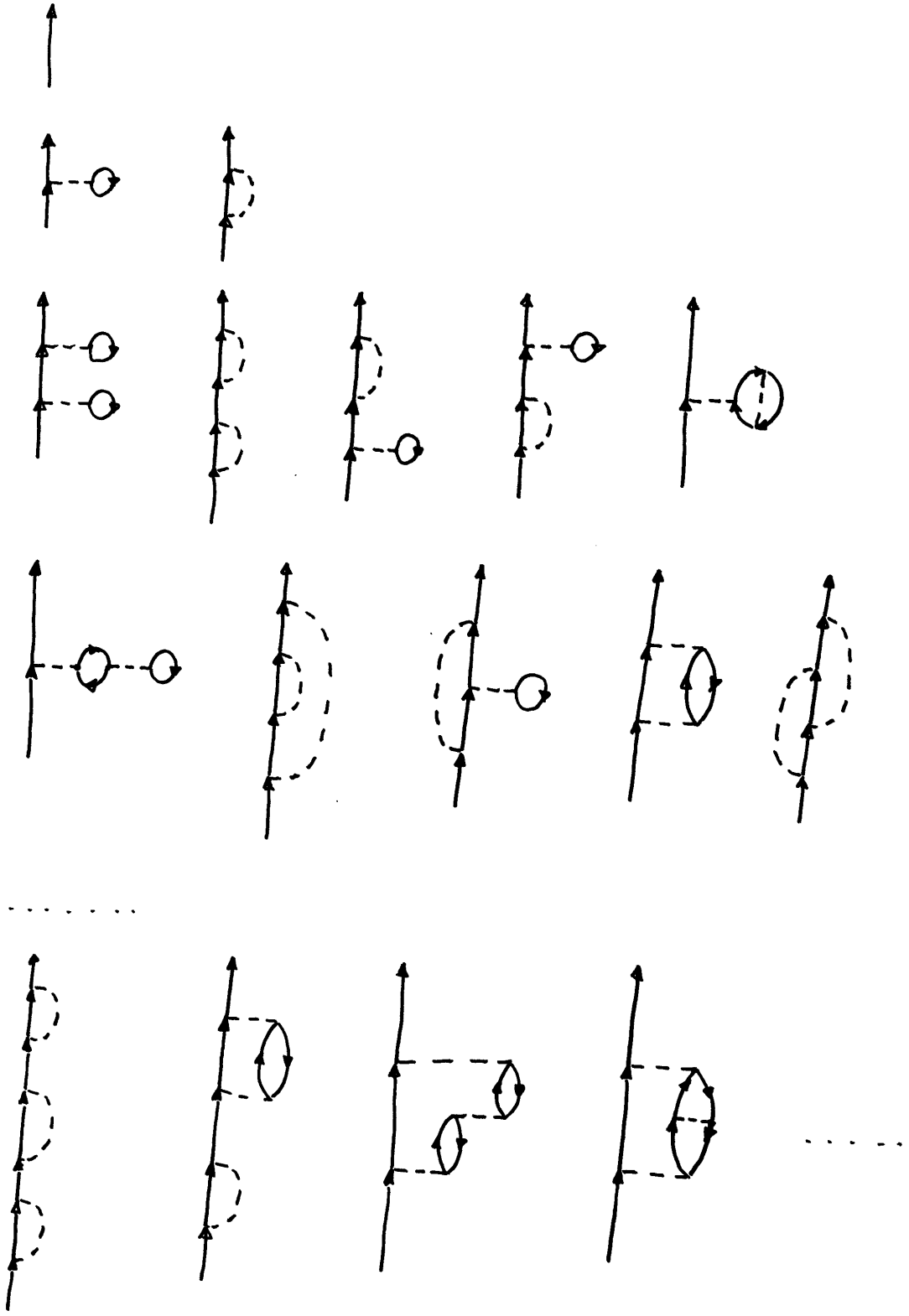
- Each particle line looking like



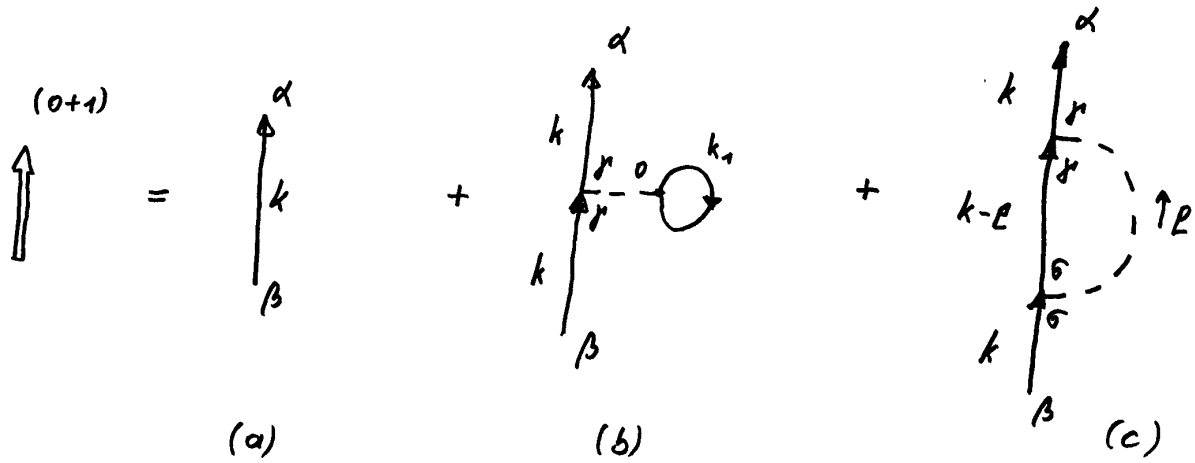
has to be replaced by $\mathrm{e}^{i\omega\eta}\, G^0(\mathbf{k},\omega)$.

- After all integrations, the limit $\eta \to 0$ has to be performed.

All Feynman diagrams of zeroth, first, and second order, and some examples of Feynman diagrams of third order:



Evaluation of the graphs of zeroth and first order:



$[(\mathbf{k},\omega) \rightarrow \quad k \qquad \text{etc.}]$

Zeroth-order term (a):

$$G^a_{\alpha\beta}(k) = \delta_{\alpha\beta} G^0(k)$$

First-order term (b):

$$G^b_{\alpha\beta}(k) = \frac{i}{\hbar}\frac{1}{(2\pi)^4}(-1)\underbrace{V(0)}_{=0}\left[G^0(k)\right]^2 \underbrace{\sum_\gamma \sum_\sigma \delta_{\alpha\gamma}\delta_{\gamma\beta}\delta_{\sigma\sigma}}_{2\delta_{\alpha\beta}} \int d^4k_1 \mathrm{e}^{i\omega_1\eta} G^0(k_1) = 0\,.$$

Tadpole terms do not contribute to the Green's function!

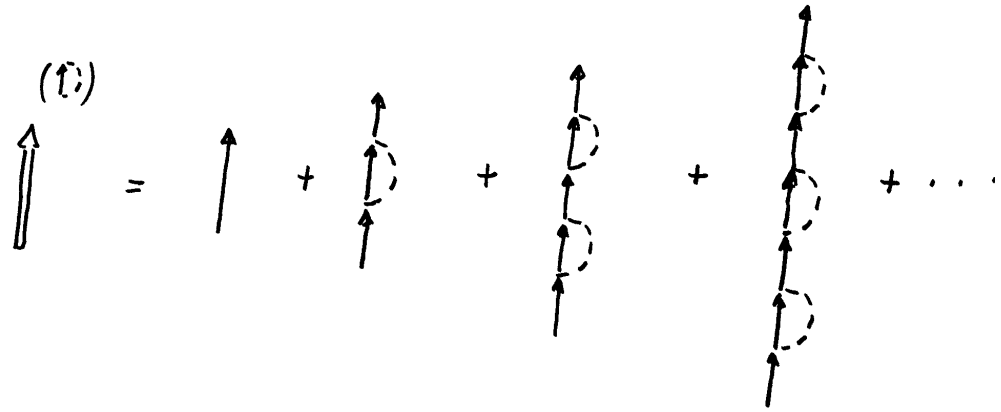
First-order term (c):

$$\begin{aligned} G^c_{\alpha\beta}(k) &= \frac{i}{\hbar}\frac{1}{(2\pi)^4}\left[G^0(k)\right]^2 \underbrace{\sum_\gamma \sum_\sigma \delta_{\alpha\gamma}\delta_{\gamma\sigma}\delta_{\sigma\beta}}_{\delta_{\alpha\beta}} \int d^4q\, V(\mathbf{q}) \mathrm{e}^{i(\omega-\omega_1)\eta} G^0(k-q) \\ &= \delta_{\alpha,\beta}\frac{i}{\hbar}\frac{1}{(2\pi)^4}\left[G^0(\mathbf{k},\omega)\right]^2 \int_\Omega d^3q\, V(\mathbf{q}) \int d\omega_1\, \mathrm{e}^{-i\omega_1\eta}\, G^0(\mathbf{k}-\mathbf{q},\omega-\omega_1)\,. \end{aligned} \tag{1}$$

Selfenergy and proper selfenergy:

→ central terms in electron theory

If we look on the table of Feynman diagrams, we see that the first-order term $G^c_{\alpha\beta}$ has *corresponding higher-order terms* leading to an infinite sum of graphs as

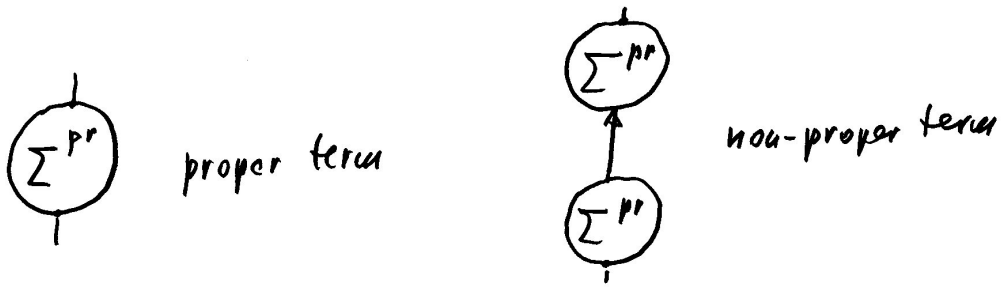


The same *type of interaction* ("Wechselwirkungs-Motiv" in German) appears zero-times, once, twice, threetimes etc. Graphically, this can be written as

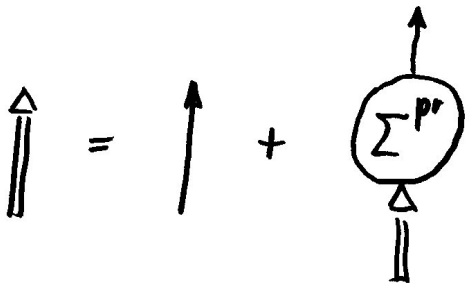
$\Sigma(M)$ is called the selfenergy insertion of the interaction type M into the electron particle line.

Please note: $\Sigma(M)$ is not "the" electron selfenergy but only the M^{th} part of it. Actually, the type of interaction shown in the above figures is the simplest approximation to the selfenergy:

The definition of a so-called proper selfenergy term can also easily be explained in a graphic way:



Such a definition of proper selfenergy terms enables the representation of the inifite sum of diagrams in a recursive way:



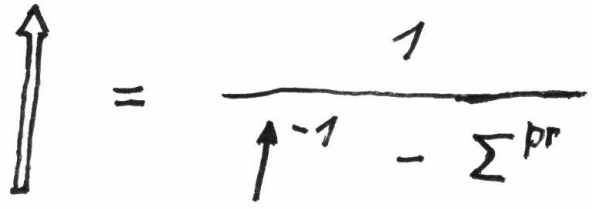
Mathematically, such a Dyson equation means an integral equation in the $\{\mathbf{r};t\}$ space and an algebraic equation in the $\{\mathbf{k};\omega\}$ space: the corresponding translation into a mathematical formula is extraordinary simple [once more: k means $(\mathbf{k},\omega)$]:

$$G(k) = G^0(k) + G(k)\,\Sigma^{pr}(k)\,G^0(k)\,.$$

This equation can easily be solved by writing

$$G(k) = \frac{G^0(k)}{1-\Sigma^{pr}(k)G^0(k)} = \frac{1}{[G^0(k)]^{-1}-\Sigma^{pr}(k)}\,,$$

or, again diagramatically:



How does the formula for the simplest approximation to Σ^{pr} look like? This can be got out by remembering that the first-order Green's function term (c) can be written in a twofold way: firstly, according to Eq. (1) of this summary, as

$$\begin{aligned} G^c(\mathbf{k},\omega) &= G^0(\mathbf{k},\omega) \\ &\times \left\{ \frac{i}{\hbar}\frac{1}{(2\pi)^4}\int_\Omega d^3q\, V(\mathbf{q}) \int d\omega_1\, \mathrm{e}^{-i\omega_1\eta}\, G^0(\mathbf{k}-\mathbf{q},\omega-\omega_1) \right\} G^0(\mathbf{k},\omega)\,. \end{aligned}$$

and secondly, directly from the corresponding diagram, as

$$G^c(\mathbf{k},\omega) = G^0(\mathbf{k},\omega)\,\Sigma^{pr(c)}(\mathbf{k},\omega)\,G^0(\mathbf{k},\omega)\,.$$

The result:

$$\Sigma^{pr(c)}(\mathbf{k},\omega) = \frac{i}{\hbar}\frac{1}{(2\pi)^4}\int_\Omega d^3q\, V(\mathbf{q}) \int d\omega_1\, \mathrm{e}^{-i\omega_1\eta}\, G^0(\mathbf{k}-\mathbf{q},\omega-\omega_1)\,. \tag{2}$$

- How can this equation be further evaluated (especially what concerns the integration over ω_1 ?
  → see the following appendix.
- What is the physical content of this equation?
  → see Sec. 1.4.