CALCULATIONS OF $1/\nu$ TRANSPORT IN AN $l = 3$
STELLARATOR MAGNETIC FIELD IN THE PRESENCE
OF MAGNETIC ISLANDS CAUSED BY MAGNETIC
SYSTEM ERRORS

VICTOR NIKOLAYEVICH KALYUZHNYI* and VICTOR VADIMOVICH NEMOV
Institute of Plasma Physics, National Science Centre "Kharkov Institute of Physics and Technology"
Akademicheskaya 1, 61108 Kharkov, Ukraine

Received November 20, 2003
Accepted for Publication March 7, 2004

The $1/\nu$ neoclassical transport (effective ripple) is calculated for the $l = 3$ Uragan-3M torsatron in the presence of magnetic islands caused by a small eccentricity in the vertical field coils. The technique used for the calculations is based on integration along magnetic field lines in a given stellarator magnetic field. For comparison, calculations are also performed for the case of negligible eccentricity.

KEYWORDS: neoclassical transport, torsatron Uragan-3M, transport computations

I. INTRODUCTION

Relatively small errors in the magnetic system of a stellarator can decrease the quality of the magnetic configuration. In particular, magnetic islands can arise for rational values of the rotational transform $\nu$. This phenomenon has been found in the $l = 3$ Uragan-3M (U-3M) torsatron.1 In numerical calculations and experiments for certain values of the vertical magnetic field, rather large magnetic islands corresponding to $\nu = \frac{1}{4}$ were found that can be explained by a small eccentricity in the vertical field coils. For the island and nonisland magnetic surfaces found numerically in real space coordinates in Ref. 1, calculations of the $1/\nu$ neoclassical transport (effective ripple) were carried out under conditions corresponding to the formation of the aforementioned magnetic islands. The technique of Ref. 2 is used for the calculations that is based on integration along magnetic field lines in a given

magnetic field. This technique allows one to make calculations using a magnetic field calculated numerically in real-space coordinates.

Numerical calculations of $1/\nu$ transport are important in assessing the general confinement properties of a device.3 In accordance with characteristic plasma parameters obtained in the U-3M experiments (see, e.g., Refs. 4 and 5) ($n \approx 2 \times 10^{12} \text{ cm}^{-3}, T \approx 300 \text{ eV}$) the mean free path $l_{\text{mfp}}$ for charged particles is $l_{\text{mfp}} \approx 10^5 \text{ cm}$. This is ten times larger than the maximum limit of $l_{\text{mfp}} \approx 10^4 \text{ cm}$ for the plateau regime corresponding to the characteristic parameters of the device ($a \approx 10 \text{ cm}, R \approx 100 \text{ cm}$, a rotational transform of 0.25 to 0.29). Therefore, the long-mean-free-path regime of neoclassical transport (particularly, the $1/\nu$ regime) can occur in U-3M. However, in the experiments this transport is disguised by rather strong anomalous losses. Nevertheless, calculations of $1/\nu$ transport are of interest in assessing the future outlook for such configurations (for higher plasma parameters). Besides, a possible eccentricity in the compensating coils and the helical field system may be a characteristic manufacturing error in various stellarator devices, and the results obtained for U-3M may be of interest when assessing the consequences of this eccentricity for different devices.

II. METHOD OF NUMERICAL INVESTIGATION

In stellarators, magnetic field asymmetry can lead to enhanced particle and heat transport in the long–mean-free-path regimes. This enhanced transport is associated with the bounce-averaged $\nabla B$ drift of ripple-trapped particles across the magnetic surfaces. According to the theory (see, e.g., Ref. 6), various transport regimes are possible in this case; however, the so-called $1/\nu$ regime turns out to be the most unfavorable because with decreasing
collision frequency \( \nu \), the transport coefficients increase. In general, this regime is characterized by the effective (or equivalent helical) ripple \( \varepsilon_{\text{eff}} \). For specific devices, the characteristic features of the specific magnetic field geometry manifest themselves in particle and heat fluxes through the factor \( \varepsilon_{\text{eff}}^2 \). For a conventional stellarator field, \( \varepsilon_{\text{eff}} \) coincides with the helical ripple \( \varepsilon_4 \). Usually, calculations of \( \varepsilon_{\text{eff}} \) for specific devices are carried out by numerical methods using Monte Carlo or DKES codes (see, e.g., Ref. 3) that require long computing times. A much faster method,\(^2\) which is based on integration along magnetic field lines, is used here to evaluate \( 1/\nu \) transport.

In Ref. 2 an arbitrary toroidal magnetic field using the solution of the banana kinetic equation, analytical formulas for particle and heat flux densities averaged over a magnetic surface are derived for the \( 1/\nu \) regime. During the derivation, integration over the magnetic surface area was transformed to integration over the volume between neighboring magnetic surfaces. After that, using the equivalence between averages over this volume and over the magnetic field line, the final expressions for the flux densities were obtained in the standard form with the factor \( \varepsilon_{\text{eff}}^2 \), which is calculated by using integration along the magnetic field line. In accordance with Ref. 2, the quantity \( \varepsilon_{\text{eff}}^2 \) for an arbitrary stellarator magnetic field can be calculated with the help of the following expressions:

\[
\varepsilon_{\text{eff}}^2 = \frac{\pi R_0^2}{\sqrt{2}} \lim_{L_1 \to \infty} \left( \int_{0}^{L_1} \frac{ds}{B} \left( \int_{0}^{L_1} \frac{ds}{B} |\nabla \psi| \right)^{-2} \right) \times \int_{\frac{B_{\text{min}}}{B_0}}^{\frac{B_{\text{max}}}{B_0}} db \sum_{j=1}^{j_{\text{max}}} \frac{\tilde{H}_j^2}{I_j} \tag{1}
\]

and

\[
\tilde{H}_j = \frac{1}{b'} \int_{b'_{\text{min}}}^{b'_{\text{max}}} \frac{db'}{B} \sqrt{b' - \frac{B}{B_0} \left( 4 \frac{B}{B_0} - 1 \right) |\nabla \psi| k_G}, \\
I_j = \int_{b'_{\text{min}}}^{b'_{\text{max}}} \frac{db'}{B} \sqrt{1 - \frac{B}{B_0 b'}}, \tag{2}
\]

where

- \( R_0 \) = major radius of the torus
- \( B_0 \) = reference magnetic field
- \( \psi \) = magnetic surfaces label
- \( k_G = (\tilde{h} \times (\tilde{h} \cdot \nabla) \tilde{h}) \cdot \nabla \psi / |\nabla \psi| \) = geodesic curvature of a magnetic field line with unit vector \( \tilde{h} = \tilde{B}/B \).

The quantity \( \varepsilon_{\text{eff}} \) is calculated by integration over the magnetic field line length \( s \), over the sufficiently large interval \( 0 \) to \( L_1 \), and by integration over the perpendicular adiabatic invariant of trapped particles \( I_j \) by means of the variable \( b' \). Here, \( B_{\text{min}} \) and \( B_{\text{max}} \) are the minimum and maximum values of \( B \) within the interval 0 to \( L_1 \). The quantities \( s_{\text{min}} \) and \( s_{\text{max}} \) within the sum over \( j \) in expressions (1) and (2) correspond to the turning points of trapped particles.

Note that expressions (1) and (2) must be supplemented with magnetic field line equations as well as with equations for the vector \( \tilde{P} = \nabla \psi \) (see Refs. 2 and 7):

\[
\frac{dp_i}{ds} = -\frac{\partial B^i}{\partial \xi^l} P_j, \tag{3}
\]

where

- \( B^i = \text{contravariant components of} \tilde{B} \text{ in real-space coordinates} \xi^l \)
- \( P_j = \partial \psi / \partial \xi^l = \text{covariant components of} \tilde{P} \).

The magnetic field of the configurations under consideration is presented in real-space coordinates. For such a presentation, the technique\(^2\) has some specific peculiarities as compared with transport calculations in magnetic coordinates. Expressions (1) and (2) are derived in Ref. 2 in an arbitrary real-space coordinate system but not in magnetic coordinates. In this case, the \( \psi \) function (magnetic surface label) represents an unambiguous integral of the magnetic field line equations but not directly a toroidal flux. For an arbitrary toroidal magnetic field without axial symmetry, this integral (magnetic surface integral) exists in regions where closed magnetic surfaces can be found, namely, intact and island surfaces. As for the \( \psi \) function, only \( \nabla \psi \) is necessary for calculating expressions (1) and (2). The value of \( \nabla \psi \) can be calculated with the help of Eq. (3), which should be solved concurrently with the magnetic field line equations. Using Eq. (3) allows one to compute \( \varepsilon_{\text{eff}}^2 \) with the help of the field line following the code only, for which there is no difference in the studies for nonisland and island magnetic surfaces. Such an approach allows one to avoid transforming the magnetic field to magnetic coordinates if this field is originally available in real-space coordinates. In real-space coordinates, the existence of the \( \psi \) integral can be confirmed by computing \( \nabla \psi \) by using integration along the magnetic field lines for a sufficiently large integration interval such that the magnetic field line entirely covers the magnetic surface. In our computations, field lines were followed 250 times around the torus and even more times for island magnetic surfaces. One can find a more detailed discussion of \( \nabla \psi \) calculation in Ref. 7.

Expression (1) can be presented as a product of two parts. The first part is \( 1/|\nabla \psi|^2 \) with \( |\nabla \psi| \) being averaged over the magnetic field line as follows:

\[
|\nabla \psi| = \lim_{L_1 \to \infty} \left( \int_{0}^{L_1} \frac{ds}{B} \right)^{-1} \int_{0}^{L_1} \frac{ds}{B} |\nabla \psi|.
\]

The second part, which is now denoted as \( \varepsilon_{\text{eff}}^2 \) (see Ref. 8), is an averaged (in an analogous way) combination
of terms containing $|\nabla \psi| / k_\omega$. For the same magnetic surface under consideration, these two parts can vary with changing the starting point of integration as well as the starting value of $\nabla \psi$ at this point. Therefore, separately, the values of these two parts do not give correct information about the effective ripple for the magnetic surface under consideration. Only the product of these parts gives the correct value for $e_{\text{eff}}^{3/2}$, which does not depend on the starting value of $\nabla \psi$ because, as it follows from expressions (1) and (2), $\nabla \psi$ enters the numerator and denominator of this product in the same power.

As it follows from Ref. 2, expressions (1) and (2) can also be used when the magnetic field is presented in magnetic coordinates. In this case, a toroidal magnetic flux can be used as the $\psi$ function. The data sets obtained in magnetic coordinates from three-dimensional magnetohydrodynamic (MHD) equilibrium codes (see, e.g., Ref. 3) should be used in this case. Naturally, these data do not contain information about islands. Therefore, using these data does not allow one to consider the effective ripple for magnetic islands. Examples of effective ripple computations with the use of this technique can be found in Refs. 2 and 9 (in real-space coordinates), Refs. 10 and 11 (in real-space coordinates and in magnetic coordinates), and Ref. 12 (in real-space coordinates).

III. MAGNETIC FIELD CONFIGURATION

Previous numerical investigations\(^1\) of the U-3M magnetic configuration allow us to draw the following conclusions. If the confining magnetic field is unperturbed, and if we assume that there exists a magnetic well and the plasma confinement region is sufficiently large, then a promising mode of operation for U-3M is the mode characterized by $B_\perp / B_0 \approx 1.2\%$, where $B_0$ is the toroidal magnetic field, $B_\perp$ is the total vertical field generated by the helical and compensating windings, and the direction of $B_\perp$ is such that the magnetic configuration is shifted toward the outer circumference of the torus (outward-shifted configuration). However, in this case, the rotational transform $\iota$ has a minimum ($\iota \approx \frac{1}{4}$) at a certain distance from the magnetic axis and increases in two directions: toward the axis of the magnetic configuration and toward its boundary. On the other hand, experimental studies of magnetic surfaces indicate that because of the presence of magnetic islands, the plasma confinement region is not as large as predicted by the above calculations. A comparison of the experimentally observed family of magnetic surfaces with that calculated numerically in Ref. 1 for the operating mode characterized by $B_\perp / B_0 \approx 1.2\%$ and by a small eccentricity $d = 2$ mm of the external compensating coils suggests that the true picture of magnetic surfaces can be attributed, in particular, to this type of asymmetry. Pronounced magnetic islands corresponding to the $\iota = \frac{1}{4}$ surfaces were observed in experiments carried out under the above conditions.\(^1\)

These conditions are used now in subsequent calculations carried out in a cylindrical coordinate system $(\rho, \varphi, z)$ whose $z$ axis coincides with the major axis of the torus for the parameters $d = 2$ mm and $B_\perp / B_0 \approx 1.2\%$. The vacuum magnetic field generated by the torsatron helical coils is modeled similarly to the way it was done in Ref. 1 for toroidal harmonic analysis of the helical magnetic field (finite series of toroidal harmonics expressed in terms of the Legendre associated functions). The only difference is that to increase the accuracy of the calculations, 33 harmonics are used instead of the 24 harmonics used in Ref. 1. The amplitudes of these harmonics were calculated by modeling each helical coil by 20 current filaments distributed in five layers. The magnetic field generated by compensating coils is modeled in the same fashion as in Ref. 1.

Figure 1 shows the Poincaré plots computed by means of the above representation of the magnetic field in the $\varphi = 0$ plane (Fig. 1a), after one-half of a helical field period (Fig. 1b), and after one-half of a circuit around the $z$ axis (Fig. 1c, $\varphi = \pi$). Good agreement is seen between the calculation results shown in Fig. 1a, and those in Fig. 15(b) of Ref. 1. With $\iota = \frac{1}{4}$, one can see two main chains of magnetic islands located inside and outside a narrow region of regular magnetic surfaces with $\iota \leq \frac{1}{4}$ (every chain consists of four island groups). Tracing the magnetic field lines shows that the outer chain with $\iota = \frac{1}{4}$ magnetic islands is enclosed by a narrow layer consisting of chains of numerous very small magnetic islands. For comparison, Fig. 2 displays the Poincaré plots of the magnetic surfaces for an unperturbed magnetic configuration ($d = 0$) in the $\varphi = 0$ plane (Fig. 2a) and after a quarter (Fig. 2b) and a half (Fig. 2c) of a helical field period. Figure 2a is close to that presented in Fig. 4(b) of Ref. 1.

IV. COMPUTATIONAL RESULTS

Results of $e_{\text{eff}}^{3/2}$ computations (except those for islands) are shown in Fig. 3. These results are presented as functions of the ratio $r/a$, where $r$ is the mean radius of a given magnetic surface and $a$ is the mean radius of the outermost magnetic surface for the unperturbed magnetic configuration ($a \approx 10.7$ cm, see Fig. 2). The thick curve in Fig. 3 corresponds to the case of a magnetic configuration in the presence of magnetic islands, and the thin curve corresponds to an unperturbed magnetic configuration. From Fig. 3 follows that the curves for the $e_{\text{eff}}^{3/2}$ factor practically coincide in those regions of the magnetic configurations where the magnetic island structure is absent. So, for the significant parts of the magnetic configurations corresponding to the $0 < r/a < 0.4$ region, the effective ripple $e_{\text{eff}}^{3/2}$ for both configurations is
Fig. 1. Magnetic surface cross section for the configuration with small eccentricity ($d = 2$ mm) of the compensating coils.

Fig. 2. Magnetic surface cross section for unperturbed magnetic configuration ($d = 0$).
approximately within the limits 0.006 to 0.013, and it is within the limits 0.019 to 0.025 for the 0.54 < r/a < 0.66 region. On the other hand, it follows from the results that the configuration with magnetic islands is characterized by increased values of the effective ripple inside the magnetic islands. The value of e_{eff}^{3/2} in this case (not shown in Fig. 3) increases to 0.03 to 0.06 and reaches values commensurable with the e_{eff}^{3/2} value near r/a = 1. This is true for the outer island chain; for the inner island chain, e_{eff}^{3/2} is somewhat smaller. As follows from the computational results for the outer region in Fig. 3, for rather large r/a values, e_{eff}^{3/2} increases to its maximum magnitude: e_{eff}^{3/2} = 0.06. So, one finds larger values in the magnetic island region of the configuration with a small eccentricity in the vertical field coils as compared with the e_{eff}^{3/2} values for the unperturbed magnetic configuration. Note that the e_{eff}^{3/2} value of 0.06 at the edge of the configurations is comparable with that for the edge of the standard Compact Helical System configuration.

Some comments should be added with respect to the results obtained. From the modulation of B along the magnetic field line, the magnetic field ripple e_{h} can be found. Using this quantity, one finds that for the unperturbed magnetic configuration, the e_{eff}^{3/2} value is 1.5 times larger than e_{h}^{3/2} near the magnetic axis to 6 times larger near the edge of the configuration. So, in this correlation, the computed e_{eff}^{3/2} values are larger than those for a conventional stellarator. This result gives a quantitative evaluation of the consequences of an increased radial \( VB \) drift of ripple-trapped particles for an outward-shifted stellarator configuration, which is the opposite case of \( \sigma \)-optimized stellarator configurations (see, e.g., Refs. 13, 14, and 15).

For a configuration with islands, for the r/a regions without islands, e_{eff}^{3/2} is practically the same as that for the unperturbed configuration. However, inside the islands, e_{eff}^{3/2} is greatly increased. This increase can be explained mainly by a change in the island surface shapes as compared with the shapes of nonisland surfaces.

Near the island separatrix, a rather large increase in e_{eff}^{3/2} follows from the calculations. It also follows from the computations that instead of a separatrix, a thin layer exists with stochastic behavior of the magnetic field lines (with a layer width of 1 to 2 mm). The existence of stochastic layers is a consequence of the nonsymmetry of the magnetic field. It is seen from Fig. 3 that near-separatrix regions with increased e_{eff}^{3/2} are rather thin. Therefore, one can conclude that for a configuration with islands, the confinement properties are determined mainly by the e_{eff}^{3/2} values for the nonisland and island magnetic surfaces (two gaps corresponding to the islands are seen for the thick curve in Fig. 3).

As was pointed out in Sec. II, conclusions about the value of the effective ripple can be made only by using expressions (1) and (2) in total. Nevertheless, an independent consideration of \( \langle |\nabla \psi| \rangle \) is also of interest since a significant change in the shape of the magnetic surfaces manifests itself as a significant change (an increase or a decrease) in the \( \langle |\nabla \psi| \rangle \) value (for the same starting values of \( |\nabla \psi| \)). Therefore, together with e_{eff}^{3/2}, the behavior of \( \langle |\nabla \psi| \rangle \) with changing r/a was also analyzed. Some of the results obtained are discussed here. As was pointed out above for such an analysis, it is important to take into account the starting points of the integration and the starting values of \( |\nabla \psi| \) in these points. First, the calculations were made for the analysis of e_{eff}^{3/2} and \( \langle |\nabla \psi| \rangle \) for the outer island chain. In these calculations, the starting points of the integration were chosen at the outer sides of the magnetic surfaces in the \( \varphi = 0 \) plane (see Fig. 1a) for \( \varphi = 0 \) with \( |\nabla \psi| = 1 \) being the starting value of \( |\nabla \psi| \). It follows from the calculations under such conditions that near the island separatrix, \( \langle |\nabla \psi| \rangle \) decreases for the outer island chain and increases for the inner island chain. It is seen from Fig. 1a that the starting points (\( \varphi = 0 \)) near the inner islands are near a rib of the separatrix, whereas the starting points near the outer islands are far from the ribs. Near the rib of a separatrix, the distances between the neighboring magnetic surfaces are larger (and \( |\nabla \psi| \) is smaller) than these distances (and \( |\nabla \psi| \)) far from the rib. For the same starting \( |\nabla \psi| \) value, this fact leads to an increased value of \( \langle |\nabla \psi| \rangle \) and explains the increased value of \( \langle |\nabla \psi| \rangle \) in the vicinity of the inner islands.

As for the outer islands, it follows from the calculations that when approaching the separatrix from the side between the ribs with the same starting value of \( |\nabla \psi| \), the value of \( \langle |\nabla \psi| \rangle \) decreases, and this decrease correlates with an increase in e_{eff}^{3/2}.

To find e_{eff}^{3/2} for islands in the inner island chain, the calculations were made with the starting points placed in the \( \varphi = \pi \) plane for \( \varphi = 0 \) (Fig. 1c). In this case, the
starting points are between the ribs of the separatrix of the inner islands. For such starting points, it is found that with approaching the separatrix of the inner island chain, $\langle |\nabla \psi| \rangle$ decreases in the same way as for the outer island chain (with the starting points placed in the $p = 0$ plane), and this decrease correlates with an increase in $e_{\text{eff}}^{3/2}$.

It follows from the analysis that near the island separatrix, a strong increase in the effective ripple correlates with a strong change in the magnetic surface shapes.

In conclusion, essentially smaller values of $e_{\text{eff}}^{3/2}$ are found for the U-3M configuration corresponding to the opposite direction of the resulting vertical magnetic field ($B_L/B_0 = -1.2\%$). In this case, $e_{\text{eff}}^{3/2}$ as well as $e_L^{3/2}$ near the magnetic axis are greatly decreased. At the edge of the configuration, $e_{\text{eff}}^{3/2}$ turns out to be approximately six times smaller than that for the configurations considered above. This case corresponds to the inward-shifted magnetic configuration, which is close to “α-optimized" configurations. From the viewpoint of MHD stability, the configuration corresponding to this case seems to be less favorable than the outward-shifted one because of a change in the magnetic well into the magnetic hill (see Ref. 1). However, such a supposition is in conflict with the recent experimental findings for the Large Helical Device (LHD) summarized in Ref. 16. From these LHD results, it follows that an inward-shifted configuration with a magnetic hill has better neoclassical confinement than outward-shifted configurations with a magnetic well. Also, from these results it follows that plasma with an average beta of 3% is stable in the inward-shifted configuration, even though the theoretical stability conditions of Mercier modes and pressure-driven low-n modes are violated. So, it follows from Ref. 16 that MHD stability and good transport properties are compatible in the inward-shifted configuration.

It is of interest also to note that increased transport for the magnetic island regions can take place not only for the $1/\nu$ regime. For example, in the study carried out in Refs. 17 and 18 for the hydrodynamic regime, essentially increased values of the Pfirsch-Schlüter factor ($\gamma_L/\gamma_0$) were found for the island regions.

V. SUMMARY

The $1/\nu$ neoclassical transport regime, in which the transport coefficients increase with decreased particle collision frequency, $\nu$, is considered for an outward-shifted magnetic configuration of an $l = 3$ stellarator magnetic field in the presence of magnetic islands caused by an eccentricity in the helical and compensating coils. To calculate the transport coefficients, a technique based on integration along magnetic field lines in a given stellarator magnetic field is used. In the presence of this inaccuracy in the magnetic system, the magnetic field periodicity is of one field period along the torus in contrast to nine helical field periods along the torus for a precisely manufactured magnetic system. The transport coefficients are presented in a standard form containing factor $e_{\text{eff}}^{3/2}$ (with $e_{\text{eff}}$ being the effective ripple) depending on the magnetic field geometry. For comparison, the effective ripple is also calculated for the case of an unperturbed magnetic configuration. The $e_{\text{eff}}^{3/2}$ values obtained for the magnetic islands and their close vicinity are substantially larger than those for a confining magnetic field unperturbed by errors in the magnetic system. Outside the islands, the $e_{\text{eff}}^{3/2}$ values are close to those for the unperturbed configuration. These values are substantially larger than those for a conventional stellarator as well as those for the inward-shifted magnetic configuration. This increase ranges from a factor 1.6 near the magnetic axis to 6 near the edge of the configuration. Also, because of the errors, the confinement region markedly decreases as compared with the unperturbed configuration. Essentially smaller values of $e_{\text{eff}}^{3/2}$ are found in the calculations for the inward-shifted U-3M configuration.

ACKNOWLEDGMENTS

The authors would like to thank E. D. Volkov and G. G. Lesnyakov for useful discussion of the results obtained.

REFERENCES


______________________________

**Victor Nikolayevich Kalyuzhnyj** (degree, physics, 1971, and PhD, physics and mathematics, 1984, Kharkov State University, Ukraine) is on the senior researcher staff at the Institute of Plasma Physics, National Science Centre, “Kharkov Institute of Physics and Technology,” where he works in the field of plasma physics and controlled nuclear fusion. His research interest is plasma confinement in stellarators.

**Victor Vadimovich Nemov** (degree, electrical engineering, Odessa Polytechnic Institute, Ukraine, 1954, PhD, physics and mathematics, Donetsk State University, Ukraine, 1972; Doctor of Sciences, physics and mathematics, Kharkov State University, Ukraine, 1994) is on the leading researcher staff at the Institute of Plasma Physics, National Science Centre, “Kharkov Institute of Physics and Technology,” where he works in the field of plasma physics and controlled nuclear fusion. His research interest is plasma confinement in stellarators.