

Hamiltonian

$$H = \sum_R (V(R) - \mu) \psi_R^\dagger \psi_R \\ - t \sum_R (\psi_R^\dagger \psi_{R+1} + h.c.) \\ + U \sum_R (\psi_R^\dagger)^2 \psi_R^2$$

$$[\psi_R, \psi_{R'}^\dagger] = \delta_{RR'} \quad (\text{cf. HARMONIC OSCILLATOR})$$

1-DIMENSIONAL LATTICE: $R = 1 \dots N$

PERIODIC BOUNDARY CONDITIONS: $R+N \equiv R$

$V(R)$ TAKE IT ARBITRARY

EXAMPLES

• HARMONIC POTENTIAL $V(R) = \alpha \left(R - \frac{N}{2}\right)^2$
(WHICH IS NON PERIODIC)

• APPROXIMATELY HARMONIC \rightarrow PERIODICITY FORCED

$$V(R) = -2\beta \cos \frac{2\pi R}{N}$$

A

$$\psi(R) = \psi(R) + b_R$$

↑ NUMBER ← BOSONIC OPERATOR

$$[b_R, b_{R'}^\dagger] = \delta_{RR'}$$

INSERT IN H

(1) CONDITION:

TERM LINEAR IN $b(R)$ VANISHES

GROSS-PITAEVSKII EQUATION

$$\left((N(R) - \mu) + U |\psi(R)|^2 \right) \psi(R) + \epsilon (\psi(R+1) + \psi(R)) = 0$$

⇒ SOLVE NUMERICALLY:

- START FROM A $\psi(R) = \psi_0$
- SOLVE EIGENVALUE EQUATION
(FOR EVAL μ , EVEC $\psi(R)$)
SMALLEST μ
- USE NEW $\psi(R)$ AND
REITERATE UNTIL CONVERGENCE

③ The part above could be already enough for a Bachelor, depending on how long it takes

Next step: Bogolubov approximation (is actually also on Books):

TAKE H WITH $N(R)=0$

$\Rightarrow \psi(R) = \psi$ CONSTANT

\Rightarrow SIMPLE EQUATION FOR ψ

- RESTRICT TO TERMS

QUADRATIC IN h
(NEGLECT h^3, h^4)

- FOURIER TRANSFORM

$$h_R = \frac{1}{\sqrt{N}} \sum_k e^{i k R} h_k$$

$$h_R^+ = \frac{1}{\sqrt{N}} \sum_k e^{-i k R} h_k^+$$

- QUADRATIC H WILL CONTAIN TERMS

$$\dots h_k^+ h_n + \dots h_n^+ h_{-n}^+ + h_n h_{-n}$$

• BOGOLUBOV TRANSFORMATION

$$b_k = M_k a_k + N_k^* a_{-k}^+$$

$$b_k^+ = M_k^* a_k^+ + N_k a_{-k}$$

$$(b_{-k}^+ = \dots \text{ JUST PUT } k \rightarrow -k)$$

TAKE M, N REAL

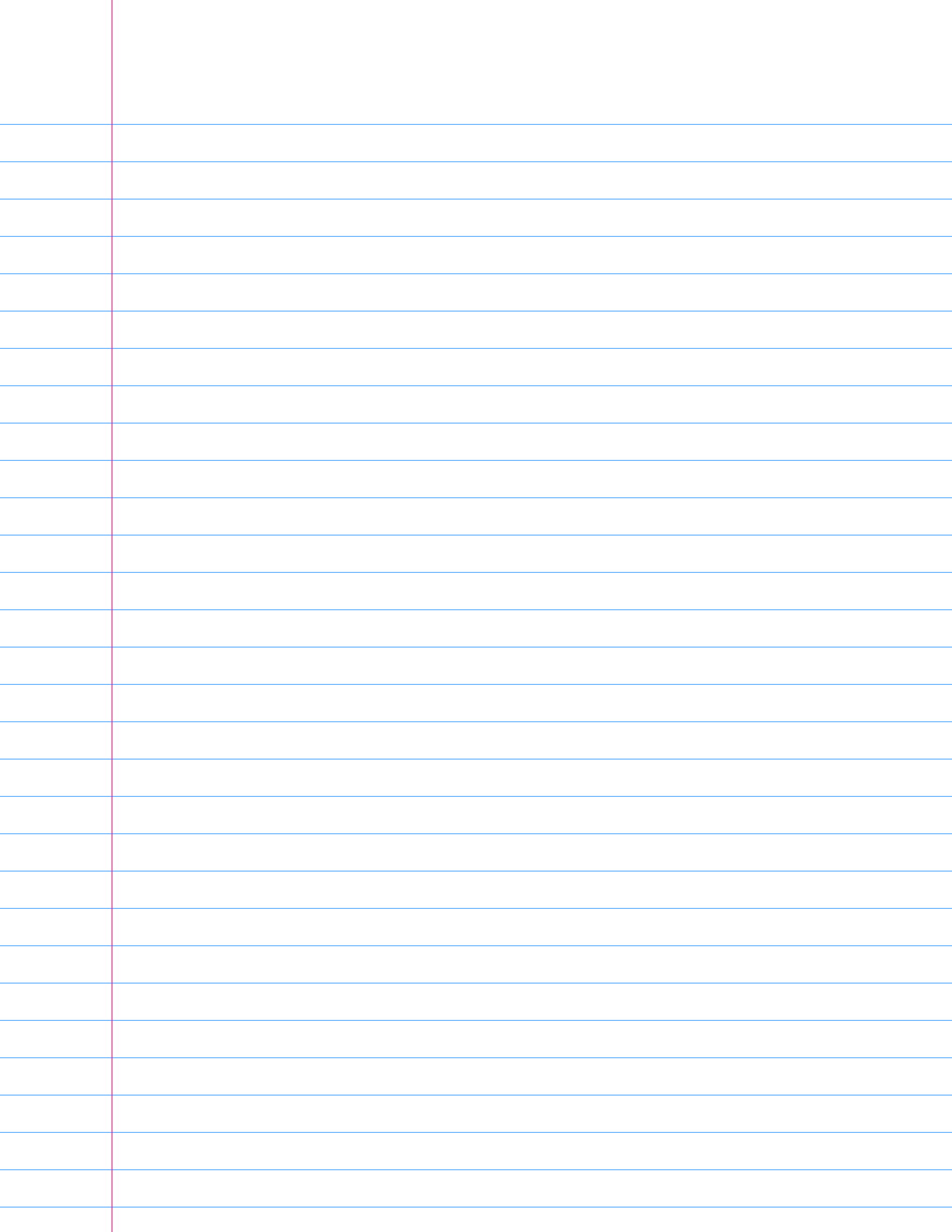
• CHOOSE M_k, N_k SUCH THAT

① COMMUTATION RULES ARE FULLFILLED

$$[a_k, a_{k'}^+] = \delta_{k, k'}$$

② H BECOMES DIAGONAL

$$\text{CONST} + \sum_k \epsilon_k a_k^+ a_k$$



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IN CASE YOU REALLY
DIDN'T HAVE ENOUGH:

CONSIDER $V(R) \neq 0$

AGAIN:

CONSIDER TERM QUADRATIC IN $b(R)$

(NEGLECT 3, 4 ORDER:

BOGOLUBOV APPROXIMATION)

NUMERICALLY SOLVE QUADRATIC HAMILTONIAN

VIA MULTI-MODE BOGOLUBOV TRANSFORMATION

• WRITE IN VECTOR NOTATION

$$B^+ = (b_1^+ \dots b_N^+ \quad b_1 \dots b_N)$$

$$B = (B^+)^+$$

$$H = \text{const.} + B^\dagger M B + \mathcal{O}(\hbar^3)$$

TRANSFORMATION

$$B = U P$$

$$P^\dagger \equiv (p_1^\dagger \dots p_n^\dagger \quad p_1 \dots p_n)$$

MUST SATISFY CORRECT COMMUTATION RULES

$$[p_i, p_j^\dagger] = \delta_{ij}$$

$$[p_i, p_j] = 0$$

IN MATRIX NOTATION

$$[B, B^\dagger] = [P, P^\dagger] = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array} \right) \equiv S$$

CONDITION

$$[B, B^\dagger] = U[P, P^\dagger]U^\dagger = U S U^\dagger \doteq S$$

2^o CONDITION :

HAMILTONIAN DIAGONAL

$$B^\dagger M B = P^\dagger U^\dagger M U P$$

$$U^\dagger M U = D \quad \text{DIAGONAL MATRIX}$$

HOW TO SOLVE : SEE KNAP
ET. AL. (STARTING BELOW
E9.75 UNTIL E9.18 :

SEE ALSO APPENDIX A

NOTICE : HERE $S' = S$)