

$$|\psi_{ST}\rangle = a(t)|0\rangle|\tilde{0}\rangle + b(t)|1\rangle|\tilde{1}\rangle + \frac{1}{\sqrt{2}}c(t)(|0\rangle|\tilde{1}\rangle + |1\rangle|\tilde{0}\rangle) + \frac{1}{\sqrt{2}}d(t)(|0\rangle|\tilde{1}\rangle - |1\rangle|\tilde{0}\rangle)$$

$$\begin{aligned} |\psi_{ST}\rangle\langle\psi_{ST}| &= a^*b^*|0\rangle\langle 0|\tilde{0}\langle\tilde{1}| + \frac{1}{\sqrt{2}}ac^*(|0\rangle\langle 0|\tilde{0}\langle\tilde{1}| + |0\rangle\langle 1|\tilde{0}\langle\tilde{0}|) + \frac{1}{\sqrt{2}}ad^*(|0\rangle\langle 0|\tilde{0}\langle\tilde{1}| - |0\rangle\langle 1|\tilde{0}\langle\tilde{0}|) \\ &+ b^*a^*|1\rangle\langle 1|\tilde{1}\langle\tilde{0}| + \frac{1}{\sqrt{2}}bc^*(|1\rangle\langle 1|\tilde{1}\langle\tilde{0}| + |1\rangle\langle 0|\tilde{1}\langle\tilde{1}|) + \frac{1}{\sqrt{2}}bd^*(|1\rangle\langle 1|\tilde{1}\langle\tilde{0}| - |1\rangle\langle 0|\tilde{1}\langle\tilde{1}|) \\ &+ \frac{1}{\sqrt{2}}ca^*(|0\rangle\langle 1|\tilde{0}\langle\tilde{0}| + |1\rangle\langle 0|\tilde{0}\langle\tilde{0}|) + \frac{1}{\sqrt{2}}cb^*(|0\rangle\langle 1|\tilde{1}\langle\tilde{1}| + |1\rangle\langle 0|\tilde{1}\langle\tilde{1}|) \\ &+ \frac{1}{2}cd^*(|0\rangle\langle 1|\tilde{1}\langle\tilde{0}| - |0\rangle\langle 1|\tilde{0}\langle\tilde{1}| + |1\rangle\langle 0|\tilde{1}\langle\tilde{0}| - |1\rangle\langle 0|\tilde{0}\langle\tilde{1}|) \\ &+ \frac{1}{\sqrt{2}}da^*(|0\rangle\langle 1|\tilde{0}\langle\tilde{1}| - |1\rangle\langle 0|\tilde{0}\langle\tilde{0}|) + \frac{1}{\sqrt{2}}db^*(|0\rangle\langle 1|\tilde{1}\langle\tilde{1}| - |1\rangle\langle 0|\tilde{1}\langle\tilde{1}|) \\ &+ \frac{1}{2}dc^*(|0\rangle\langle 1|\tilde{1}\langle\tilde{0}| + |0\rangle\langle 1|\tilde{0}\langle\tilde{1}| - |1\rangle\langle 0|\tilde{1}\langle\tilde{0}| - |1\rangle\langle 0|\tilde{0}\langle\tilde{1}|) \\ &+ |a|^2|0\rangle\langle 0|\tilde{0}\langle\tilde{0}| + |b|^2|1\rangle\langle 1|\tilde{1}\langle\tilde{1}| + |c|^2\frac{1}{2}(|0\rangle\langle 1|\tilde{1}\langle\tilde{0}| + |0\rangle\langle 1|\tilde{0}\langle\tilde{1}| + |1\rangle\langle 0|\tilde{1}\langle\tilde{0}| \\ &+ |1\rangle\langle 0|\tilde{0}\langle\tilde{1}|) + |d|^2\frac{1}{2}(|0\rangle\langle 1|\tilde{1}\langle\tilde{0}| - |0\rangle\langle 1|\tilde{0}\langle\tilde{1}| - |1\rangle\langle 0|\tilde{1}\langle\tilde{0}| + |1\rangle\langle 0|\tilde{0}\langle\tilde{1}|) \end{aligned}$$

$$\hat{\rho}_S = \sum_e \langle \phi_e | \psi_{ST} \psi_{ST}^\dagger | \phi_e \rangle$$

$$\begin{aligned} \hat{\rho}_S^{(0)} &= \frac{1}{\sqrt{2}}ac^*|0\rangle\langle 1| - \frac{1}{\sqrt{2}}ad^*|0\rangle\langle 1| + \frac{1}{\sqrt{2}}ca^*|1\rangle\langle 0| - \frac{1}{2}cd^*|1\rangle\langle 1| - \frac{1}{\sqrt{2}}dc^*|1\rangle\langle 0| - \frac{1}{2}dc^*|1\rangle\langle 1| \\ &+ |a|^2|0\rangle\langle 0| + \frac{1}{2}|c|^2|1\rangle\langle 1| + \frac{1}{2}|d|^2|1\rangle\langle 1| \end{aligned}$$

$$\begin{aligned} \hat{\rho}_S^{(1)} &= \frac{1}{\sqrt{2}}bc^*|1\rangle\langle 0| + \frac{1}{\sqrt{2}}bd^*|1\rangle\langle 0| + \frac{1}{\sqrt{2}}cb^*|0\rangle\langle 1| + \frac{1}{2}cd^*|0\rangle\langle 0| + \frac{1}{\sqrt{2}}db^*|0\rangle\langle 1| \\ &+ \frac{1}{2}dc^*|0\rangle\langle 0| + |b|^2|1\rangle\langle 1| + \frac{1}{2}|c|^2|0\rangle\langle 0| + \frac{1}{2}|d|^2|0\rangle\langle 0| \end{aligned}$$

$$\rho_S = \begin{pmatrix} |a|^2 + \frac{1}{2}cd^* + \frac{1}{2}dc^* + \frac{1}{2}(|c|^2 + |d|^2) & \frac{1}{\sqrt{2}}ac^* - \frac{1}{\sqrt{2}}ad^* + \frac{1}{\sqrt{2}}cb^* + \frac{1}{\sqrt{2}}db^* \\ \frac{1}{\sqrt{2}}ca^* - \frac{1}{\sqrt{2}}da^* + \frac{1}{\sqrt{2}}bc^* + \frac{1}{\sqrt{2}}bd^* & -\frac{1}{2}cd^* - \frac{1}{2}dc^* + \frac{1}{2}(|c|^2 + |d|^2) + |b|^2 \end{pmatrix}$$