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THE MECHANICS OF ONE-, TWO- AND THREE-BODY PROBLEMS

BERTRAND'S THEOREM AND LAGRANGE POINTS

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Contents

1 Introduction

The main goal of this bachelor thesis is to shed some light on the vast theory of one- and more-body problems, the various difficulties they have posed to mathematicians and physicists in the 18^{th} century and all the way up until the present day, as well as some methods that have been used to get quantitative result.

To start with, I will try to give an introduction of the necessary prerequisites that are needed to be able to dive into the matter. In the following, I will give a brief overview of one-body and two-body problems along with some surprising mathematical results that go along with them. Finally, I will present the notorious three-body problem of which the general case has up until now defied any attempt of coming up with a solution in closed analytical form. I will therefore present some special solutions and in the end discuss one of them more in depth from an analytical point of view.

However, as the field of many-body problems is way too big to be comprehensively covered within one bachelor thesis, some major topics such as perturbation theory will not even be touched on. Other topics that are covered could also be much more elaborated, given that the matter has stumped scientists and mathematicians over several centuries and there is a huge accumulation of ideas and theorems that have arisen over that time. What I nonetheless would like to reach with this thesis is to give a broad insight into an area of research that goes considerably beyond what is taught in a bachelor's class on theoretical mechanics and every here and there also go into detail by discussing some rigorous analytical derivations of the subject at hand.

2 Prerequisites in theoretical mechanics

As I generally like the notion of building up knowledge on pre-existing knowledge and I dislike the idea of certain theorems or ideas just seemingly falling out of the sky and being presented to the reader without any justification for their validity, I would like to start my thesis from the very basis of physics and try to work my way up to my actual topic of investigation primarily building on what I have already written before. Of course, this is a quite ambitious endeavour and bound to failure at some point, as a thorough and detailed analysis of all the concepts and mathematics needed to get to three-body-problems, let alone to one-body-problems, would fill books of a size way bigger than the expected scope of this thesis. Nonetheless, it is often possible to get from one point to another using quite stringent and consistent argumentation even without presenting the whole picture of what is going on. Whenever possible, I would therefore like to try and take this route [1] and whenever I feel that this is not possible, I will try to direct the interested reader to some useful references that can help him catch up on the necessary concepts to be able to move on without a big gap in understanding.

So let me start with the very beginning. Physics is the study of nature and as such, it tries to make observations and measurements of the world around us, deduce any regularities and formulate laws describing them. The language it uses to do so is usually the one of mathematics and so one of the first things that have to be dealt with when describing measurements in a mathematical setting is the question of units. When you measure the power of a light source to be 500 watt or when you measure the wavelength of the emitted light to be 500 nano metres, you are using the same number to describe two properties of the light source. In both cases it is the number 500, but the concepts entailed are of course very different. Just as important as the number is therefore also the unit that goes along with it (in this case watt or nano meters respectively). It may seem as if there is a seemingly unlimited number of different such units, but quite astonishingly it can really be broken down to just a handful. The International System of units (abbreviated SI from the french Système International (d'unités)) lists only seven different units, every other one can be written as a combination thereof. I want to take three of them as a starting point for my thesis: the metre denoted by "m" as a unit of length, the kilogram denoted by "kg" as a unit of weight and the second denoted by "s" as a unit of time [2]. To be very exact, I would actually have to state their official definition, but as some of them are more cumbersome than one might imagine and as I suppose that anyone reading this will have a good idea of the concepts these units represent anyway, I decided to start from there.

2.1 The force law and the gravitational law

The first thing to do with these units is to combine them to give more complicated units. The simplest such combination would be the measure of metres per second (m/s) as a unit of speed. It tells you how many metres an object will have covered before one second has passed. If an object continuously changes its speed, the next question to ask would be how fast this change of speed is brought about. How much does the speed of an object change over the course of one second? The answer to this question is given in units of speed per second, that is metres per second per second or metres per second squared (m/s/s=m/s^2). The quantity this unit describes is called acceleration. Next, we are interested in the idea of force and this is where the third elementary unit, the kilogram, is brought in. On the surface of the earth, a free falling object is accelerated by roughly $9,81$ m/s^2[3]. Now, if you hold in your hand say an apple, you have to withstand the force that would naturally accelerate it to the ground. If you cut that apple into six equal pieces and only

[1] However, this is only true for the physical arguments that are brought forward. Concerning the mathematical part of my thesis, I unfortunately figured out while writing that I would probably have to quadruple the length of this thesis if I really wanted to explain and justify every operation that is not taught on a pre-university level.

[2] In fact, these three units alone would suffice to define the remaining four units of the SI, the others were just added on for reasons of practicability.

[3] In theory, this is true for an elephant as well as for a feather. The reason for the feather actually accelerating much slower is merely the air resistance hindering it to pick up speed and being way more efficient in doing so with a feather than with an elephant.

keep one of them in your hand, that force drops to one sixth of the original force, the reason being that the mass that you are holding drops with equal proportions. If you go to the moon however, the force of the entire apple also drops to about a sixth of the force on earth. What is happening now? While the mass of the apple remains the same (it is still made up of the same atoms), a free falling object on the moon is only accelerated by roughly 1.62 m/s^2, about one sixth of the acceleration on earth. Apparently, both of these factors together, meaning mass and acceleration, account for the force. The important thing is that they do so exclusively, meaning there is no other factor influencing it (this last sentence is an empirical finding, but probably one of the most experimentally substantiated one in all of physics). If you double the mass, the resulting force doubles, if you double the acceleration, the resulting force also doubles, if you double both mass and acceleration, the resulting force quadruples. It follows that the force is proportional to both mass and acceleration and that it can be written as a product thereof.

$$F = ma \tag{1}$$

In equation 1, F stands for force, m stands for mass and a stands for acceleration. The unit of force is a multiplication of the units of mass and acceleration: $[F] = [m][a] = \text{kg} \cdot \text{m/s}^2$ (the square brackets denote that what is meant is the unit of whatever is written inside the brackets). The fact that with time it can get quite cumbersome to always spell out "$\text{kg} \cdot \text{m/s}^2$" made people come up with a different name for this unit. It is called Newton or in short form N in honour of Sir Isaac Newton who first formulated this relationship in mathematical terms in the late 17^{th} century. The equation is so astonishingly simple and was yet so ground breaking in the development of the natural sciences that it can hardly be beaten therein by any other equation that the whole realm of physics has since then brought forth.

The takeaway up until now is this: if we know the force acting on an object, we can calculate its acceleration by merely dividing by its mass. But how can we now determine the force in the first place? Using a somewhat circular argument, one could answer by saying that you simply have to measure the acceleration of the object and then deduce the force by multiplying with its mass. But this is not the answer we are looking for. In the case of many-body problems, we want to be able to predict the objects' motion rather than measure it and only get the force in retrospect. In the following, I would like to therefore give an explanation of the theory of gravitation, which will answer the question in a more satisfactory way. The depiction will be somewhat unorthodox and the analogy used might in future turn out to be inadequate, but as I quite like the parallels that can be drawn therewith and as the current situation is such that the deeper origin of gravitation is far from being well understood anyway, I thought I might just as well go for it. It uses the picture of the sun and its radiating heat to illustrate the concept and goes as follows:
Let's imagine the radiation emanating from the sun as a collection of photons, the latter being discrete bundles of light, each carrying the same quantized amount of energy. Let us furthermore suppose that the emission of these photons is radially symmetrical, meaning it is evenly distributed in all directions. If the total power leaving the sun P_{tot} is known, that is if we know the number of photons that is emitted every second, how could you determine the intensity of photons that is to be expected a distance r away from the sun? Assuming that all of the photons leaving the sun travel off in a straight line out into infinity, we would be justified in reasoning that the same number of them leaving the sun in a set time interval would also have to transverse the surface of a hypothetical sphere centred at the position of the sun with radius r during that same time interval [4] (if that was not true and the sun emitted more photons than would leave the sphere, there would be a pile-up of photons somewhere inside the sphere with ever more and more photons, a trend that could not physically be sustained indefinitely) . Due to the radial symmetry of the situation, the density of photons can be expected to be identical on every point of the spheres surface, such that the power going through a unit of area perpendicular to the sun can be written as P_{tot} divided by the surface area of our fictitious sphere, that is:

$$P(r) = \frac{P_{tot}}{4r^2\pi}. \tag{2}$$

[4] In fact, this would hold for any sphere with any radius, as long as the sun is contained in its inside.

In just the same way, the gravitational force exerted by one point mass [5] on another is directly proportional to the inverse square of the distance between them, as if there were "gravitational photons" or gravitational waves that thin out over distance in precisely such a way that they can be thought of as emanating from the point mass and travelling out in a straight line. The bigger the mass in question, the more photons, also in perfect proportionality. The only difference to the sun analogy is the fact that in the latter, the power arriving at a certain point in space is solely dependent on the sun and not the receiver of its power, whereas in the case of gravitational force, it is proportional to both masses involved (notice the interchangeability of perspective: both masses can be thought of as the "emitter" or the "receiver" and in a sense they fulfil both descriptions simultaneously). And that's about it, all that is left is a factor of proportionality, which has to be figured out empirically and is called the universal gravitational constant G. In short [1, p. 64 and 72]:

$$F = G\frac{m_1 m_2}{r^2}, \qquad G = 6.67 \cdot 10^{-11}\frac{\mathrm{m}^3}{\mathrm{kg\ s}^2}, \tag{3}$$

where m_1 and m_2 are the two point masses attracting each other. The direction of the force is along the line connecting the point masses, so written in vector form, the gravitational law reads:

$$\vec{F} = \pm G\frac{m_1 m_2}{r^3}\vec{r}_{12}. \tag{4}$$

The vector $\vec{r}_{12}$ [6] in equation 4 is the vector pointing from the position of the first point mass to that of the second one and the ± sign reflects the fact that the direction of the force depends on whether you are talking about the force acting on point mass 1 (+) or point mass 2 (-).

With equation 1 and equation 4 at hand, the basic endeavour the topic of my thesis is all about is this: you have a certain number of objects or point masses situated somewhere in space. You then work out the gravitational forces that act on each object due to all the other ones. Finally, you divide the force acting on each object by its mass, yielding its acceleration and thereby allowing you to determine its subsequent trajectory. And in principle, that's it. This is all the physics that is necessary to understand everything about three-body-problems or even n-body problems, all the rest is just mathematics. One could lock oneself into a closet with a piece of paper and a pen and derive everything else there is to know, as there are no more experiments necessary to obtain empirical results. However, the further proceedings are still not as simple as one might suspect from my description at the beginning of the paragraph. The catch of it all is this: while it is really easy to calculate the acceleration for a certain object at a particular point in time, the distance from all the other objects that cause the gravitational force is changing the moment you let one object move, meaning you permanently have to adapt your calculated acceleration to the current position of all the other objects. This is the real difficulty, which in the general case of arbitrarily many objects has not yet been resolved in closed analytical form [7] and even in the case of one or two objects cannot easily be resolved without some more general contemplations on the nature of n-body problems. This is what the rest of this chapter will deal with before diving straight into one-, two-and three-body problems in chapters 3 through 5.

2.2 Conservation laws

In the previous subsection I started off with the three elementary units metre, second and kilogram and then combined them to give more complicated units. Metres and seconds were combined to give metres per

[5] A quick note on the use of the terms mass, point mass and body of mass. While the mathematical equations used in this thesis are only correct for point masses or spherical bodies of mass with uniform mass density, the distance between two of the bodies of mass considered is usually so large in comparison to their volume that the exact shape of that volume and the mass density within it can be neglected. In this thesis, the terms mass, point mass and body of mass will therefore be used interchangeably.

[6] Throughout this thesis, a vector will be represented by an arrow on top of the respective variable, while the length of that vector will be written without the arrow (e.g. $\vec{r}_{12}$ denotes a vector quantity, while r_{12} denotes a scalar quantity).

[7] Simulations are of course always possible and they can be made with quite high precision, but that will have to be the topic of some other thesis.

second (m/s), which then turned to metres per second squared (m/s^2) and eventually to newtons (kg $\cdot$ m/s^2 = N). Let me take this one step further. Suppose you have a mass of 1 kilogram sitting on top of a table. If the table wasn't there, it would be accelerated towards the ground with 9.81m/s^2, so it is pressing down on the table with a force of 9.81 N. Next to it lies a second object with a mass of 2 kilograms, so pressing down on the table with 2 kg $\cdot$ 9.81 m/s^2 = 19.62 N. If someone takes the one kilogram object and lifts it up by one metre, he has to use some energy to do so. If he lifts it by two meters, the necessary energy doubles. However, it turns out that if he lifts the second object by only one metre, the required energy again doubles in comparison to the original case. To describe the energy put into lifting the objects, we can therefore simply multiply the force against which one has to come up with the distance over which this is done. The resulting unit is called Joules (kg m/s$^2\cdot$ m = N $\cdot$ m = J)[8].

The type of energy I was describing in the previous paragraph, the one that an object acquires when it is lifted up against the gravitational field of the earth (or against any other conservative force field [9]), is called **potential energy**. A second kind of energy that is of prime importance is the so-called **kinetic energy**, describing the energy an object has due to the speed with which it is moving. Suppose an object with 100 kg is moving with a speed of v = 20 m/s. What is its kinetic energy? The first necessary observation to be made is that it does not matter how the object reached its final speed, whether it accelerated from 10 m/s or whether it was previously even going faster and decelerated to 20 m/s. The current speed is all that matters for the current kinetic energy, so we might just as well assume for reasons of simplicity that the object was initially at rest and then picked up speed with constant acceleration until it reached v_{final} = 20 m/s. With that assumption, we can now use the definition from above, namely that the energy is found by the force that has to be applied to the object to make it accelerate multiplied by the distance over which it is applied. From equation 1, we know that $F = ma = m\ v_{final}/\tau$ if τ describes the time period it took the object to accelerate [10]. Next, we need to find the distance over which this force was applied. This can either be done by coming up with a linear function describing the time-dependent speed of the object and then integrating over time or by noticing that the distance covered when accelerating with constant acceleration from 0 m/s to 20 m/s is the same as the distance covered by constantly going with 10 m/s (the average of 0 m/s and 20 m/s) over the same time period. But that distance is just $\tau \cdot v_{final}/2$, so the energy we are looking for is simply:

$$E_{kin} = \text{force} \cdot \text{distance} = m\frac{v_{final}}{\tau} \cdot \tau\frac{v_{final}}{2} = \frac{mv_{final}^2}{2}. \tag{5}$$

It is these two types of energy, kinetic and potential, that are important to us when discussing n-body problems and the realisation that their sum is a quantity that is always conserved is of an importance whose significance in not only the theory of n-body problems, but in virtually any branch of physics cannot be underestimated [11]. As a matter of fact, the way I introduced the concept of kinetic energy is almost by definition already a half proof of the conservation of energy. Just imagine lifting an object against the gravitational field of the earth up from the floor, thereby increasing its potential energy. If you then let go of the object and calculate the speed it will have picked up by the time it returns to the floor using the argumentation from the previous paragraph, you will find the equivalence straight away. However, for the sake of being more general and also because it will be needed for the subsequent chapters of the thesis, I will first introduce another concept about force fields and only then derive the conservation of energy in a more general form.

[8] One might argue that this is more empirical experiments that I am using, in contrary to me saying in the previous subsection that no more empiricism is needed. I still stand by the statement that the force law and the gravitational law are in principle enough to solve any n-body problem, but introducing the concept of energy just makes things a lot easier and allows me to not take the route of the lagrange equations of the second kind, which would be a lot harder to derive.

[9] the defining features of a conservative force field will be explained in the following paragraphs.

[10] Notice that this force is constant over time, as we assumed a constant acceleration.

[11] The concept of conservation of energy in its most general form is actually much more far-reaching than this, stating that energy in whatever form imaginable can never be created or destroyed, it can only be converted to other forms. The trade-off between potential and kinetic energy of the subject at hand is a very special case in which only those two forms of energy are involved and no considerations about frictional effects or other forms of energy have to be made.

A force field in its most general sense is a function that ascribes a certain force with magnitude and direction to every point in space. In the case of gravity, it is described by equation 4. An important property about a force field is whether it is conservative, meaning whether the energy necessary to move an object from point 1 to point 2 is independent of the trajectory taken. As the energy is a multiplication of force and distance and in the case of continuously changing forces has to be written using an integral, the mathematical equivalent of the last sentence is

$$\vec{F} \text{ is conservative} \quad \Rightarrow \quad \int_{1_{(A)}}^{2} \vec{F} d\vec{r} = \int_{1_{(B)}}^{2} \vec{F} d\vec{r}, \tag{6}$$

where the first integral is taken over path (A) and the second one over path (B), both trajectories starting at 1 and ending at 2 but taking different routes to get there. Some simple rearrangements lead to the equivalent condition that any loop integral has to vanish in order for a force field to be conservative [2, p. 256].

$$\int_{1_{(A)}}^{2} \vec{F} d\vec{r} - \int_{1_{(B)}}^{2} \vec{F} d\vec{r} = 0 \quad \Rightarrow \quad \int_{1_{(A)}}^{2} \vec{F} d\vec{r} + \int_{2_{(B)}}^{1} \vec{F} d\vec{r} = 0 \quad \Rightarrow \quad \oint \vec{F} d\vec{r} = 0 \tag{7}$$

What equation 7 is in effect saying is that if you have a conservative force field and you position an object at an arbitrary location therein, you can move it around in whatever way you like, as soon as the object returns to its original position, the net energy expenses for its movement are zero. You can picture it like a small toy car that is powered by a battery: while at some points of the trajectory the battery is needed to keep the car moving, at other points the car will automatically accumulate kinetic energy which can be used to recharge the battery. A conservative force field then implies that whenever returning to its original position, the battery will be equally charged [12]. It is therefore possible to ascribe a scalar value to every point in space describing how much energy is necessary to get there from a certain reference point (or describing how empty the battery of the car would be at that position). In physical terms, such a function is called a potential, building the link to potential energy explained earlier. The potential difference between two points should be such that it exactly equals the amount of energy necessary to move an object between those points, or in other words that it equals the gain in potential energy that an object being moved between these points would experience. But that gain in potential energy is described by the integral of force over distance. In mathematical terms, we therefore demand that [2, p. 256]

$$\int_{\vec{O}}^{\vec{P}} -\vec{F}(\vec{r}) d\vec{r} = V(\vec{P}) \quad \Rightarrow \quad \vec{F}(\vec{r}) = -\vec{\nabla} V(\vec{P}), \tag{8}$$

where V is the function describing the potential, $\vec{P}$ is an arbitrary point in space and $\vec{O}$ is the reference point that we choose as the "zero potential energy point" (or the position where the toy car starts off with its battery fully charged). The minus sign in equation 8 is there to account for the fact that $\vec{F}(\vec{r})$ describes the force exerted on our toy car by the force field. However, we want the potential to account for the energy that has to be used by the battery to keep the toy car moving and the necessary forces to do so have to be directed **against** $\vec{F}(\vec{r})$ in order to overcome its pull on the toy car. That being said, the existence of a potential $V(\vec{P})$ that fulfills equation 8 for a given force field $\vec{F}(\vec{r})$ is one more necessary and sufficient condition (in addition to equation 7) for $\vec{F}(\vec{r})$ being a conservative force. In the case of two point masses attracting each other with gravitational forces, the potential reads:

$$V(r) = -G\frac{m_1 m_2}{r}. \tag{9}$$

as can be easily checked by applying the $\vec{\nabla}$-operator[13], multiplying by minus one and comparing the result with equation 4. Of course, adding a constant to the potential would just as well be acceptable, as any

[12] At least this is true when neglecting frictional forces, which are of course present in reality and part of the reason why you do have to refuel you car every now and then, even though the gravitational force field of the earth is actually conservative.

[13] The $\vec{\nabla}$-operator in spherical coordinates is $\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\varphi \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \varphi}$. However, as equation 9 only has a radial component, the second and the third term vanish and so all that's left to do is to differentiate with respect to r.

constant vanishes when being differentiated. It just reflects the fact that we are free to choose a point of reference that we agree on to be the zero potential. The most convenient constant in the current case is zero, meaning that we are choosing the zero potential to be reached when the two point masses are separated by an infinite distance (notice that as r goes to infinity in equation 9, $V(r)$ goes to zero).

Now finally we are ready to prove the conservation of energy. Let's imagine again that we have a point mass sitting somewhere in a conservative force field and we calculate how its potential and kinetic energy change as a result of the forces acting on the point mass due to the force field [14] [3, p. 10].

$$\begin{aligned} -dE_{pot} = -dV = \vec{F}d\vec{r} = m\vec{a}\vec{v}dt = m\frac{d\vec{v}}{dt}\vec{v}dt = m\vec{v}d\vec{v} = d(\frac{1}{2}m\vec{v}^2) = dE_{kin} \\ \Rightarrow \quad d(E_{pot} + E_{kin}) = 0 \quad \Rightarrow \quad E_{pot} + E_{kin} = E_{tot} = const. \end{aligned} \tag{10}$$

This is the first of three conservation laws covered in this chapter (though by far the one spent most lines on). The next concerns the movement of the center of mass of a system of point masses which only interacts with itself by gravitational attraction and is otherwise a closed system, meaning no external forces are at play. First, we need to define the total mass of the system M, as well as its centre of mass $\vec{R}$. We will take as a given the existence of n point masses with masses labelled m_1 through m_n as well as their corresponding position vectors labelled $\vec{r}_1$ through $\vec{r}_n$.

$$M \coloneqq \sum_{i=1}^{n} m_i, \quad \vec{R} \coloneqq \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{M} \tag{11}$$

Next, we calculate the second derivative of $M\vec{R}$ [15] [4, p. 25-26].

$$\begin{aligned} M\ddot{\vec{R}} = \sum_{i=1}^{n} m_i\ddot{\vec{r}}_i = \sum_{i=1}^{n} \vec{F}_i = \sum_{i=1}^{n}\sum_{k=1,\, k\neq i}^{n} \vec{F}_{ki} = \sum_{i=1}^{n}\sum_{k<i}^{n} \vec{F}_{ki} + \sum_{i=1}^{n}\sum_{k>i}^{n} \vec{F}_{ki} = \sum_{i=1}^{n}\sum_{k<i}^{n} \vec{F}_{ki} - \sum_{k=1}^{n}\sum_{i>k}^{n} \vec{F}_{ki} = \\ \sum_{i=1}^{n}\sum_{k<i}^{n} \vec{F}_{ki} - \sum_{i=1}^{n}\sum_{i>k}^{n} \vec{F}_{ki} = 0 \\ \Rightarrow M\dot{\vec{R}} = const. \end{aligned} \tag{12}$$

In the upper equation, $\vec{F}_i$ is the total force acting on the i^{th} point mass, whereas $\vec{F}_{ki}$ is the force acting on the i^{th} point mass due to the k^{th} point mass. In the last step of the first line, the running indices of the second term were swapped, meaning that where you previously had $\vec{F}_{35}$, you would now have $\vec{F}_{53}$, but these two forces are equal in magnitude and opposite in direction, which is why there is a minus sign. The rearrangements altogether may look a little cumbersome at first, but actually the concept is quite simple. the expression $M\ddot{\vec{R}}$ is the same as the sum of all the forces acting from every single point mass to all other point masses. However, for each force acting from point mass i to point mass k, there is an equal and opposite force acting from point mass k to point mass i, which is why there is nothing left in the end. The conclusion is that $M\ddot{\vec{R}} = 0$, ergo $M\dot{\vec{R}} = const.$ or in other words, the sum of the point masses momenta (where the momentum of one point mass is defined as $\vec{p}_i \coloneqq m_i\vec{v}_i$) is conserved.

This is the second conservation law dealt with in this chapter, and now lets go for the third. Linear momentum having just been defined as the product of mass and velocity, there is a similar definition for the angular momentum $\vec{l}_i$ of a point mass: $\vec{l}_i = \vec{r}_i \times \vec{p}_i$. The angular momentum is crucially dependent on the choice of your reference point for the zero vector $\vec{0}$, as it is a measure of the spin of the point mass about that

[14] In the last but one expression in the first line of equation 10, I made use of the fact that for a function $f(x)$, an infinitesimal change $df(x)$ can be written as $df(x) = f'(x)dx$, where $f'(x)$ is the derivative of $f(x)$.

[15] Throughout this thesis, the time derivative is symbolised either with $\frac{d}{dt}$ or with a dot on top of the respective variable.

point. Nonetheless, independent of that choice, the sum of angular momenta of all point masses is another quantity that is always conserved if we assume no external forces. Let's first define that sum.

$$\vec{L} \coloneqq \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i, \quad \vec{p}_i = m\vec{v}_i \tag{13}$$

Now, let's work out its derivative [4, p. 26-27].

$$\begin{aligned} \dot{\vec{L}} = \sum_{i=1}^{n} \vec{r}_i \times \dot{\vec{p}}_i + \underbrace{\sum_{i=1}^{n} \dot{\vec{r}}_i \times \vec{p}_i}_{=0} = \sum_{i=1}^{n} \sum_{k=1,\ k\neq i}^{n} \vec{r}_i \times \vec{F}_{ki} = \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1,\ k\neq i}^{n} (\vec{r}_i \times \vec{F}_{ki} + \vec{r}_k \times \vec{F}_{ik}) = \\ \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1,\ k\neq i}^{n} (\vec{r}_i - \vec{r}_k) \times \vec{F}_{ki} = 0 \quad \Rightarrow \quad \vec{L} = const. \end{aligned} \tag{14}$$

In the first step, the second term can immediately be set zero, because you are building a cross product between two parallel vectors which will always give zero. A similar strategy was used for the first term also. At first, it was used that $\dot{\vec{p}}_i = m\ddot{\vec{v}}_i = \sum_{k=1,\ k\neq i}^{n} \vec{F}_{ki}$. Next, the whole sum was written twice with a factor $1/2$ up front (notice that changing the indices doesn't affect the sum, as the latter goes over all i and all k anyway). Finally the relationship $\vec{F}_{ik} = -\vec{F}_{ki}$ was again made use of and only cross products between parallel vectors were left.

And that's it for this chapter, the takeaway of it all is actually quite simple. We discussed two empirical laws and three conservation laws.

$$F = ma, \quad \vec{F}_{grav.} = \pm G\frac{m_1 m_2}{r^3}\vec{r}_{12}, \quad \text{conserved quantities: energy, momentum, angular momentum} \tag{15}$$

Even though many of the introduced concepts were just quickly mentioned and not really elaborated, the knowledge presented in this chapter should be enough to keep going. The next thing to do is to look at one-body problems [16].

[16] I am fully aware that for someone never having heard of any of the concepts mentioned in this chapter, this introduction will have most likely been way too quick, leaving many questions unanswered. However, as the main topic of my thesis will only be presented in the next chapters, I decided to only mention and explain what is absolutely necessary for the further development of the thesis in order to avoid losing too many pages before actually getting to the heart of the matter.

3 The one-body problem

In the case of point masses interacting through gravitational forces only, a one-body problem is a little contrived and unrealistic, as with only one mass point, there isn't any other mass to exert force on. However, in order to comprehend the mechanics behind a two-body problem, it is vital to have a good understanding of the one-body problem nonetheless. To do so, we will postulate the existence of a radially symmetrical potential and the one of a point-mass moving therein. Where the potential comes from is quite irrelevant for now. To describe the position of the moving point-mass with respect to time t, we will make use of spherical coordinates illustrated in figure 1 (the z-axis of a superimposed cartesian coordinate system would be aligned with the vertical dotted line, the x-axis with the horizontal dotted line going diagonally to the left and the y-axis would be orientated such that it is in the plane of the two horizontal dotted lines and that the right-hand rule is fulfilled [17]).

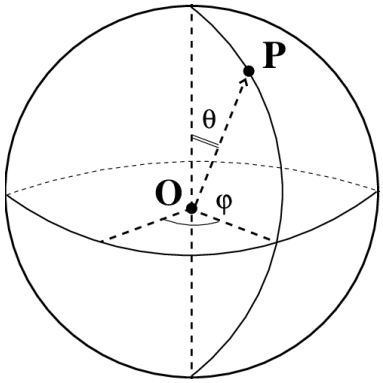

Figure 1: Spherical coordinates [5]

With the two angles φ and θ plus the radial distance from the origin r, any position in space and thus the trajectory of the point-mass $\vec{r}(t)$ can be uniquely identified. This is done using the three orthonormal [18] spherical basis vectors [2, p. 165].

$$\vec{e}_r = \begin{bmatrix} \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\theta) \end{bmatrix}, \quad \vec{e}_\varphi = \begin{bmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{bmatrix}, \quad \vec{e}_\theta = \begin{bmatrix} \cos(\theta))\cos(\varphi) \\ \cos(\theta)\sin(\varphi) \\ -\sin(\theta) \end{bmatrix}, \quad \varphi \in [0, 2\pi],\ \theta \in [0, \pi] \tag{16}$$

the position of the point mass can now be described by

$$\vec{r}(t) = r(t)\vec{e}_r(t). \tag{17}$$

As the force exerted by a radially symmetrical potential always points towards the centre of the coordinate system [19], the movement of the point mass will be restricted to a single plane slicing through three-dimensional space. It is therefore possible to re-orientate the coordinate system in such a way as to keep the φ-component of the point masses position constant over time, restricting the whole problem to identifying only the two remaining parameters as a function of time, that is $r(t)$ and $\theta(t)$. When calculating the point masses velocity given by the time derivative of its position, all derivatives of φ can now be cancelled.

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r(\dot{\theta}\vec{e}_\theta + \sin(\theta)\dot{\varphi}\vec{e}_\theta) = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \tag{18}$$

[17] The right-hand rule states that if you use the thumb, index finger and middle finger of your right hand and point them in directions that are perpendicular to one another, then if the thumb is aligned with the x-axis and the middle finger with the z-axis, the index finger must be aligned with the y-axis.

[18] In three dimensional space, an orthonormal basis is just a set of three vectors that each have a length of one and where the angle between any two of the three vecotors is exactly 90°.

[19] To be exact, if we allow any arbitrary radially symmetrical potential, it could conceivably also yield forces directed away from the coordinate centre. But in that case any resulting trajectory would not rotate about the coordinate centre but rather fly off to infinity. I will therefore only deal with forces directed towards the coordinate origin in the following derivations. I will come back to that point when dealing with Bertrand's theorem.

The fact that $\dot{\vec{e}}_r$ can be written as $\dot{\vec{e}}_r = \dot{\theta}\vec{e}_\theta + \sin(\theta)\dot{\varphi}\vec{e}_\theta$ can simply be checked by inserting the definitions from equation 16. The magnitude of the velocity now follows as:

$$|\vec{v}| = v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}. \tag{19}$$

Next, let's make use of the conserved quantities derived in the previous chapter. For the angular momentum, we have:

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} = mr\vec{e}_r \times (\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta) = mr^2\dot{\theta}\vec{e}_\varphi, \quad L = mr^2\dot{\theta}. \tag{20}$$

A second conserved quantity, the total Energy E, is composed of the kinetic energy of the point mass T and its potential energy V [3, equations 21 - 28 were motivated by p. 159-161 and p. 164].

$$E = T + V = \frac{mv^2}{2} + V(r) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = const. \tag{21}$$

As this thesis is dealing with gravitational problems, we could just as well substitute the actual gravitational potential for $V(r)$, but as the following deductions work for any potential as long as it is radially symmetrical, I decided to stick with $V(r)$ for the sake of having a more compact notation and also for the sake of being more general. I will write about the special case of gravitational potentials in the next subsection.

Now that we have two parameters which we would like to know the time dependence of along with two independent conserved quantities written in terms of these two parameters, we are ready to get going. Taking a closer look a the expression for the energy E, one can see that the second term involving $\dot{\theta}$ can be rewritten using L.

$$E = \frac{m\dot{r}^2}{2} + \frac{m^2r^4\dot{\theta}^2}{2mr^2} + V(r) = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} + V(r) \tag{22}$$

This step is the crucial one to take, taking away another variable and leaving r as the only unknown variable left. If we unite the second and the third term of the energy into one expression, let's call it $V_{eff}(r)$ for effective potential, we get:

$$E = \frac{m\dot{r}^2}{2} + V_{eff}(r), \quad V_{eff}(r) = \frac{L^2}{2mr^2} + V(r). \tag{23}$$

The second term of V_{eff} is referred to as centrifugal potential, as its effect can be related to that of the centrifugal force in circular motion, pointing radially outward. Equation 23 can now be interpreted as representing a point mass moving in one dimension under the influence of a potential which is also defined in just one dimension. Rearranging for $\dot{r} = \frac{dr}{dt}$ yields

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}(E - V_{eff}(r))} \quad \Rightarrow \quad dt = \frac{dr}{\pm\sqrt{\frac{2}{m}(E - V_{eff}(r))}}. \tag{24}$$

Integration finally yields an expression for the relationship between time t and the distance from the coordinate origin r.

$$t(r) = \pm \int \frac{dr}{\sqrt{\frac{2}{m}(E - V_{eff}(r))}} + C_1 \tag{25}$$

The $\pm$ sign in front of the integral reflects the fact that the laws of physics do not dictate the direction of the motion of the point mass. If it got from point A at t_1 to point B at t_2, it could just as well have gotten from point B at t_1 to point A at t_2 by merely altering the initial conditions. In other words, if you were to film the motion of the point mass in any arbitrary potential and you played the film backwards, there would be no way to judge which clip is the real one by merely studying the point masses trajectory. The integration constant C_1, just like the sign ambiguity, has to be figured out by considering the initial conditions.

Just as it was possible to find a relationship between t and r, similar rearrangements yield expressions relating t and θ as well as θ and r, as shown in equations 26 and 27.

$$L = mr^2\dot{\theta} \quad \Rightarrow \quad \frac{d\theta}{dt} = \frac{L}{mr^2} \quad \Rightarrow \quad \theta(t) = \int \frac{L}{mr^2(t)}dt + C_2 \tag{26}$$

$$d\theta = \frac{L}{mr^2}dt = \frac{L}{mr^2}\frac{dr}{\pm\sqrt{\frac{2}{m}(E - V_{eff}(r))}} \quad \Rightarrow \quad \theta(r) = \pm\int \frac{L}{r^2\sqrt{2m(E - V_{eff}(r))}}dr + C_3 \tag{27}$$

Another possibility to derive these formulas would be to take the Lagrange equations of the second kind as a starting point, but as the latter play no important role in the further development of this thesis and as a complete justification of their validity would be a quite lengthy exercise, I decided to hereby merely mention their existence and move on with the next subsection.

3.1 Special case: gravitational potential

In the case of a gravitational potential, which is a specific example of the more general case of a radially symmetrical potential, the potential is such as it would be if it were created by a second point mass situated in the origin of a clockwise cartesian coordinate system (if you wish to, you can actually think of a real point mass sitting there, it would only have to be "glued" to that exact position, withstanding any force of the other point mass trying to move it away). From equation 9 and equation 23, the actual and the effective potential follow as:

$$V(r) = -\frac{\alpha}{r}, \quad \alpha > 0 \in \mathbb{R} \quad \Rightarrow V_{eff}(r) = -\frac{\alpha}{r} + \frac{L^2}{2mr^2}. \tag{28}$$

In figure 2 such a potential is plotted. For small r less than 1, the second term prevails over the first, making the potential positively divergent, whereas for r greater than one, the negative contribution of the first term is dominant. This allows for a qualitative description of the path a point mass with a certain energy would take.

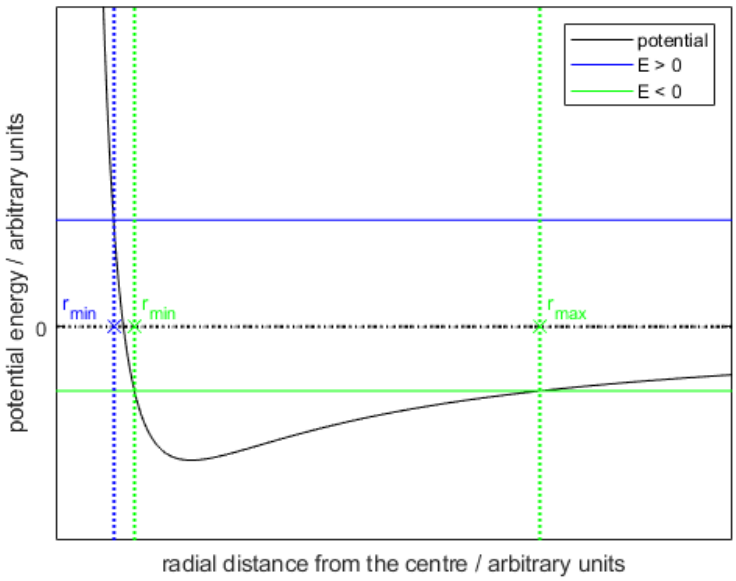

Figure 2: Effective potential for gravitational forces

If the energy is above zero (shown by the horizontal blue line in figure 2), the movement of the point mass will be unbound, meaning that even though there is a minimum distance from the origin that it will never penetrate, there is no restriction to the other side and sooner or later its radial component will diverge to

infinity. If the energy is below zero (shown by the horizontal green line in figure 2), the radial component is bound by the two values r_{min} and r_{max} and will oscillate between them. However, even though the radial velocity at the marginal points is always zero, this does not mean that the point mass is stationary, as it will have an angular velocity unequal to zero:

$$T(r = r_{min}) = \underbrace{\frac{m\dot{r}^2}{2}|_{r=r_{min}}}_{=0} + \underbrace{\frac{mr_{min}^2\dot{\theta}^2}{2}}_{\neq 0}. \tag{29}$$

In the case of a gravitational potential, the point mass will never reach the coordinate origin, it can never "fall" into the center (unless $L = 0$, but that would make the second term of $V_{eff}(r)$ vanish, the potential would not look as depicted in figure 2 and the trajectory taken would be a straight line, so not really demonstrating an interesting scenario to be investigated). However, if the potential drops faster than $1/r^2$ rises for small r, such a fall into the center would be possible (such a potential could be e.g. $V(r) = -1/r^3$).

An interesting question to ask would be which potentials will lead to closed trajectories, meaning orbits such that an object will continuously pass through the same positions in space with the same velocity as oppose to orbits where an object will never return to a location that it has passed in the past, or at least not so with identical velocity. The result is quite surprising and will be derived in the next subsection.

3.2 Bertrand's theorem

This section can be skipped without loss of continuity, as its result is barely influential on any contemplations concerning the two- or the three-body-problem. I still decided to incorporate it into my thesis because it is an example of how sometimes certain nifty mathematical rearrangements can lead to statements that one might have otherwise never deemed possible to be made. For the subject at hand, it is this: **Within all possible radially symmetrical potentials, potentials of the form $V = -\alpha/r$ or $V = \alpha r^2$ (α being a positive real number) guarantee that all bound orbits are also closed orbits [20]. Furthermore, these are the only two potentials with that property** [6]. At least to me, it was at first quite startling to imagine that such an exclusive statement about all radially symmetrical potentials you can possibly think of can ever be justified. However, it turns out that it can, and here is why [21]:

First, a quick note on the second part of the theorem. While apart from the two mentioned potentials it is not possible to come up with yet another potential that **guarantees** a closed orbit, that doesn't mean that closed orbits are impossible. In fact, any radially symmetrical potential corresponding to forces that are directed towards the coordinate centre can give rise to a circular orbit, as the requirement for the latter is merely that the centripetal force needed to keep an object on a circular track is exactly accounted for [22]. But

[20] The first of the two potentials is the gravitational one which we already encountered in the previous sections of this thesis, the second one is the so-called spring potential, as it models the behaviour of an idealised spring where the restoring force is always proportional to the deviation from the equilibrium position.

[21] To be fair, the following reasoning can only be applied to potentials that can be expanded into a taylor series, but that is a rather weak restriction as almost any conceivable physically meaningful function can be approximated to arbitrary precision by a taylor expansion.

[22] Some readers might be wondering why I added the specification that the potential must correspond to forces that are directed **towards** the coordinate centre. While at first sight it might look like I just restated the definition of a radially symmetrical potential, this is not entirely correct, as you could also come up with a radially symmetrical potential that corresponds to forces being directed **away** from the coordinate centre. If you had to deal with such a potential, a circular orbit would not be possible, in fact there would be no bound orbit at all, as all trajectories would be pushed out into infinity. Because of this circumstance, I think that Bertrand's theorem is actually slightly misstated, as such a potential would also (in addition to the gravitational and the spring potential) fulfill all the necessary conditions of the theorem. Despite not yielding a single closed orbit, it is still true that all bound orbits are closed because of the inexistence of bound orbits. This may be a very pedantic point to make, but I just wanted to clarify it anyways and say that in the following derivations, I will only consider such potentials that do have bound orbits. One last remark: I could imagine that the theorem is nonetheless well stated and that the incosistency I just pointed out merely stems from historically non-uniform definitions of the vocabulary originally used to formulate it.

this can easily be achieved, as the centripetal force is given by $F_c = mv^2/R$, so if the potential provides a certain force F, you just have to choose the velocity of your object such that $F = F_c$.

This being said, let's think through an adequate strategy to derive the theorem. What we are looking for is a trajectory such that after n orbits about the center (n being a natural number) it repeats itself. in the previous subsection, we derived the equations of motion for a one-body problem using three variables, namely t for the time, θ for the angle and r for the radius. While the time dependence of r and θ is of little use here, what we are hoping to find is a function $r(\theta)$ such that $r(\theta) = r(\theta + 2\pi n)$. Once we find that function, we would then look at what potential would be necessary to obtain it. With this in mind, I will try to argue why certain steps on the way are taken by referring to this rough strategy. However, considering that it took mathematicians over three centuries to crack the problem, this might not always be perfectly possible as sometimes certain transformations are just applied as a result of trial and error. My take on it is therefore this that as long as each step is understood, the end result can be appreciated and that's the whole goal of this exercise. So here we go:

Seeking to find a function $r(\theta)$, one might be tempted to start from equation 27 and then rearrange for r. However, as this equation contains an integral over a fraction in which we find powers of the variable r in the denominator as well as underneath a square root, this endeavour is almost certainly bound to failure. Instead, we will go for the first equation in the previous subsection that contains all the information of equation 27 but no nasty integrals. This is equation 22 and it is restated here:

$$E = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} + V(r). \tag{30}$$

Next, we differentiate with respect to r. This allows us to get rid of the energy E and only work with functions of r [23].

$$\begin{aligned} 0 = \frac{d}{dr}E = \frac{m}{2}\frac{d}{dr}(\dot{r}^2) - \frac{L^2}{mr^3} + \frac{dV}{dr} \quad &\Rightarrow \\ \frac{L^2}{mr^3} - \frac{dV}{dr} = \frac{m}{2}2\dot{r}\frac{d}{dr}(\dot{r}) = m\dot{r}\frac{d}{dt}(\dot{r})\frac{dt}{dr} = m\frac{dr}{dt}\frac{d}{dt}(\frac{dr}{dt})\frac{dt}{dr} = m\frac{d^2r}{dt^2} \quad &\Rightarrow \\ m\frac{d^2r}{dt^2} - \frac{L^2}{mr^3} = -\frac{dV}{dr} \end{aligned} \tag{31}$$

As argued above, we are not so much interested in the time dependence of r, but rather in its dependence on θ. We therefore use the transformation from equation 27 to change the independent variable from t to θ [6, equations 32 - 43 were motivated by this source] [24].

$$\frac{d}{dt} = \frac{L}{mr^2}\frac{d}{d\theta} \quad \Rightarrow \quad \frac{L}{r^2}\frac{d}{d\theta}(\frac{L}{mr^2}\frac{dr}{d\theta}) - \frac{L^2}{mr^3} = -\frac{dV}{dr} \tag{32}$$

Next, we notice that all the powers of r appear in the denominators of the fractions, so an educated guess would be that the transformation $u := 1/r$ takes the variables up to the numerator and makes the equation

[23] What this actually yields is one of the two lagrange equations of the second kind for one-body problems. Usually, this is taken as a starting point for deriving Bertrand's theorem, but as I did not introduce them in this thesis, I decided to take another route to get there.

[24] I am aware that reference [6] is a wikipedia article and should therefore usually not be cited. However, the reliability of the following equations is not based on the trustworthiness of the cited source, but on the soundness of the mathematical argumentation. I did find quite some mistakes in the source which I tried to correct in this thesis. Nonetheless, the origin of the equations is the wikipedia article and because I did not find alternative sources that offered a similar approach to proving the theorem, I decided to cite it anyways. A somewhat different approach to the theorem from a more reliable source is presented in reference [7].

easier to solve.

$$\begin{aligned} Lu^2\frac{d}{d\theta}(\frac{L}{m}u^2\frac{d(1/u)}{d\theta}) - \frac{L^2u^3}{m} = -\frac{dV}{d(1/u)} \quad &\Rightarrow \\ \frac{L^2}{m}u^2\frac{d}{d\theta}(u^2\frac{d(1/u)}{du}\frac{du}{d\theta}) - \frac{L^2u^3}{m} = -\frac{dV}{du}\frac{du}{d(1/u)} \quad &\Rightarrow \\ \frac{L^2}{m}u^2\frac{d}{d\theta}(-\frac{du}{d\theta}) - \frac{L^2u^3}{m} = \frac{dV}{du}u^2 \quad &\Rightarrow \\ \boxed{\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}\frac{dV}{du}} \end{aligned} \tag{33}$$

In the third line, the derivative of u with respect to $1/u$ changed to $-u^2$ because using the substitution $a = 1/u$ it follows that $du/d(1/u) = d(1/a)/da = -a^{-2} = -u^2$. Finally, in the last step the left and the right side were multiplied with $-m/(L^2u^2)$.

Next, let's call the right hand side $J(u)$ and make use of the fact that in radially symmetrical potentials, the force acting on a point mass is always in a radial direction and its magnitude is therefore given by $F = -\frac{d}{dr}V$.

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}\frac{dV}{dr}\frac{dr}{du} = -\frac{m}{L^2u^2}F(\frac{1}{u}) = J(u) \tag{34}$$

In a perfectly circular orbit, r would remain constant with respect to θ, so u would also be constant. We therefore get the circular solution when the first term of equation 34 vanishes. Let's call the radius where this happens r_0 and the corresponding u-value u_0.

$$u_0 = J(u_0) = -\frac{m}{L^2u_0^2}F(\frac{1}{u_0}) \tag{35}$$

What we are looking for is a potential and thus a force that gives rise to closed orbits independent of initial conditions. But that means that we are free to choose any set of initial conditions that we feel comfortable working with and then demand that the orbit be closed. A convenient choice would be to look at orbits that deviate only slightly from the circular orbit, for which we already found a solution above. If we define $\eta = u - u_0$ to be the deviation from the circular orbit, we can expand J into a Taylor series with higher order terms yielding increasingly negligible contributions.

$$J(u) \approx J(u_0) + \eta J'(u_0) + \eta^2\frac{1}{2!}J''(u_0) + \eta^3\frac{1}{3!}J'''(u_0) + \dots \tag{36}$$

Substiting equation 36 into equation 34 yields:

$$\begin{aligned} \frac{d^2(\eta+u_0)}{d\theta^2} + \eta + u_0 = J(u_0) + \eta J'(u_0) + \eta^2\frac{1}{2!}J''(u_0) + \eta^3\frac{1}{3!}J'''(u_0) + \dots \quad &\Rightarrow \\ \frac{d^2\eta}{d\theta^2} + \eta = \eta J'(u_0) + \eta^2\frac{1}{2!}J''(u_0) + \eta^3\frac{1}{3!}J'''(u_0) + \dots \quad &\Rightarrow \\ \frac{d^2\eta}{d\theta^2} + \beta^2\eta = \eta^2\frac{1}{2!}J''(u_0) + \eta^3\frac{1}{3!}J'''(u_0) + \dots . \end{aligned} \tag{37}$$

In the second line, the equality $u_0 = J(u_0)$ was used to cancel the two terms, whereas in the third line the variable $\beta^2 = 1 - J'(u_0)$ was introduced. As the higher order terms on the right-hand side can justifiably be neglected for small enough perturbations, it follows that β^2 must be a positive real number, as else the solution to the resulting differential equation would be an exponential function (in contradiction to us only considering small deviations from circularity, as any exponential function would grow indefinitely). With $\beta^2 > 0$, the differential equation is solved by trigonometric functions.

$$\frac{d^2\eta}{d\theta^2} + \beta^2\eta = 0 \quad \Rightarrow \quad \eta(\theta) = a_1\cos(\beta\theta) + a_2\sin(\beta\theta) \tag{38}$$

For the orbit to be closed, β must be rational, meaning there must exist two integers p and q such that $\beta = p/q$. Only then can the criterion for closed orbits ($r(\theta) = r(\theta + 2\pi n);\ n \in \mathbb{N}$) be fulfilled with the period $\theta = 2\pi|q|$. But we can say yet a lot more about β. To do so, Let us explicitely differentiate $J(u)$ at u_0, using its definition from equation 34. Remember that in the following equation, the argument of F is always $1/u$ or $1/u_0$ and not u or u_0. This is not highly significant, because I don't explicitly differentiate F (which is also why I din't write the arguments to increase the readibility), but it's still good to know what one is talking about.

$$J'(u_0) = \frac{d}{du}(-\frac{m}{L^2u^2}F)|_{u_0} = (2\frac{m}{L^2u^3}F - \frac{m}{L^2u^2}\frac{dF}{du})|_{u_0} = -\frac{2}{u_0}[-\frac{m}{L^2u_0^2}F] + [-\frac{m}{L^2u_0^2}F]\frac{1}{F}\frac{dF}{du}|_{u_0} = \\ -2 + \frac{u_0}{F}\frac{dF}{du}|_{u_0} = 1 - \beta^2 \tag{39}$$

Now comes the key observation of the whole theorem, which to me is all the more ingenious, as it relies on an area of mathematics seemingly completely disparate to what we are doing here, namely topology. Equations 38 and 39 both depend on $u_0 \coloneqq 1/r_0$. In equation 39, this is completely obvious because of the explicit appearance of u_0, but also equation 38 involves u_0 because of the definition of η as $\eta = u - u_0$. Put in words, what equation 38 is saying is that if you have a fixed circular orbit and thus a fixed variable u_0, there exists a corresponding closed orbit that deviates only slightly from the circular one and contains the rational parameter β. However, as explained at the beginning of chapter 3.2, a circular orbit can be created at an arbitrary radius, meaning that for every possible choice of u_0, equation 38 yields a solution and $\beta(u_0)$ must always be rational. Furthermore, β is defined as $\beta \coloneqq 1 - J'(u_0)$ and $J'(u)$ must be a continuous function as it must be differentiable (see equation 36). Because of the fact that the rational numbers are totally disconnected, it follows that $\beta(u_0)$ must be a constant and therefore independent of the value of u_0.

This result is crucial for the further development of the argument, and because of that I would like to spend some more lines on it. First of all, I would like to clarify the implications of the set of rational numbers being totally disconnected. It implies that if you choose two arbitrary but distinct rational numbers x_1 and x_2, you can always find an irrational number x_3 such that $min(x_1, x_2) < x_3 < max(x_1, x_2)$, no matter how close x_1 and x_2 are to one another. With this remark, it is easy to prove the above statement by contradiction. Let's assume that $\beta(u_0)$ is not a constant and let's pick two distinct values, $\beta_1 = \beta(u_{0_1})$ and $\beta_2 = \beta(u_{0_2})$. Next, let's choose an irrational number x such that $min(\beta_1, \beta_2) < x < max(\beta_1, \beta_2)$. As $\beta(u_0)$ is a continuous function of u_0, the intermediate value theorem [25] ensures the existence of $u_x \in [\min(u_{0_1}, u_{0_2}), \max(u_{0_1}, u_{0_2})]$ such that $\beta(u_x) = x$. But that would be in contradiction to the fact that β must be rational, completing the proof by contradiction and showing that β must indeed be a constant.

With this result in our pocket, we can rewrite equation 39, but this time we can leave out the subscript zero for u_0, as we just established that β is independent of u_0 and that the equation must be correct for every u_0.

$$J'(u) = ... = -2 + \frac{u}{F}\frac{dF}{du} = 1 - \beta^2 \tag{40}$$

Taking the last equivalence and substituting back to the radius r yields:

$$3 - \beta^2 = \frac{1}{rF}\frac{dF}{dr}\frac{dr}{du} = \frac{1}{rF}\frac{dF}{dr}\frac{d(1/u)}{du} = \frac{-1}{u^2rF}\frac{dF}{dr} = \frac{-r}{F}\frac{dF}{dr}. \tag{41}$$

The rest is a relatively easy exercise of solving ordinary differential equations.

$$\int \frac{1}{F}\frac{dF}{dr}dr = \int \frac{\beta^2 - 3}{r}dr \quad \Rightarrow \\ \log(|F|) = (\beta^2 - 3)\log(|r|) + \log(c) \quad \Rightarrow \\ F(r) = cr^{\beta^2 - 3} \tag{42}$$

[25] The intermediate value theorem states that if f is a continuous function whose domain contains the interval [a, b], then it takes on any given value between f(a) and f(b) at some point within the interval [8, p. 87].

Integrating the force $F(r)$ to obtain the potential $V(r)$ and setting the integration constant zero results in:

$$V(r) = \frac{c}{\beta^2 - 3} r^{\beta^2 - 2} = \tilde{c} r^{\beta^2 - 2}. \tag{43}$$

This is still not the desired result of Bertrand's theorem, but we have already gone a very long way from knowing nothing whatsoever about the form of the potential we are looking for to being able to say that it must take on precisely the form of equation 43, with the only degree of freedom left being the choice of β [26]. What is left to do now is to show that β can only take on the values 1 for the gravitational potential ($V = -\alpha/r$, $\alpha > 0$) or 2 for the spring potential ($V = \alpha r^2$, $\alpha > 0$) and that all other values of β are out of question.

To do so, we must also consider orbits that deviate more considerably from the circular orbit, such that more higher order terms on the right side of equation 36 must be considered. As an Ansatz for a closed orbit solution, we can choose a fourier series (which by its nature can describe any periodic function) whose fundamental term is the one from equation 38, as the latter must be recovered in the limit of approaching the circular orbit. Furthermore, if we choose the initial condition that $\eta'(\theta) = 0$, we can leave out all the sine terms, which helps reduce the writing effort in what is to come. In equation 44, I copied the last line of equation 37, which is the differential equation that our solution $\eta(\theta)$ must fulfill, keeping terms up to third order [27]. Underneath, also in equation 44, I put the Ansatz for the solution.

$$\begin{aligned} \frac{d^2\eta}{d\theta^2} + \beta^2\eta &= \eta^2 \frac{1}{2!} J''(u_0) + \eta^3 \frac{1}{3!} J'''(u_0) \\ \eta(\theta) &= a_0 + a_1 \cos(\beta\theta) + a_2 \cos(2\beta\theta) + a_3 \cos(3\beta\theta) \end{aligned} \tag{44}$$

In the second line [9, p. 5], I again did not write down the complete series but only used the first few terms. This time, the reason is that these first terms will already turn out to yield a solution to the differential equation, meaning that if I had added more terms, their coefficients would have all been zero anyways.

Now, we have reached a point where no more general contemplations on the problem at hand are necessary. All that is left to do is inserting the second line of equation 44 into the first line and noticing that we end up having cosine terms of different frequencies on the left as well as on the right hand side. For the equation to be correct for every single choice of θ, it must therefore hold that the coefficients in front of cosines with the same frequency must be the same on both sides. We can thus equate those coefficients and end up having a number of algebraic equations involving different derivatives of $J(u)$. These derivatives can however be expressed as a function of β using equation 34 for the definition of $J(u)$ and equation 42 for the expression of $F(\frac{1}{u})$ as a function of β. The algebraic equations can then be inserted into one another to solve for β.

As the process described in the last paragraph does not yield any new physical or mathematical insight but is quite lengthy (notice that in the first line of equation 44 we have to compute η^3 where η is the sum of cosines of four different frequencies), I decided to not write it out here and instead put the corresponding reference where all these calculations are carried out into the footnote [28]. At this point, I merely want to state the resulting equation that pops out at the end [9, p. 7].

$$\beta^4(1 - \beta^2)(4 - \beta^2) = 0 \tag{45}$$

[26] Of course, the value of $\tilde{c}$ is also not determined yet, but as it just scales the whole potential and does not alter its form, its precise value is of little interest.

[27] Again, the disregard of even higher order terms indirectly implies that we are now only looking at orbits that are close enough to the circular one for these terms to vanish. It may feel like an arbitrary choice to only include the first order in the previous pages and now truncate at the third order, but as pointed out at the top of page 15, we are free to choose initial conditions requiring the consideration of arbitrarily many terms in the taylor expansion, as Bertrand's theorem deals with potentials that yield closed orbits for **all** bound orbits.

[28] The part of the caclculation that is not presented in my thesis starts at page 5 of reference [9].

It follows that β^2 is either 0, 1 or 2. Using equation 43, the only candidates left to produce closed orbits independent of initial conditions are therefore:

$$\begin{aligned} V(r) &= \frac{\alpha}{r^2} \\ V(r) &= \frac{\alpha}{r} \\ V(r) &= \alpha r^2. \end{aligned} \tag{46}$$

Keep in mind that nowhere in our above derivations we made claims of the sort "if the potential fulfils this or that equation or property, all bound orbits are guaranteed to also be closed", but rather we made claims of the sort "in order for a potential to yield closed orbits for all bound orbits, it **must** fulfil those properties". This means that the above potentials still have to be examined to find out whether they meet our requirements.

The first one can quite easily be dismissed by going back to chapter 3.1 and looking at what the corresponding effective potential would look like ($V_{eff}(r) = V(r) + L^2/(2mr^2)$). If α was smaller than $-L^2/(2mr^2)$, it would diverge to negative infinity for small values of r, so every orbit would fall into the centre. For $\alpha = -L^2/(2mr^2)$, it would vanish completely, equation 23 would turn into $E = m\dot{r}^2$ and as this quantity is conserved, r would diverge into infinity. Finally, for $\alpha > -L^2/(2mr^2)$ the effective potential has no local minimum and instead decreases asymptotically towards high values of r, again yielding only divergent and therefore unbound trajectories. Incidentally, $\beta^2 = 0 \Leftrightarrow \beta = 0$ also merely returns the perfectly circular orbit in equation 38 which can be obtained by any potential with $d/dr(V(r)) > 0$.

The second candidate also has to be inspected for different values of α separately. For $\alpha > 0$, there is again no minimum of the effective potential and all trajectories diverge into infinity. For $\alpha < 0$ however, we obtain the gravitational potential, for which we will still show in the next chapter that all bound orbits are closed [29].

Finally, the third potential diverges to minus infinity and so produces divergent trajectories for $\alpha < 0$ and yields the so-called spring potential for $\alpha > 0$. As spring potentials are not central to this thesis, I will leave out the proof that this potential indeed yields closed orbits, but it is actually not very complicated and can be found in reference [6].

This completes the section on Bertrand's theorem and before going on to two-body-problems, I would like to state the theorem once more to wrap things up:

Within all possible radially symmetrical potentials, potentials of the form $V = -\alpha/r$ or $V = \alpha r^2$ (α being a positive real number) guarantee that all bound orbits are also closed orbits. Furthermore, these are the only two potentials with that property.

[29] The completion of the proof will then be presented in the appendix, but a majority of the way there will be gone through in section 4.1.

4 The two-body problem

Just as in the case of the one-body problem, where we managed to get a solution by eliminating one of the variables and solving the remaining equation in one dimension at a time, a similar strategy will help us to get hold of the two-body problem. Also in analogy to the one-body problem, it is helpful to make use of conserved quantities. If $\vec{r}_1$ and $\vec{r}_2$ describe the position of the two point masses with respect to time, the energy follows as:

$$E = T + V = \frac{m_1 \dot{\vec{r}}_1^2}{2} + \frac{m_2 \dot{\vec{r}}_2^2}{2} + V(\vec{r}_1, \vec{r}_2). \tag{47}$$

In the one-body problem, we assumed the existence of a potential without really questioning where it comes from, as postulating a second point mass for its origin would immediately make the problem at hand fall outside its own definition of a **one**-body problem. Now that we have two point masses to account for, this problem is gone and we can ascribe the force experienced by one point mass to the presence of the other. We will again work with radially symmetrical potentials in the sense that forces only act along the connecting line between the point masses and their magnitude is merely dependent on the mutual distance between $\vec{r}_1$ and $\vec{r}_1$. If we now define a new variable $\vec{r} \coloneqq \vec{r}_1 - \vec{r}_2$, the potential can be written as a function of the scalar quantity of this one variable only.

$$V(\vec{r}_1, \vec{r}_2) = V(r) \tag{48}$$

Furthermore, if we define $\vec{R}$ to be the center of mass of the system, $\vec{r}_1$ and $\vec{r}_2$ both can be written as a function of the new variables $\vec{R}$ and $\vec{r}$, as can easily be checked by just inserting the definitions (of course it is necessary for at least one of the two masses m_1 and m_2 to be greater than zero for $\vec{R}$ to be well defined, but this is a rather theoretical restriction, especially as we will later only be dealing with gravitational attraction, but even for the general case a scenario where this would not hold would be quite contrived) [4, equations 49 - 51 were motivated by p. 132 and p. 134].

$$\vec{R} \coloneqq \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \quad \Rightarrow \quad \vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r}, \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r} \tag{49}$$

If we make two more definitions, we can find an alternative formulation for equation 47 and the solution to the two-body problem becomes surprisingly straightforward without necessitating new concepts or methods that we have not already used for the one-body problem. The total mass of the system will be assigned to the capital M and the reduced mass to the Greek μ.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad M = m_1 + m_2 \tag{50}$$

To wrap up all the definitions we have made so far, they are summarised in table 1.

Table 1: definitions of variables in the two-body problem

variable	description	definition
$\vec{r}_1$	position vector of point mass one	-
$\vec{r}_2$	position vector of point mass two	-
m_1	mass of point mass one	-
m_2	mass of point mass two	-
$\vec{r}$	position vector of point mass one relative to point mass two	$\vec{r}_1 - \vec{r}_2$
$\vec{R}$	position vector of the center of mass of the two point masses	$(m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$
M	total mass	$m_1 + m_2$
μ	reduced mass	$(m_1 m_2)/(m_1 + m_2)$

Equipped with all of these variables, we can express the total energy contained in the system another way. Again, the correctness of equation 51 can be checked by inserting the definitions from table 1 and comparing

the result with equation 47.

$$E = \frac{M\dot{\vec{R}}^2}{2} + \frac{\mu\dot{\vec{r}}^2}{2} + V(r) \tag{51}$$

What we have actually done by rewriting the energy is we have separated the movement of the center of mass of the system (the first term) from that of the relative motion of the two point masses (terms two and three). As we know from section 2.2, the center of mass will move with constant velocity in a straight line, so the contribution of the first term to the energy is not changing over time. Therefore, if you compare equation 51 with equation 21, you will find an exact equivalence in their layout, the only difference being the naming of the constants. E changes to $E - M\dot{\vec{R}}^2$ (which is still a constant!, as argued just above) and m changes to μ. Strictly speaking, v^2 also changes to $\dot{\vec{r}}^2$, but that is of course the same thing anyway, just a different notation. Therefore, as the angular momentum of the system is again conserved (see section 2.2), the exact same arguments and solutions from the one body problem can be carried over. We are so to say solving a one-body problem for a fictitious mass point with a fictitious mass and are practically done with the whole two-body problem!

All that is left to do is to put the different pieces together. You can solve for the motion of the center of mass of the system by merely working with the initial conditions (remember that $\dot{\vec{R}}$ stays constant over time), you can solve for the relative motion between the two point masses using the methods presented in chapter 3 and you can finally determine the time-dependent position of $\vec{r}_1$ and $\vec{r}_2$ using equation 49.

A specific example of this procedure is given in the next section, where $V(r)$ is the gravitational potential (In practice, hardly any other potential will be relevant, as in the only other prominent force that can arise for two-body problems, namely the electrostatic force between charged particles, the same inverse square relationship holds as in the case of gravity.).

4.1 The main result of the Kepler problem

There is a lot that could be said about the Kepler problem, such as how the three famous Kepler laws for planetary motion can be derived from it. However, in order to be able to proceed to the three-body problem as quickly as possible and also because there are numerous good resources where these derivations can be checked (for instance p. 141 - 150 of reference [4]), I decided to only go through the parts that will be absolutely necessary for following the further development of this thesis.

Our task is to solve equation 51. We can pick the coordinate system in which we want to describe all of the trajectories taken such that the centre of mass ($\vec{R}$) remains stationary within it. If we do that, the first term after the equals sign in equation 51 vanishes (it would otherwise be a constant that could be moved to the left side of the equation as explained above) and what we are left with is [4, p. 141]:

$$E = \frac{\mu\dot{\vec{r}}^2}{2} - \frac{\alpha}{r} = \frac{\mu\dot{r}^2}{2} + \underbrace{\frac{L^2}{2\mu r^2} - \frac{\alpha}{r}}_{V_{eff}}. \tag{52}$$

In the first step I exchanged the potential from equation 51 with the gravitational potential and in the second step I used equations 22 and 23 to write the energy as a function of r only. Let's quickly check that the angular momentum L can really be inserted in just the same way as it could be in the one-body problem. Back in equation 20, I calculated L to be equal to $mr^2\dot{\theta}$. If we can now show that $L = \mu r^2\dot{\theta}$, then the rearrangements of equation 22 can be carried over and we are good to go. So lets work out the angular momentum of our two-body problem, keeping in mind that we aligned the centre of our coordinate system

with the centre of mass of the system.

$$\begin{aligned}\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = m_1 r_1^2 \dot{\theta}_1 \vec{e}_{\varphi_1} + m_2 r_2^2 \dot{\theta}_2 \vec{e}_{\varphi_2} = (m_1 r_1^2 + m_2 r_2^2)\dot{\theta}\vec{e}_\varphi = \\ (m_1(\frac{m_2}{m_1+m_2}\vec{r})^2 + m_2(-\frac{m_1}{m_1+m_2}\vec{r})^2)\dot{\theta}\vec{e}_\varphi = \\ (m_1\frac{m_2^2}{(m_1+m_2)^2} + m_2\frac{m_1^2}{(m_1+m_2)^2})r^2\dot{\theta}\vec{e}_\varphi = \\ (m_2+m_1)\frac{m_1 m_2}{(m_1+m_2)^2}r^2\dot{\theta}\vec{e}_\varphi = \\ \frac{m_1 m_2}{m_1+m_2}r^2\dot{\vec{\theta}}\vec{e}_\varphi = \\ \mu r^2 \dot{\theta}\vec{e}_\varphi \\ \Rightarrow\ L = \mu r^2 \dot{\theta}\end{aligned} \tag{53}$$

In the last step of the first line I used $\vec{e}_\varphi \coloneqq \vec{e}_{\varphi_1} = \vec{e}_{\varphi_2}$ and $\dot{\theta} \coloneqq \dot{\theta}_1 = \dot{\theta}_2$. The two unit vectors in the φ_1- and φ_2-direction are identical because they are perpendicular to the plane of motion which is the same for both $\vec{r}_1$ and $\vec{r}_2$. $\dot{\theta_1} = \dot{\theta_2}$ is true because we aligned the origin of our coordinate system with the centre of mass ($\Leftrightarrow \vec{R} = 0$). Therefore, if θ_1 changes then θ_2 must change by the same amount, as otherwise the center of mass (which always lies on the connecting line between $\vec{r}_1$ and $\vec{r}_2$) would move away from the origin. In the second line, I expressed $\vec{r}_1$ and $\vec{r}_2$ by the relative coordinates (see equation 49), again making use of $\vec{R} = 0$. In the last but one line, I finally used the definition of μ from equation 50, showing that the angular momentum does indeed take on the value $\mu r^2 \dot{\theta}$.

The solution to equation 52 was already derived in equations 25 through 27 with the only difference that we now have to replace m with μ. As what I want to derive in this section is the trajectory taken and not really the time dependence of it, the relevant solution is the one where the parameter r is related to the parameter θ (that is equation 27).

$$\theta(r) = \pm \int \frac{L}{r^2\sqrt{2\mu(E - V_{eff}(r))}}dr + C \tag{54}$$

Essentially, equation 55 is equation 27 rewritten with m replaced with μ. It already constitutes the solution we were looking for and in most cases where you have integrals that look as nasty as this one, one would just take a computer and tell it to come up with a nice polar plot to visualise it. Even though in this case it is possible to simplify the integral analytically, I decided to just state the end result here and put the step by step calculation into the appendix, as it involves a number of substitutions and is a little lengthy [4, equations 55 - 58 were motivated by p. 141].

$$\theta(r) = \pm \int \frac{L}{r^2\sqrt{2\mu(E - (\frac{L^2}{2\mu r^2} - \frac{\alpha}{r}))}}dr + C = \ \ldots\ = \pm \arccos(\frac{\frac{L}{r} - \frac{\mu\alpha}{L}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}}) + C \tag{55}$$

At first glance, this result still doesn't look very handy, but its appearance can still be significantly improved. First, we can ignore the integration constant C, since it just adds a constant to every θ-value and we can always just rotate our coordinate system into the θ-direction until that constant vanishes. We can also ignore the $\pm$ sign and make a plus out of it, since it just makes a statement about the direction of the motion. But as we are merely interested in the path of the trajectory and not if it is followed from left to right or vice versa we don't have to worry about it. By taking the cosine on both sides, we can now get rid of the arccosine.

$$\cos(\theta) = \frac{\frac{L}{r} - \frac{\mu\alpha}{L}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}} = \frac{\frac{1}{r} - \frac{\mu\alpha}{L^2}}{\sqrt{\frac{2EL^2}{\mu\alpha^2} + 1}\frac{\mu\alpha}{L^2}} \tag{56}$$

In the second step, I pulled the expression $\frac{\mu^2\alpha^2}{L^2}$ out of the root in the denominator and divided both numerator

and denominator by L. We can now define two new parameters to further simplify the expression.

$$\epsilon \coloneqq \sqrt{1 + \frac{2EL^2}{\mu\alpha^2}} \qquad p \coloneqq \frac{L^2}{\mu\alpha} \tag{57}$$

Using these definitions turns equation 56 into:

$$\cos(\theta) = \frac{\frac{1}{r} - \frac{1}{p}}{\epsilon\frac{1}{p}} \Leftrightarrow \epsilon\cos(\theta) = \frac{p}{r} - 1 \Leftrightarrow \boxed{\frac{p}{r} = \epsilon\cos(\theta) + 1}. \tag{58}$$

The equation in the box now looks quite simple and it allows us to investigate what the trajectory will look like. A very crucial role is played by the parameter ϵ which is also called the eccentricity of the orbit. We know from figure 2 in section 3.1. that if the energy of the system is below zero there will be a bound orbit and if it is above zero the orbit will be unbound. This is also reflected in equation 58. Suppose $E > 0$. Then it follows that $\epsilon > 1$ from equation 57 (as L^2, μ and α^2 are all greater than zero). But this would mean that the right hand side of the equation in the box goes to zero at a certain angle θ, so also the left side must go to zero at that angle. However, that can only be if r goes to infinity, which is equivalent to saying that the orbit is unbound. The advantage of the boxed equation as oppose to figure 2 is that it allows you to calculate the exact orbit. For $\epsilon > 1$ this orbit turns out to be a hyperbola and for $\epsilon = 1$ it is a parabola. For $\epsilon < 1$ the orbit is bound. This can either be argued by noticing that $\epsilon < 1$ implies $E < 0$ or by again looking at the equation in the box. The maximum value of r is reached whenever the right side of the equation reaches its minimum. But for $\epsilon < 1$ this minimum will necessarily be greater than zero, meaning that r_{max} must be smaller than infinity. The exact shape of the orbit turns out to be an ellipse. As for $\epsilon > 1$ and $\epsilon = 1$, I will not present the proof thereof in this section, but because it is important for validating Bertrand's theorem (see section 3.2), I put it into the appendix. A special kind of ellipse occurs when $\epsilon = 0$. In that case, the equation in the box can be rearranged to $r = p$. But p is a constant (see equation 57), meaning that r is also constant and that what you end up with is a perfect circle. This $\epsilon = 0$ orbit will become important again in chapter 5.2 when dealing with the restricted three-body problem.

5 Selected aspects of the three-body problem

It was possible to solve the one-body problem by eliminating one of the two variables, it was possible to solve the two-body problem by reducing it to a one-body problem and now it wouldn't seem too far fetched to assume that in just the same way a three-body problem could be solved by reducing it to a two-body problem and an n-body problem by reducing it to an (n-1)-body problem. However, despite the most brilliant thinkers of the last three centuries having spent numerous hours puzzling their heads over it, no general solution to the three-body problem could yet be found.

In equation 59, I put the defining equations of motion for three bodies of mass under mutual gravitational attraction.

$$\begin{aligned} m_1 \frac{d^2\vec{r}_1}{dt^2} &= Gm_1m_2 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} + Gm_1m_3 \frac{\vec{r}_3 - \vec{r}_1}{|\vec{r}_3 - \vec{r}_1|^3} \\ m_2 \frac{d^2\vec{r}_2}{dt^2} &= Gm_2m_1 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} + Gm_2m_3 \frac{\vec{r}_3 - \vec{r}_2}{|\vec{r}_3 - \vec{r}_2|^3} \\ m_3 \frac{d^2\vec{r}_3}{dt^2} &= Gm_3m_1 \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3} + Gm_3m_2 \frac{\vec{r}_2 - \vec{r}_3}{|\vec{r}_2 - \vec{r}_3|^3} \end{aligned} \tag{59}$$

In the equations above $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ represent the position vectors of the three bodies of mass, m_1, m_2 and m_3 represent their masses and G is the universal gravitational constant. Notice that there are in fact nine scalar equations to be solved, as for each of the three vector equations above the x-, the y- and the z-component are themselves scalar equations that have to be fulfilled.

What I would like to do next is to quickly recap the key ideas and insights that helped us solve the one- and the two-body problem and see if any of the ideas can be carried over to the three-body problem. In the one-body problem, we noticed that the radial symmetry of the potential necessarily reduced the trajectory taken to a two-dimensional plane slicing through three dimensional space, which allowed us to describe the position of the object with two variables only, namely r and θ. Next, we made use of the fact that the energy and the angular momentum are conserved quantities and we could insert the angular momentum into the expression for the energy, thereby getting rid of the parameter θ and leaving r as the only parameter left to find the time dependence of. The rest was a comparatively easy task of solving the remaining equation for r and finally expressing θ as a function of r and t. In the two-body problem, we used relative coordinates to express the energy and found that the motion of the center of mass of the system decoupled from the relative motion of the two bodies. Furthermore, the equation describing the relative motion was perfectly equivalent to the one describing the one-body problem (with the only difference that the involved constants had different values), allowing us to carry over the solution from there.

One of the key differences in the three body problem that significantly complicates the issue is the fact that we can no longer reduce the motion of the three bodies to a single two-dimensional plane. If that were the case, we could describe every body of mass with two parameters only, but as in the general case a three-dimensional description is necessary, we are still left with all nine parameters (three per body of mass) to find the time dependence of. The strategy of the two-body problem of coming up with relative coordinates can nonetheless be pursued and is shown in figure 3.

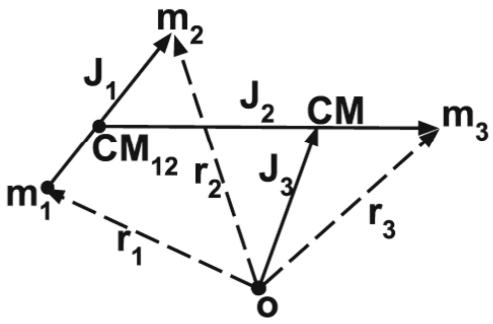

Figure 3: Relative coordinates for three bodies of mass [10, p. 6]
m_1, m_2, m_3 ... masses of objects one, two and three
r_1, r_2, r_3 ... position vectors of objects one, two and three
CM ... centre of mass of the three masses
CM_{12} ... centre of mass of masses one and two
O ... origin of the coordinate system
J_1, J_2, J_3 ... Jacobi vectors

In the figure above, the so-called Jacobi vectors are the generalised coordinate vectors that can be used instead of $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_3}$ to simplify the mathematical description. $\vec{J_1}$ is the vector going from $\vec{r_1}$ to $\vec{r_2}$, $\vec{J_2}$ goes from the centre of mass of masses one and two (CM_{12}) to $\vec{r_3}$ and $\vec{J_3}$ goes from the origin to the centre of mass of all three masses (CM). Equation 60 defines the Jacobi vectors in mathematical terms [10, equations 60 - 62 were motivated by p. 6-7].

$$\vec{J_1} := \vec{r_2} - \vec{r_1} \qquad \vec{J_2} := \vec{r_3} - \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2} \qquad \vec{J_3} := \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3}}{m_1 + m_2 + m_3} \tag{60}$$

Now, let's define three new masses, similar to the reduced mass in the two-body problem.

$$M_1 := \frac{m_1 m_2}{m_1 + m_2} \qquad M_2 := \frac{(m_1 + m_2)m_3}{m_1 + m_2 + m_3} \qquad M_3 := m_1 + m_2 + m_3 \tag{61}$$

Here, M_1 is the reduced mass of the masses one and two, M_2 is the reduced mass of mass three and the sum of masses one and two and M_3 is the sum of all the three masses. In analogy to equation 51, where we managed to express the total energy of the system merely in terms of the reduced mass and the relative coordinates, we can now do the same thing for the three-body problem.

$$E = T + V = \underbrace{\frac{1}{2}\sum_{i=1}^{3} M_i \dot{\vec{J}}_i^2}_{T} \underbrace{-G\frac{m_1 m_2}{|\vec{J_1}|} - G\frac{m_2 m_3}{|\vec{J_2} - \frac{M_1}{m_2}\vec{J_1}|} - G\frac{m_3 m_1}{|\vec{J_2} + \frac{M_1}{m_1}\vec{J_1}|}}_{V} \tag{62}$$

The correctness of equation 62 can easily be checked by inserting the definitions from equations 60 and 61. Doing so does take two or three lines, because you have to square all the jacobi vectors and then look for terms that cancel out, but in the end you are left with exactly the right terms. The kinetic part (T) reduces to $1/2(m_1\dot{\vec{r}}_1^2 + m_2\dot{\vec{r}}_2^2 + m_3\dot{\vec{r}}_3^2)$ and the second part simplifies to the three terms describing the potential between two of the three masses each. So far, things look quite similar to when we looked at the two-body problem. What's more, the third Jacobi vector ($\vec{J_3}$) describing the position of the centre of mass of the whole system only appears in differentiated form in the kinetic term and because of chapter 2.2 (specifically equation 12) we know that its contribution is just a constant, since the centre of mass of a closed system does not accelerate. Therefore, the only parametres we still have to find the time dependence of (the time dependence of $\vec{J_3}$ is determined by the initial conditions) are the components of $\vec{J_2}$ and $\vec{J_3}$, meaning we have again successfully decoupled the motion of the centre of mass from the relative motion between the masses.

This is the point where the complexity of the problem does not allow to proceed further just the way we did in the two-body problem. In the two-body problem, the next thing we did was realise that the equation we were left with was identical to that of the one-body problem and in the one-body problem we then made use of the conservation of angular momentum to get rid of one of the remaining parametres. While one might have hoped that the current form of the three-body problem resembled that of the two-body problem and so allowed us to find a solution by reducing that two-body problem further to a one-body problem, this is quite obviously not the case as can be easily seen when comparing equation 62 to equation 47. But what about angular momentum? While even in the three-body problem it is of course conserved (remember that the conservation laws in chapter 2.2 were derived for arbitrarily many bodies of mass), its expression in terms of the components of $\vec{J}_1$ and $\vec{J}_2$ does not coincide with any of the terms in equation 62, so also angular momentum cannot be used as effectively as it could be used in the one-body problem.

In general, the energy equation cannot really be put into a simpler form than that which we have already derived just above without making some further restrictions on the motion of the three bodies of mass and thereby just looking at special cases of the general three-body problem. Nonetheless, this does not mean that physicists are hopelessly lost when confronted with a three-body problem with arbitrary initial conditions. What can always be done is making use of simulations and computing the solutions to the equations of motion (equation 59) by means of numerical methods. These methods can also quite successfully be applied to problems involving many more bodies of mass than just three, but as they are not the topic of my thesis, I want to hereby merely mention their existence and proceed with problems that can be tackled analytically.

As mentioned in the last paragraph, analytical solutions can only be obtained when we impose some restrictions on our three-body problems and so only consider special cases of the latter. In many cases, equation 62 can then be simplified enough to yield solutions in closed form, in other cases the approach taken is entirely different. In the following paragraphs, I will mention a number of solutions that have been discovered over the last centuries, but I will not derive all of them, as that would fill too many pages and go beyond the expected scope of this theses. Instead, I will present the analytical approach of merely one of the special cases in section 5.2.

5.1 Special solutions

One of the most obvious special cases to consider is the one where the movement of the three masses is restricted to a two-dimensional plane. Under this restriction, Euler (1767) and Lagrange (1772) found two different periodic special solutions which are depicted in figure 4. In the Euler solution, the three masses are collinear, meaning they are always connected by a straight line which therefore also always contains the centre of mass of the system. Furthermore, each mass by itself follows a keplerian orbit about the centre of mass (see section 4.1. for keplerian orbits). In the Lagrange solution, the three masses are positioned at the vertices of an equilateral triangle which rotates around the centre of mass and may also change its size as a function of time, as shown in figure 4 (b) [10, p. 7-8].

Notice that in the Euler solution as well as in the Lagrange solution, the masses are labelled with subscripts 1 through 3 and not merely with the letter m, implying that the masses need not be identical and can take on arbitrary ratios. Also, both solutions constitute so-called central configurations, meaning that the force experienced by each mass is always directed towards the centre of mass and proportional to its distance from it. Incidentally, the Euler and the Lagrange solutions turn out to be the only central configurations in the three-body problem [10, p. 8-9].

Figure 4: Collinear Euler and Lagrange solutions [10, p. 7]
(a) ... Euler solution
(b) ... Lagrange solution
m_1, m_2, m_3 ... masses one. two and three
CM .. centre of mass

Another type of restriction one could impose on the three-body problem results in the so-called restricted three-body problem. It is considerably more complex than the restriction to a 2D plane, but it is also way more realistic and has got quite some interesting practical applications. It assumes that one of the three masses is so much lighter than the other two that its gravitational effect on the big masses can be neglected. An example of where this situation might occur is when you consider the earth and the moon and then you want to send a sattelite into orbit around the earth. While the sattelite experiences gravitational forces both from the earth as well as from the moon, its own mass is so little that it is more than save to say that its gravitational pull on the earth and the moon can be neglected without diminishing the quality of the resulting calculations. The approach is therefore the following that you solve the two-body problem for the two big masses, and then you solve the one-body problem for the light mass, where the potential is defined by the time-dependent position of the two big masses. Notice that this potential is not radially symmetrical, so you cannot just go back to chapter 3 and use the argumentation presented there (which only applies to radially symmetrical potentials). Unfortunately, the resulting equations are already too complex to allow for an analytical expression of the general solution. However, a quite interesting and practically relevant result can still be obtained analytically and will be the topic of section 5.3.
If you further simplify the restricted three-body problem to the situation that the position of the two big masses is fixed in place and does not move and if you additionally restrict the motion of the third mass to a single plane you arrive at the planar Euler three-body problem. In this case the problem can again be solved analytically as Euler showed in 1760 [10, p. 11-12 and p. 15].

Before continuing with an analytical analysis of the restricted three-body problem in chapter 5.2, I want to mention one more special solution which was discovered not so long ago in 1993 by Chris Moore. In this solution, every body of mass is tracing out the same trajectory which looks like a figure eight, hence it is often referred to as the "figure-8 solution". It requires all three masses to be the same (despite the potentially misleading subscripts in figure 5), to be equally spaced in time and to produce a vanishing total angular momentum. Even though the figure-8 solution was shown to be a stable configuration (meaning small deviations in initial conditions still yield essentially the same outcome), the probability of finding such a configuration in outer space was calculated to be extremely low (between one per galaxy and one per universe) [10, p. 21-22] [11, p. 6].

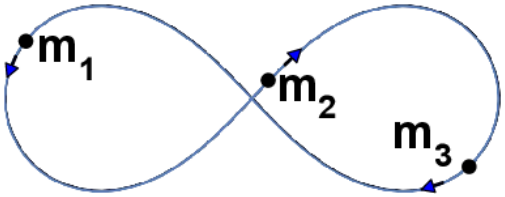

Figure 5: Figure eight solution [12]
$m_1 = m_2 = m_3$

5.2 The restricted three-body problem

As already explained in section 5.1, the endeavour in the restricted three-body problem is to determine the trajectory of a body of mass (let's call it m) that moves within the time-dependent gravitational potential generated by two other bodies of mass (let's call them m_1 and m_2). These two other bodies of mass must be a lot heavier than the third one and their own trajectory is determined by the two-body problem for gravitational attraction solved in section 4.1.

For the sake of simplifying the mathematical description, it is helpful to describe the positions of the three mass points in a rotating coordinate system which is oriented such that its origin coincides with the center of mass of m_1 and m_2, its z-axis is parallel to the angular momentum vector $\vec{L}$ [30] and the y-axis passes through both masses m_1 and m_2 (such a coordinate system is illustrated in figure 7). Of course, the rotating coordinate system also comes with a cost, namely that the equations of motion have to include so-called inertial forces, but this is a price well worth paying in light of how much easier the resulting equations can be analysed. So let's look at how these inertial forces are brought about.

5.2.1 The Centripetal force and the Coriolis force

Before deriving the formulas for any inertial forces, there are two important concepts that need to be understood. I will use figure 6 to try to explain them.

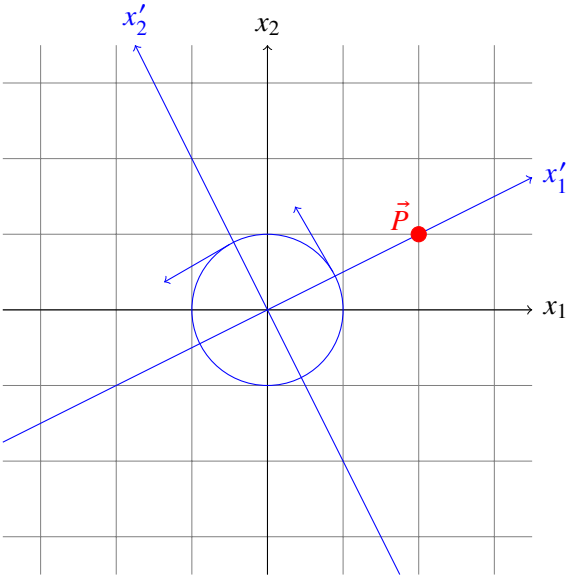

Figure 6: Rotated coordinate systems
The distance between the gray grid lines represents one unit

[30] The angular momentum vector itself is perpendicular to the vectors going from the centre of mass of m_1 and m_2 to either of the two masses because of the definition $L = \vec{r_1} \times \vec{p_1} + \vec{r_2} \times \vec{p_2}$ and the fact that the centre of mass coincides with the coordinate origin.

In black, I plotted a conventional two-dimensional coordinate system with the axes labelled x_1 and x_2 along with some gray gridlines. On top, I put a blue coordinate system that is rotated 30° counter-clockwise with respect to the black one and which has axis labels x_1' and x_2' (the blue circle with the two arrows will become important later). In addition, I marked one of the intersections of two of the grid lines with a red dot and called the position $\vec{P}$ [31]. The question that arises is the following: Which are the appropriate coordinates of the position $\vec{P}$?

If you ignored the blue coordinate system, the answer would be quite obvious: $\vec{P} = (2, 1)$. But for someone thinking in terms of the blue coordinate system, $\vec{P}$ is a point somewhere on the x_1'-axis. He could even determine the exact value of the x_1'-coordinate from the Pythagorean theorem to be $\sqrt{2^2 + 1^2} = \sqrt{5}$, so from his perspective the correct location of $\vec{P}$ would be $\vec{P} = (\sqrt{5}, 0)$. Most obviously $(2, 1) \neq (\sqrt{5}, 0)$ even though we just argued that both coordinates represent the exact same position in space. Therefore just offering numeric values of two coordinates doesn't have any meaning unless we specify which coordinate system we are talking about (or there is only one coordinate system in question). I will now present two ways of fixing the problem that will both be used in the following derivations. The first one is to call the black coordinate system the k-system and the blue one the k'-system and to always add this label. Using this convention, we could write $\vec{P}|_k = (2, 0)$ as well as $\vec{P}|_{k'} = (\sqrt{5}, 0)$ without being unclear about what we really mean. The second possibility would be to consider every vector as a linear combination of some basis vectors. In a two dimensional coordinate system there are two basis vectors, namely the unit vector in x_1-direction $\hat{x}_1$ (or $\hat{x}_1'$ for the blue system) and the unit vector in x_2-direction $\hat{x}_2$ (or $\hat{x}_2'$ for the blue system) [32]. This way, we can write: $\vec{P} = 2\hat{x}_1 + 1\hat{x}_2 = \sqrt{5}\hat{x}_1' + 0\hat{x}_2'$. This is the first of the two concepts that need to be introduced.

The second one concerns the description of time-dependent positions in different coordinate systems. Imagine that the blue coordinate system in figure 6 starts rotating around its origin. Furthermore, imagine that the red dot is glued to the blue coordinate system at the position $\sqrt{5}\hat{x}_1' + 0\hat{x}_2'$, so it is moving around along with the blue system. Now, what is the vector indicating the velocity of the red dot [33]? With respect to the blue system it is zero, since it always sits at the same location on the x_1'-axis. However, with respect to the black system it is obviously not zero, as the blue system is moving with respect to the black one. Notice that while in the last paragraph we were describing the **same** vector $\vec{P}$ in two different systems and came up with representations that looked quite different ($(2, 1)$ vs. $(\sqrt{5}, 0)$), we are now looking at a vector that not only looks different depending on the representation we choose, but which really **is** different (in the blue system, the red dot is stationary, whereas in the black one it is moving)[34].
Now, lets add one more step and lets allow the red dot to move within the blue coordinate system. At the time when it leaves the position $\sqrt{5}\hat{x}_1' + 0\hat{x}_2'$, what is its velocity in the black system? It is a superposition of the velocity of the red dot within the blue system and the velocity of the position $\sqrt{5}\hat{x}_1' + 0\hat{x}_2'$ with respect to

[31] This has got nothing to do with the momentum $\vec{P}$ but really only describes the position of the red dot. Unfortunately, all Roman letters have the one or the other physical meaning, so it is almost impossible to choose a letter that does not have the potential to be misinterpreted.

[32] In fact, one can choose any two vectors (as long as they are not parallel), define them as ones basis and express every other vector as a linear combination thereof. However, $\hat{x}_1$ and $\hat{x}_2$ are the most obvious choices, so I will stick with those.

[33] The vector representing the velocity of an object always points in the direction in which the object is moving and its length is equal to the speed with which it is moving.

[34] If the blue coordinate system was allowed to move freely with respect to the black one (meaning that also its origin could move away from the black origin), then not only the velocity of the red dot would be described by different vectors but also its position. One would then introduce a vector $\vec{R}_0$ that points from the origin of the black system to that of the blue system. If we called the position of the red dot with respect to the black system $\vec{P}$ and with respect to the blue system $\vec{P}'$, we could then write $\vec{P} = \vec{R}_0 + \vec{P}'$. For describing the most general case, the vector $\vec{R}_0$ would have to be carried along in all of the following calculations. This would not really make it more difficult to derive all of the necessary equations, but for the sake of a shorter notation and also because I will later on only talk about inertial forces that arise in purely rotational systems, I decided not to do so.

the black system [35]. Mathematically, this looks like this:

$$(\text{velocity of } \vec{P})|_k = (\frac{d}{dt}\vec{P})|_k = (\frac{d}{dt}\vec{P})|_{k'} + (\frac{d}{dt}(\sqrt{5}\hat{x}_1' + 0\hat{x}_2'))|_k = (\frac{d}{dt}\vec{P})|_{k'} + (\sqrt{5}\frac{d}{dt}\hat{x}_1' + 0\frac{d}{dt}\hat{x}_2')|_k. \tag{63}$$

Notice that equation 63 is only correct at the point in time where $\vec{P}$ leaves the position $\sqrt{5}\hat{x}_1' + 0\hat{x}_2'$. A split second later, the position of $\vec{P}$ has changed with respect to the black system and so the expression $\sqrt{5}\hat{x}_1' + 0\hat{x}_2'$ in the last term in equation 63 has to be exchanged by that new position. In order to get a more general formulation, we therefore need to substitute the values $\sqrt{5}$ and 0 with general terms. Let's call these general terms p_1 and p_2 in the black system and p_1' and p_2' in the blue system. With $\vec{P} = p_1\hat{x}_1 + p_2\hat{x}_2 = p_1'\hat{x}_1' + p_2'\hat{x}_2'$, equation 63 can be reformulated [4, equations 64 - 67 were motivated by p. 43-44].

$$\begin{aligned}(\frac{d}{dt}\vec{P})|_k = (\frac{d}{dt}(p_1'\hat{x}_1' + p_2'\hat{x}_2'))|_{k'} + (p_1'\frac{d}{dt}\hat{x}_1' + p_2'\frac{d}{dt}\hat{x}_2')|_k = \\ \hat{x}_1'\frac{d}{dt}p_1' + \hat{x}_2'\frac{d}{dt}p_2' + (p_1'\frac{d}{dt}\hat{x}_1' + p_2'\frac{d}{dt}\hat{x}_2')|_k = \\ \underbrace{\sum_i \hat{x_i}'\frac{d}{dt}p_i'}_{=(\frac{d}{dt}\vec{P})|_{k'}} + \sum_i p_i'\frac{d}{dt}\hat{x}_i'\end{aligned} \tag{64}$$

In the last line, I introduced a notation with two sums but without specifying the boundaries for the index i. I hope to have given a plausible explanation for why this formula is correct in two dimensions (that is for $i = 1, 2$). However, the exact same reasoning can be applied to three dimensions and then i would take on the values one, two and three, where the third direction can be thought of as sticking out of the plane of the paper in figure 6. One last remark before going on: even though I think it is a good exercise to try and comprehend the arguments I presented in the last paragraphs in specific and the physical arguments behind a physical phenomenon in general, it is often quite remarkable to me how almost trivial certain derivations can become if you just use the rules of mathematics alone. For the subject at hand, if we write $\vec{P} = \sum_i p_i\hat{x}_i$ and then we would ask for the time derivative of $\vec{P}$, one could immediately write down the last line of equation 64 by simply making use of the product rule for differentiation.

Now we are ready to derive the inertial forces. Let's again take figure 6 as an illustration, but lets imagine a third axis which coincides for the black and the blue system and sticks out of the paper in a 90° angle to all of the other coordinate axes (let's call this axis x_3-axis or x_3'-axis interchangeably). Let's assume that the black system is not accelerating, meaning that the force law $F = ma$ holds within in. The blue system should rotate counter-clockwise with respect to the black system with an angular velocity of ω. The vector describing this angular velocity is then called $\vec{\omega}$ and is parallel to the x_3-axis. Now let's introduce an object with mass m that moves along a trajectory $\vec{r}(t)$ and let's calculate its velocity in the black system using the last line of equation 64. Let's first think about the term at the very end describing the velocity of the unit vectors in the blue system: $d/dt(\hat{x}_i')$. To simplify the imagination, I drew a blue unit circle centred at the origin of the coordinate systems in figure 6. Notice that the tip of the unit vectors $\hat{x}_1'$ and $\hat{x_2}'$ will always lie on that unit circle, so that their velocity will always be tangential to it. I indicated the direction of this velocity with the two blue arrows (of course, as the blue system rotates, these two arrows will move along and their direction will change). Now what about the magnitude of the velocity? The angular velocity ω describes by how many radians the blue system rotates every second. But this is exactly equal to the arclength of a unit circle enclosing the angle ω, in other words it is equal to the distance the vectors $\hat{x}_1'$ and $\hat{x}_2'$ move within one second. The velocity of the unit vectors of the blue system can therefore be written as $d/dt(\hat{x}_i') = \omega \times \hat{x}_i'$, as this cross product exactly fulfills the properties described above. It produces vectors

[35] The way I imagine how this superposition is brought about is by discretising space and time. When asking about the velocity of $\vec{P}$, I am really asking how its position changes from one time step to the next. And this change in position can quite easily be thought of as the change of position within the blue system plus the change in position of $\sqrt{5}\hat{x}_1' + 0\hat{x}_2'$ within the black system.

pointing in the directions of the blue arrows and their magnitude is equal to the area of the rectangle formed by $\vec{\omega}$ and $\hat{x}_i'$ (this follows from the properties of the cross product). But this area is equal to the magnitude of $\vec{\omega}$, since the magnitude of $\hat{x}_i'$ is one. Now, we can write:

$$\vec{v} := (\frac{d}{dt}\vec{r})|_k = (\frac{d}{dt}\vec{r})|_{k'} + \sum_i r_i'(\vec{\omega} \times \hat{x}_i') = \vec{v}' + \vec{\omega} \times (\sum_i r_i' \hat{x}_i') = \vec{v}' + \vec{\omega} \times \vec{r}. \tag{65}$$

I replaced $(\frac{d}{dt}\vec{r})|_k$ with $\vec{v}$ and $(\frac{d}{dt}\vec{r})|_{k'}$ with $\vec{v}'$ to represent the velocities in the two systems. Next, let's think about the acceleration ($\vec{a}$) in the black system. To do so, we have to take the vector $\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}$ and look at how it changes with respect to time. How do we do that? Exactly the same way we figured out how $\vec{r}$ changed with time, by making use of equation 64 [36][37].

$$\begin{aligned} \vec{a} := (\frac{d}{dt}\vec{v})|_k = \frac{d}{dt}(\vec{v}' + \vec{\omega} \times \vec{r})|_k = \frac{d}{dt}(\vec{v}' + \vec{\omega} \times \vec{r})|_{k'} + \sum_i (\vec{v}' + \vec{\omega} \times \vec{r})_i'(\vec{\omega} \times \hat{x}_i') = \\ \frac{d}{dt}(\vec{v}')|_{k'} + \vec{\omega} \times \frac{d}{dt}(\vec{r})|_{k'} + \vec{\omega} \times (\sum_i (\vec{v}' + \vec{\omega} \times \vec{r})_i' \hat{x}_i') = \\ \vec{a}' + \vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{v}' + \vec{\omega} \times \vec{r}) = \\ \vec{a}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \end{aligned} \tag{66}$$

Now, we are almost done. Multiplying with m to get an expression for how the forces relate yields:

$$\begin{aligned} \overbrace{m\vec{a}}^{=\vec{F}} = \overbrace{m\vec{a}'}^{=\vec{F}'} + 2m\vec{\omega} \times \vec{v}' + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \Rightarrow \\ \boxed{\vec{F}' = \vec{F} \underbrace{-2m\vec{\omega} \times \vec{v}'}_{\text{coriolis force}} \underbrace{- m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{centripetal force}}}. \end{aligned} \tag{67}$$

Equation 67 is the main result that we have been working for in all of section 5.2.1. It applies whenever you deal with rotating coordinate systems such as the surface of the earth or a roundabout on a playground (the latter being an example where large forces are more easily generated) [38]. What equation 67 is saying is that in such a coordinate system, the force equation $\vec{F} = m\vec{a}$ no longer holds. Instead, for each force that you measure in your system ($\vec{F}'$), you have to add/subtract two so-called inertial forces, namely the coriolis force and the centripetal force.

The centripetal force always points towards the centre of rotation and can be experienced in a roundabout by the fact that the back rest of the seat exerts ever larger forces on your back the faster you rotate. However,

[36] In the derivation of equation 64, I always mentioned that the vector we want to find the time dependence of represents a position in space. However, now that the vector we differentiate represents a velocity, we can still make use of the same equation as the whole derivation thereof could just as well be done with no reference to what the vector we are investigating actually represents.

[37] When writing these pages, my original idea about explaining the acceleration in rotating systems was actually to say that just like we got the speed of the red dot in figure 6 by adding its speed within the blue system to the speed of the location $\sqrt{5}\hat{x}_1'$ within the black system, the acceleration of the red dot could also be obtained by adding its acceleration within the blue system to the acceleration of $\sqrt{5}\hat{x}_1'$ within the black system. It took me quite some time to see why this is wrong and here is why: While it is true that whenever you have multiple forces acting on an object you can just add them together one by one, the part of the acceleration that is only seen in the black system (that is the part that comes from the movement of the blue system) depends on the actual position of the red dot within the blue system and because $\vec{v}' \neq 0$ it is continuously changing. If the movement of the blue system with respect to the black one was purely translational, my original approach would have been valid, but because it is rotational the second acceleration term changes whenever you change your position in the blue system and you have to take the route that I explained in the main text.

[38] If you are in a roundabout on a playground, you are strictly speaking within a rotating coordinate system which itself is embedded in another rotating coordinate system, namely the earth. However, as the coriolis force and the centripetal force both scale with ω (which in the case of the earth is only about 2π/the number of seconds per day $= 2\pi/86400$), you can neglect the rotation of the earth in a first approximation when investigating the roundabout.

for someone sitting inside the roundabout it does not really feel as if the seat is pushing towards the center, but rather as if "something else" is pushing away from the centre and that the seat is just there to withstand that force. This "something else" is referred to as centrifugal force and it is directed away from the rotation centre. Notice that in equation 67 I included the minus sign in the bracket for the coriolis force whereas I did not include it in the bracket for the centripetal force. Along with the minus sign the expression would be called centrifugal force, so you can either add the centrifugal force or equivalently subtract the centripetal force and get the same result.

The coriolis force only appears when you move within the rotating system and its direction is perpendicular both to $\vec{\omega}$ and to $\vec{v}'$. You can experience it the following way. Imagine standing in the centre of a big roundabout where you just rotate around an axis going straight through your body. Let's say the roundabout is rotating counter-clockwise. If you now pick any direction and you make a big step forward, the motion of the roundabout will pull you to the left, because while you were only rotating about your axis before, you are now part of a circular trajectory around the centre of the roundabout. It is not even necessary that your starting point is the centre of the roundabout. If you stand on an arbitrary spot and you make a step away from the centre, you will still experience a force to the left as your new location has a bigger velocity, since it has to cover the circumference of a bigger circle within an unaltered time period. Likewise, if you make a step towards the centre of the roundabout you will experience a force to the right and if you make a step in an arbitrary other direction you will always experience a force that is perpendicular to the direction of the step you are taking.

This is pretty much all there is to say about inertial forces that appear in rotating coordinate systems. Now we are ready to tackle the restricted three-body problem.

5.2.2 The Jacobi integral

Figure 7 shows the set-up with which we are going to approach the restricted three-body problem.

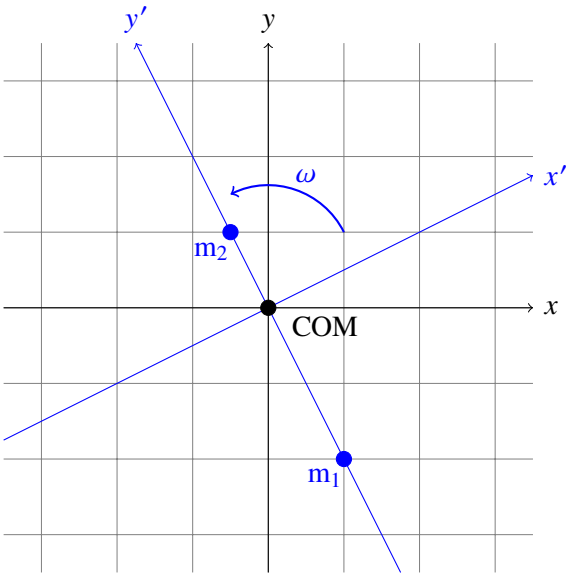

Figure 7: Coordinate system for the restricted three-body problem
The distance between the gray grid lines represents one unit
m_1, m_2 ... masses one and two
COM ... centre of mass of m_1 and m_2
black system ... resting coordinate system k in which $F = ma$ is valid
blue system ... rotating coordinate system k' in which m_1 and m_2 are stationary
ω ... angular velocity with which the blue system rotates relative to the black one
The z-axis is not shown here, but it coincides with the z'-axis and sticks out of the plane of the paper, being perpendicular to the x-y-plane (the vector $\vec{\omega}$ is parallel to it)

Let me first clarify a notational issue. Notice that while I used axis labels x_1 and x_2 (or x_1' and x_2' for the blue system respectively) in the previous section, I now labelled them x and y (or x' and y' for the blue system). Both versions are commonly used in physics and I would have preferred sticking with one of them. However, I needed the first version in section 5.2.1 in order to be able to introduce the sum notation where I summed over the index i (see equation 64), whereas I need the second version now in order to not get confused with too many subscripts. While in the previous section we were just concerned with a single trajectory we now have to keep track of three, and in this section I will differentiate between them using subscripts. Whenever you see a subscript one it means that I am talking about the first body of mass, subscript two indicates the second body of mass and if there is no subscript at all, it means that I am talking about the third mass, the one that we are mainly interested in. The trajectory of masses one and two is already known, because they behave according to the Kepler problem explained in section 4.1. The blue coordinate system rotates in such a way that these masses remain stationary within it. Notice that this implies another restriction on the problem, as we are assuming an eccentricity of zero. If the eccentricity was any different, the trajectories of m_1 and m_2 would not describe circles in the black system, so while we could still come up with a rotating system whose y'-axis permanently passes through both masses, they would periodically move closer together and further apart on this axis. This would further imply that the angular velocity of that system becomes time dependent. We don't want all of that, so we just assume an eccentricity of zero. Even though that significantly simplifies the mathematical analysis it is still useful in reality, as the eccentricity of the orbit of most of the planets orbiting the sun is indeed quite close to zero.

This being said, let's move on with the subject. What I will do in section 5.2.2 is derive the so-called

Jacobi integral. It is a conserved quantity in the restricted three-body problem, just like the energy was in the one- and the two-body problem [39]. First, let's look at the potential generated by m_1 and m_2. It is given in equation 68 as the sum of the gravitational potentials of the individual masses [13, equations 68 - 77 were motivated by p. 165-166].

$$V(\vec{r}) = -G\frac{m_1 m}{|\vec{r} - \vec{r}_1|} - G\frac{m_2 m}{|\vec{r} - \vec{r}_2|} \tag{68}$$

In the equation above, $\vec{r}$ represents the position of the third mass (m) which we are actually interested in, $\vec{r}_1$ describes the position of m_1 and $\vec{r}_2$ the one of m_2. Now, let's use $\vec{F} = -\nabla V$ as well as the expression of $\vec{F}$ in terms of $\vec{F}'$ and the inertial forces.

$$-\nabla V(\vec{r}) = \vec{F} = m(\vec{a}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r})) \tag{69}$$

The centripetal force (the last term in the expression above) can be written as the gradient of a scalar function.

$$V_c(\vec{r}) = -m\frac{(\vec{\omega} \times \vec{r})^2}{2} \quad \Rightarrow \nabla V_c(\vec{r}) = m(\vec{\omega} \times (\vec{\omega} \times \vec{r})) \tag{70}$$

I put the proof of equation 70 into the footnote [40]. We can now define another function V_+ to be the sum of $V(\vec{r})$ and $V_c(\vec{r})$.

$$V_+(\vec{r}) = V(\vec{r}) + V_z(\vec{r}) = -Gm\frac{m_1}{|\vec{r} - \vec{r}_1|} - Gm\frac{m_2}{|\vec{r} - \vec{r}_2|} - m\frac{(\vec{\omega} \times \vec{r})^2}{2} \tag{72}$$

With this definition, equation 69 can be written as:

$$-\nabla V_+(\vec{r}) = m(\vec{a}' + 2\vec{\omega} \times \vec{v}'). \tag{73}$$

To derive the Jacobi integral, we now have to perform an operation of which the use can probably not be seen immediately, but it will become apparent one equation later. This operation is to multiply both sides with $\vec{v}'$ and to integrate with respect to t.

$$\int \vec{v}' m\vec{a}' dt + \int \vec{v}' 2m(\vec{\omega} \times \vec{v}')dt = -\int \vec{v}' \nabla V_+(\vec{r})dt \tag{74}$$

The first term in equation 74 can now be trivially integrated, as $\vec{v}' = \dot{\vec{r}}'$ and $\vec{a}' = \ddot{\vec{r}}'$ and $d/dt(\dot{\vec{r}}'^2/2) = \dot{\vec{r}}\ddot{\vec{r}} = \vec{v}'\vec{a}'$. This means that $\int \vec{v}'\vec{a}'dt = \int d/dt(\dot{\vec{r}}'^2/2)dt = \dot{\vec{r}}'^2/2 + c = \vec{v}'^2/2 + c$. This way, we now no longer have double derivatives to worry about and we are only left with $\vec{r}'$ and $\vec{v}'$. Also, the integral on the right hand side can be simplified by writing the dot product between $\vec{v}'$ and $\nabla V(\vec{r})$ in component notation.

$$\frac{m\vec{v}'^2}{2} + \int 2m\vec{v}'(\vec{\omega} \times \vec{v}')dt + c = \int \sum_{i=1}^{3} \frac{\partial r_i'}{\partial t}\frac{\partial V_+(\vec{r})}{\partial r_i'}dt \tag{75}$$

Instead of $\vec{v}_i'$ on the right hand side I wrote the time derivative of $\vec{r}_i$, which is of course equivalent. However, writing it this way you can see that the $\partial r_i'$ cancel out and that what we are left with is the time derivative of $V_+(\vec{r})$, yielding another trivial integral. Already, the seemingly arbitrary operation in equation 74 more

[39] Of course, the energy is also conserved in the restricted three-body problem, but we cannot really use that to our advantage when working in the blue system.

[40]

$$V_c(\vec{r}) = -m\frac{(\vec{\omega} \times \vec{r})^2}{2} = -\frac{m}{2}(\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} \times \begin{bmatrix}x\\ y\\ z\end{bmatrix})^2 = -\frac{m}{2}\begin{bmatrix}\omega_y z - \omega_z y\\ \omega_z x - \omega_x z\\ \omega_x y - \omega_y x\end{bmatrix}^2 = -\frac{m}{2}((\omega_y z - \omega_z y)^2 + (\omega_z x - \omega_x z)^2 + (\omega_x y - \omega_y x)^2) \Rightarrow$$
$$\nabla V_c(\vec{r}) = m\begin{bmatrix}(\omega_x y - \omega_y x)\omega_y - (\omega_z x - \omega_x z)\omega_z\\ (\omega_y z - \omega_z y)\omega_z - (\omega_x y - \omega_y x)\omega_x\\ (\omega_z x - \omega_x z)\omega_x - (\omega_y z - \omega_z y)\omega_y\end{bmatrix} = m\vec{\omega} \times \begin{bmatrix}\omega_y z - \omega_z y\\ \omega_z x - \omega_x z\\ \omega_x y - \omega_y x\end{bmatrix} = m(\vec{\omega} \times (\vec{\omega} \times \vec{r})) \tag{71}$$

than pays off. We have something that looks just like the expression for kinetic energy on the left side and we have $V_+(\vec{r})$ on the right side. But it gets even better when we investigate the last integral left. If we use equation 65 to rewrite $\vec{v}'$ as $\vec{v}' = \vec{v} - \vec{\omega} \times \vec{r}$ we can show that the integrand vanishes.

$$\begin{aligned} 2m\vec{v}'(\vec{\omega} \times \vec{v}') = 2m(\vec{v} - \vec{\omega} \times \vec{r})(\vec{\omega} \times (\vec{v} - \vec{\omega} \times \vec{r})) = 2m(\vec{v} - \vec{\omega} \times \vec{r})(\vec{\omega} \times \vec{v} - \vec{\omega} \times (\vec{\omega} \times \vec{r})) = \\ 2m(\underbrace{\vec{v}(\vec{\omega} \times \vec{v})}_{=0} - \vec{v}(\vec{\omega} \times (\vec{\omega} \times \vec{r})) - (\vec{\omega} \times \vec{r})(\vec{\omega} \times \vec{v}) + \underbrace{(\vec{\omega} \times \vec{r})(\vec{\omega} \times (\vec{\omega} \times \vec{r}))}_{=0}) = \\ -2m((\vec{\omega} \times \vec{r})(\vec{v} \times \vec{\omega}) + (\vec{\omega} \times \vec{r})(\vec{\omega} \times \vec{v})) = 0 \end{aligned} \tag{76}$$

The two terms that I set zero in the second line vanish because they constitute a dot product between orthogonal vectors. In the last line, I used the fact that triple products of the form $\vec{a}(\vec{b} \times \vec{c})$ remain unchanged under a circular shift (i.e. $\vec{a}(\vec{b} \times \vec{c}) = \vec{c}(\vec{a} \times \vec{b}) = \vec{b}(\vec{c} \times \vec{a})$) to rewrite the first of the remaining terms. Now, the two terms look identical, with the exception that the order in one of the cross products is changed. However, undoing that change produces a minus sign, showing that the integrand and therefore the whole integral vanishes. What we are left with is:

$$\frac{m\vec{v}'^2}{2} + c = \int \frac{\partial V_+(\vec{r})}{\partial t} dt = V_+(\vec{r}) + \tilde{c} \quad \Rightarrow \quad \boxed{\frac{m\vec{v}'^2}{2} + (-V_+(\vec{r})) = C}. \tag{77}$$

And this is it, the equation in the box is referred to as the Jacobi integral and it states that the sum of $\vec{v}'^2 m/2$ and $-V_+(\vec{r})$ is a conserved quantity. The C on the right hand side is an integration constant and can take on arbitrary values depending on the initial conditions. It is important to note that equation 77 does not state the conservation of energy. On the one hand, $\vec{v}'^2$ is not the real speed but the speed measured in the blue system meaning that the first term is not the kinetic energy and on the other hand $-V_+(\vec{r})$ is not the real potential but the sum of the real potential and $V_c(\vec{r})$ times minus one (see equation 72 for the definition of $V_+(\vec{r})$), so the second part is not the potential energy. Additionally, the equation only describes one of the three objects of a three-body system where the other two masses are actually way larger than the third one and so constitute by far the bigger contributions to the total energy. Nonetheless, we can use the equation to make some important statements about the trajectory the third mass will take.

Firstly, if we know the value of C (which we can get from the initial conditions), we can already restrict the space in which the third object is allowed to move by looking at where the value of $-V_+(\vec{r})$ equals that of C. All of these points put together will form one or more planes that are warped within 3D space (provided $C < max(-V_+(\vec{r}))$ and they form a barrier for the motion of the third object. Usually, the value of $-V_+(\vec{r})$ will be greater than C on one side of that barrier and smaller than C on the other side. If the third oject is on the side where $-V_+(\vec{r}) < C$, then it will have a finite velocity such that equation 77 is fullfilled. However, in light of that very equation its velocity will decrease the closer it gets to the barrier [41] and will be zero when it reaches it, making it impossible to penetrate it.

What I can also do is to search for minima of $-V_+(\vec{r})$. If minima happen to exist, then they must represent stable equilibrium positions where the third object can rest stationary in the blue system. This is because the value of C can be arbitrarily chosen, so it must be possible to choose it such that it is only slightly bigger or even equal to one of the minima of $-V_+(\vec{r})$. But then if the object is located at the respective minimum it must be trapped there since its velocity must go to zero in order to fulfill equation 77. Looking for these minima is the task of the next subsection.

[41] It does not necessarily have to decrease instantaneously, but because $-V_+(\vec{r})$ is a continuous function there must come a point where it does decrease.

5.2.3 Introduction to Lagrange points

To begin, let's rewrite the definition of $-V_+(\vec{r})$ to know what the function we want to find the minima of explicitly looks like.

$$-V_+(\vec{r}) = -V(\vec{r}) - V_c(\vec{r}) = Gm\frac{m_1}{|\vec{r}-\vec{r_1}|} + Gm\frac{m_2}{|\vec{r}-\vec{r_2}|} + m\frac{(\vec{\omega}\times\vec{r})^2}{2} \tag{78}$$

Next, we can argue that any minimum must be located in the plane of the orbits of the two bigger masses. This can be seen using an argument by contradiction. Let's suppose that the third object was actually motionless in the blue system with a non-vanishing z'-coordinate. Then there would necessarily have to be a net force with a non-vanishing component towards the $z' = 0$ plane due to the gravitational attraction of m_1 and m_2. But there would be nothing to withstand that force component, so the third object would indeed start moving towards the $z' = 0$ plane. But then the alleged minimum cannot really be a minimum, because otherwise the object would have to stay there and not move away, which I just argued that it would. Ergo we can justifiably restrict our search of minima to the $z' = 0$ plane. This allows us to significantly simplify equation 78 if we additionally notice that $\vec{\omega}$ just has a z'-component [14, p. 20].

$$\begin{aligned} \frac{m}{2}(\begin{bmatrix}0\\0\\\omega\end{bmatrix}\times\begin{bmatrix}x'\\y'\\0\end{bmatrix})^2 = \frac{m}{2}\begin{bmatrix}-\omega y'\\\omega x'\\0\end{bmatrix}^2 = \frac{m}{2}\omega^2(x'^2+y'^2) \Rightarrow \\ -V_+(x',y',z'=0) = Gm\frac{m_1}{\sqrt{x'^2+(y'-y_1')^2}} + Gm\frac{m_2}{\sqrt{x'^2+(y'-y_2')^2}} + \frac{m}{2}\omega^2(x'^2+y'^2) \end{aligned} \tag{79}$$

To find the minima, we now have to differentiate with respect to x' and with respect to y' and look at where both yield zero [14, p. 22].

$$\begin{aligned} \frac{1}{m}\frac{\partial}{\partial x'}(-V_+(x',y',z'=0)) &= -G\frac{m_1x'}{(x'^2+(y'-y_1')^2)^{\frac{3}{2}}} - G\frac{m_2x'}{(x'^2+(y'-y_2')^2)^{\frac{3}{2}}} + \omega^2x' \stackrel{!}{=} 0 \\ \frac{1}{m}\frac{\partial}{\partial y'}(-V_+(x',y',z'=0)) &= -G\frac{m_1(y'-y_1')}{(x'^2+(y'-y_1')^2)^{\frac{3}{2}}} - G\frac{m_2(y'-y_2')}{(x'^2+(y'-y_2')^2)^{\frac{3}{2}}} + \omega^2y' \stackrel{!}{=} 0 \end{aligned} \tag{80}$$

Unfortunately, equation 80 can neither be analytically solved for x' nor for y'. At first glance, one might think that y' could be isolated by multiplying by the term in the denominator and then moving all the exponents to the other side. However, the denominators of the two fractions are slightly different (the y_1' in the first one is replaced by y_2' in the second one), making this approach impossible. What I therefore did was analyse the solution graphically.

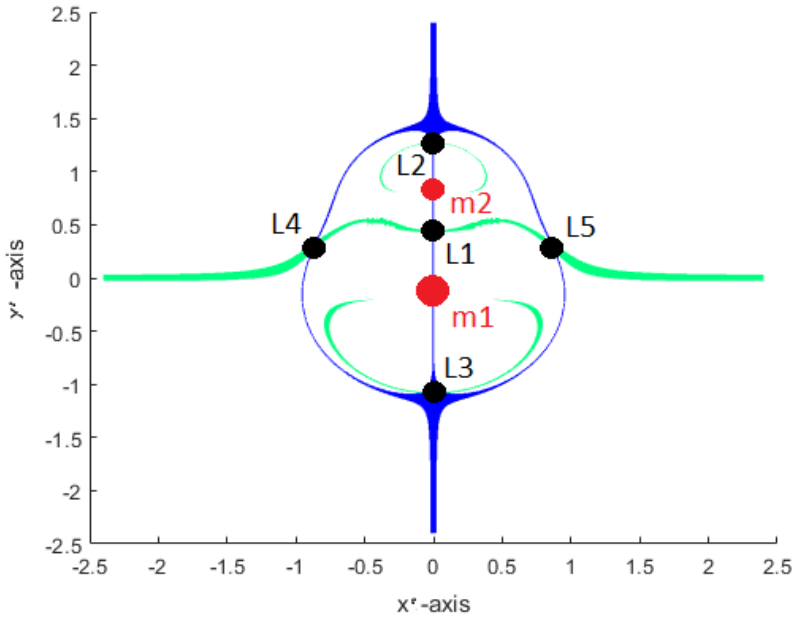

Figure 8: Lagrange points (created independently, but motivated by p. 27 of [14])
green contour ... $\frac{\partial}{\partial y'}(-V_+(x', y', z' = 0)) = 0$
blue contour ... $\frac{\partial}{\partial x'}(-V_+(x', y', z' = 0)) = 0$
black dots ... Lagrange points one through five
$m_1 = 0.8$ ... big red dot
$y_1' = -0.2$ ... y'-component of m_1
$m_2 = 0.2$ ... small red dot
$y_2' = 0.8$ ... y'-component of m_2
$\omega = 1$
$G = 1$

Let me explain how I went about generating figure 8. Initially, I tried working with the parameters that you would have in the earth-sun system. However, it turned out that the lagrange points L1 and L2 were so close together that in the visualisation they completely overlapped with m_2. I therefore tried picking values that allowed for an effective visualisation of the most important concepts. I set G and ω equal to one and chose $m_1 = 0.8$ as well as $m_2 = 0.2$. Notice that for none of these parameters I added any units. This is because the absolute scaling is irrelevant now and if I had added units, I would have had to adapt them to one another (e.g. if you know m_1, m_2 and ω, that fixes the distance between the two masses, but if you leave out the units you don't have to worry about any of these issues). Setting $y_1' = -0.2$ and $y_2' = 0.8$ puts the centre of mass into the origin (as $m_1 y_1 + m_2 y_2 = 0$), making the setup identical to that of figure 7.

What I did next was divide the square bounded by $x' = -2.4$, $x' = 2.4$, $y' = -2.4$, $y' = 2.4$ and $z' = 0$ into one thousand intervals along the x'-axis and one thousand intervals along the y'-axis, creating one million evenly spaced pixels. Then, I used matlab to compute the derivative of $-V_+(\vec{r})$ with respect to x' and with respect to y' for every single pixel (i.e. I computed equation 80) and deleted all the non-zero values. I plotted the pixels that were left after that in blue for $\partial/\partial x'$ and in green for $\partial/\partial y'$. Notice that the blue and green contours are not lines with uniform thickness. This is because strictly speaking I did not delete all non-zero values, but I deleted all values outside the interval [-0.02,0.02]. Had I deleted all non-zero values, then none (or at least very few) of the pixels would have survived, as the computation was performed on one of the corners of each pixel and so even if the pixel as a whole covered a spot where the derivative was equal to zero, the probability that this spot is located precisely at the corner where the computation was done is

very low. Therefore, I experimented with different limits above which to delete my pixels and found 0.02 to be an appropriate one. The non-uniform thickness actually yields additional information, as it tells you that the thicker the line is, the slower the change in the derivative of $-V_+(\vec{r})$. At this point, I have to admit that I slightly cheated in two locations. In between L4 and L5, the green line going through L1 became so thin that it was hardly visible anymore, so I varied the limit for deletion in that region, reaching a maximum of 0.2 right at L1. Also, I manually increased the thickness of the blue line between L2 and L3 (fussing around with your data in that way is something you are always warned from doing, but I hope that now that I explained what I did and given that the graph just serves qualitative purposes I will be forgiven for having done so nonetheless).

This is basically how I generated figure 8. Next, we have to talk about how to interpret it. Remember that what we set out to do at the very beginning of section 5.2.3 was trying to find out if $-V_+(\vec{r})$ had any minima. A necessary condition for a minimum to exist is that both the derivative with respect to x' and the derivative with respect to y' vanish. Well, the blue contour in figure 8 shows where the first condition is met and the green contour shows where the second one is met, so at the five points where they meet (labelled L1 through L5) they are both fulfilled simultaneously. In honour of the one who first derived their positions, they are called the five Lagrange points [42]. What we still need to do is to check whether the lagrange points are really minima of $-V_+(\vec{r})$, after all they could also be maxima or saddle points. To do so, we would have to differentiate $-V_+(\vec{r})$ a second time and check the values at the positions of the lagrange points. However, partly because I did not determine the positions to scale and partly because it would further increase the length of this thesis, I decided not do to so explicitly and instead direct the interested reader to a more detailed explanation in reference [15].

The Lagrange points one through three turn out to be saddle points [15, p. 6-7] and therefore do not fulfill the condition for which we argued that if it was met, they would constitute stable equilibrium points (namely that they are minima of $-V_+(\vec{r})$). They are all found on a straight line going through m_1 and m_2 and indeed they are unstable, which can also be understood qualitatively. Let's imagine the rotation of the earth around the sun. The distance between the earth and the sun is exactly such that the centripetal force necessary to keep the earth on a circular track is provided by the gravitational pull of the sun. Moreover, this distance is not dependent on the mass of the earth, meaning that any other mass (provided it is still much smaller than the mass of the sun) moving with the same speed as the earth would adopt the very same distance as its orbital radius around the sun. If it went any closer to the sun and kept the same angular velocity, it would spiral into the latter and if it went further away it would at least momentarily drift away from the sun. The lagrange points L1 and L2 are exactly those positions where the gravitational pull of m_2 (in our example the earth) exactly compensates for the created imbalance. Without the earth, an object at L1 would spiral into sun but because of the earth it experiences an additional force directed away from the sun and so it does not. One can also imagine how the instability of that equilibrium position is brought about because if the third mass moves only slightly away from L1, it would either end up crashing into the earth or into the sun depending on the direction in which it is moving. Similarily, an object at L2 would usually drift away from the sun but because of the extra pull from the earth is kept on a circular track. Again, if it moves only slightly away from L2, it either drifts away from the sun or crashes into the earth. A very similar reasoning can be applied to L3. Without the presence of the earth, it would have the same orbital radius as the earth. However, as the gravitational pull from the earth adds to that of the sun, the L3 point moves a little further away from the sun until the increased centripetal force of the circular motion exactly matches the increased pull towards the sun. It is also an unstable equilibrium point, causing an object to either drift away from the sun or spiral into it when slightly moving away from L3.

Unfortunately, the L4 and L5 points are also not minima of $-V_+(\vec{r})$, so we again need to investigate further to find out about their behaviour. It turns out that they even represent local maxima [15, p. 7], so one would

[42] Strictly speaking, there are two more locations where the green and the blue contour meet, namely the positions of m_1 and m_2. Most obviously though, they are not very interesting since as soon as you place the third body of mass on top of (or within) one of the other two, you are left with a two-body problem which we already solved earlier.

be tempted to think that they are also unstable. However, one needs to be very cautious here. $-V(\vec{r})$ in equation 77 does not take on the same role as $V_{eff}(r)$ did in equation 23 in the one-body problem. Back in the one-body problem, we managed to reduce the three-dimensional problem to a two-dimensional one by noticing that the motion must happen within a single 2D plane. However, we then managed to further reduce it to a one-dimensional problem by replacing the θ-dependence with an expression involving the angular momentum L and the radius r. But the situation in equation 77 is different because the θ-dependence is still hidden within $\vec{v}'^2$. Remember from equation 19 that $\vec{v}'^2$ can be written as $\vec{v}'^2 = \dot{r}'^2 + r'^2\dot{\theta}'^2$. We could incorporate the second term in $V_+(\vec{r})$ to only leave behind $m\dot{r}'^2/2$ in equation 77 and thereby produce a problem involving a one-dimensional kinetic energy term plus a potential energy term. But that potential energy term would now be dependent on the velocity via $\dot{\theta}$, meaning we can no longer ascribe a scalar value to every point in space describing the potential at that point because that value changes dependent on the speed of the object we want to investigate. By the way, this does not invalidate my reasoning about having stationary equilibrium points for minima in $-V_+(\vec{r})$, because the integration constant C in equation 77 can always be chosen such that an object at that minimum has zero velocity. However, it does make a difference with respect to the stability of L4 and L5, because while an object will initially move away from it, the potential it sees will change the moment it picks up speed and it turns out that the way it changes is exactly such that it makes the object rotate around the L4 point (or the L5 point respectively. I do not treat the two of them separately, because they are exactly symmetrical with respect to m_1 and m_2). Unfortunately, there is no time for going through the reasoning for why this is the case here in this thesis, but an explanation can be found on page 7 of reference [15].

All in all, it became apparent throughout this chapter that we had to make simplifications all along the way in order to be able to come up with an analytical analysis of what is happening. Initially, we set out to describe the three-body problem and discovered that the general case is way to complex for a closed form solution. We thus looked at some special cases where analytical solutions do exist and then concentrated on the restricted three-body problem where the gravitational influence of the third mass on the other two is neglected. We further simplified the problem by assuming that the eccentricity of the orbits of m_1 and m_2 was zero and even then we could not describe the trajectory of the third mass but could only calculate equilibrium points. That not being enough, I didn't even derive these equilibrium points analytically and instead determined them numerically (see figure 8) by setting the derivatives of $-V_+(\vec{r})$ equal to zero [43]. The point I want to make with all of that is that for an effective modelling of reality it is inevitable to heavily rely on numerical investigations. This should by no means devalue the effort of trying to understand as many phenomena as possible on an analytical basis, as I think that a lot of physical and mathematical understanding can be built from doing exactly that. However, it may be some sort of conclusion that one could draw from this thesis that while everything looks nice and clean when looking at one or two bodies of mass, the complexity explodes when adding a third one, let alone a fourth one or even a million more. For these cases, the stage is open for numerical mathematics, which might be the topic of some other thesis. This being said, I think this is a good place to end. I hope that whoever gets to read this thesis will find the one or the other notion or concept that is new to him or her and profit from it as much as I did when writing about it.

[43] Initially, the derivation was of course done analytically by Lagrange, but with considerably more effort.

6 Summary and conclusion

Following the introduction in chapter one, I started by introducing some concepts that are necessary for understanding the one-, the two- and the three-body problem in chapter two. Specifically, I discussed two empirical laws (the force law and the gravitational law) and three conserved quantities (the energy, the momentum and the angular momentum).

In chapter three, I tackled the one-body problem by coming up with a mathematical expression for the conservation of energy. I then inserted the angular momentum in order to get rid of one unknown parameter and reduced the resulting equation to a one-dimensional problem that could be straightforwardly solved analytically. I went through the specific example of gravitational potentials (section 3.1) and then discussed Bertrand's theorem in section 3.2. It states that there are exactly two potentials that guarantee that all bound orbits are closed and its derivation uses the fact that rational numbers are totally disconnected. From this, one can learn that no matter how distant from reality a mathematical theorem may seem, it often pops up in the most unexpected circumstances imaginable and helps solve problems that would otherwise seem impossible to be cracked.

Moving on, I introduced relative coordinates in chapter four to solve the two-body problem. This helped decoupling the motion of the center of mass of the system from the relative motion of the two point masses. The resulting equation could then be mapped onto a one-body problem and solved with the methods developed in chapter three. I explicitly solved the two-body problem for a gravitational potential (i.e. the Kepler problem) in chapter 4.1.

In chapter five, I finally introduced the three-body problem and showed that reducing it to a two-body problem the same way one could reduce the two-body problem to a one-body problem was not possible. Nonetheless, the equations of motion could be simplified by again decoupling the motion of the centre of mass (see equation 62). A further analytical analysis was only possible when imposing some restrictions on the general three-body problem and only looking at special cases. I introduced some of the special solutions (see section 5.1) found over the course of history and then focused on an analytical approach to the restricted three-body problem (where the mass of one of the two bodies of mass is so little in comparison to the other two that its gravitational effect on the latter can be neglected) in section 5.2. In order to be able to work in a rotating coordinate system, I derived and explained the inertial forces (coriolis force and centripetal force) arising in such systems and then used the knowledge gathered to discuss the five lagrange points. These are equilibrium positions for the light mass within the rotating system in which the two heavy masses are stationary.

Throughout the thesis, the mathematical complexity for analysing the problems at hand was steadily increasing. However, while during chapters two through four, the analytical approach was fully successful in depicting what was going on, I had to make numerous simplifications in the three-body problem to be able to make any reasonable arguments at all using a purely analytical approach. It became evident that at the latest when adding a fourth point mass (but also when trying to describe the general three-body-problem), one has to inevitably make use of numerical methods to solve for the individual trajectories. Nonetheless, going as far as possible by analytical means definitely proved to be helpful in improving the mathematical and physical understanding of the phenomena investigated.

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7 Appendix

7.1 The integral of the Kepler problem

In this section, I will carry out the step by step calculation for the integral from equation 55, which is given again in equation 81.

$$\theta(r) = \pm \int \frac{L}{r^2\sqrt{2\mu(E - (\frac{L^2}{2\mu r^2} - \frac{\alpha}{r}))}} dr + C \tag{81}$$

The first thing to do is to get rid of all the brackets inside the root.

$$\theta(r) = \pm \int \frac{\frac{L}{r^2}}{\sqrt{2\mu E + \frac{2\mu\alpha}{r} - \frac{L^2}{r^2}}} dr + C \tag{82}$$

Next, we have to make a number of substitutions. To do so, we have to have an idea in mind of what form we wish to "morph" the integrand into by applying the substitutions. I already stated the solution of the whole integral in the main text and it turns out to be an arccosine. This means that the integrand in equation 82 needs to be transormed into the derivative of the arccosine, which I stated in equation 83.

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1 - x^2}} \tag{83}$$

Comparing the right-hand side of equation 83 with the integrand in equation 81 makes it plausible to substitute for $\frac{L}{r}$. Lets call the substituting variable a.

$$a \coloneqq \frac{L}{r} \Rightarrow \frac{da}{dr} = \frac{d}{dr}(\frac{L}{r}) = -\frac{L}{r^2} = -\frac{a^2}{L} \Rightarrow dr = -\frac{L}{a^2} da \tag{84}$$

Applying the substitution yields:

$$\begin{aligned}\theta(r) = \pm \int \frac{\frac{a^2}{L}}{\sqrt{2\mu E + \frac{2\mu\alpha a}{L} - a^2}}(-\frac{L}{a^2} da) + C = \pm \int -\frac{1}{\sqrt{2\mu E - (\frac{\mu\alpha}{L} - a)^2 + \frac{\mu^2\alpha^2}{L^2}}} da + C = \\ \pm \int -\frac{1}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}} \frac{1}{\sqrt{1 - \frac{(\frac{\mu\alpha}{L} - a)^2}{2\mu E + \frac{\mu^2\alpha^2}{L^2}}}} da + C.\end{aligned} \tag{85}$$

At the end of the first line of equation 85 I completed the square inside the root and in the second line I pulled a factor out of the root. The integrand now takes on more and more the form of the right-hand side of equation 83. To increase the readability, let's now replace the factor I pulled out of the root with the letter u.

$$u \coloneqq \sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}} \Rightarrow \theta(r) = \pm \int -\frac{1}{u} \frac{1}{\sqrt{1 - (\frac{\frac{\mu\alpha}{L} - a}{u})^2}} da + C \tag{86}$$

If we now substitute the expression within the brackets in the root, we have exactly the form we need to get to the arccosine in the end. Let's call the substituting parameter b.

$$b \coloneqq \frac{\mu\alpha}{Lu} - \frac{a}{u} \Rightarrow \frac{db}{da} = \frac{d}{da}(\frac{\mu\alpha}{Lu} - \frac{a}{u}) = -\frac{1}{u} \Rightarrow da = -u\, db \tag{87}$$

Let's now apply the substitution.

$$\theta(r) = \pm \int -\frac{1}{u} \frac{1}{\sqrt{1 - b^2}}(-u db) + C = \pm \int -\frac{d}{db}(\arccos(b))db + C = \mp \arccos(b) + C \tag{88}$$

Now, all that's left to do is use equations 84, 86 and 87 to substitute back to the original parameters.

$$\begin{aligned}\theta(r) = \mp\arccos(b) + C \overset{\text{equ. } 87}{=} \mp\arccos(\frac{\mu\alpha}{Lu} - \frac{a}{u}) + C = \mp\arccos(\frac{\frac{\mu\alpha}{L} - a}{u}) + C \overset{\text{equ. } 86}{=} \\ \mp\arccos(\frac{\frac{\mu\alpha}{L} - a}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}}) + C \overset{\text{equ. } 84}{=} \\ \mp\arccos(\frac{\frac{\mu\alpha}{L} - \frac{L}{r}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}}) + C\end{aligned} \tag{89}$$

The last line of equation 89 is almost the solution given in equation 55 in the main text. The only differences are that the $\pm$ sign turned to a $\mp$ sign and that the numerator of the fraction inside the arccosine got multiplied by minus one. However, that discrepancy just represents a different choice in the orientation of the coordinate system and can be easily fixed. If we had defined b in equation 87 to be minus one times the definition we did use, then that minus sign would have stayed inside the integrand until the very end. However, the expression da would have also changed to $da = u\ db$ (instead of $da = -u\ db$), which would have produced another minus sign that could have been pulled outside of the integral, thus fixing the second change by converting $\mp$ to $\pm$. Therefore, the solution offered in equation 55 is indeed correct.

7.2 The shape of Kepler orbits for $\epsilon < 1$

In this section, I will prove that the Kepler orbit for eccentricities smaller than one is indeed an ellipse and therefore a closed orbit. This shows that Bertrand's theorem (see section 3.2) really holds for the case of gravitational potentials [44]. Back in the main text of chapter 4.1, I said that the orbit turns out to be an ellipse and that this can be deduced from equation 58. In order to prove that, we need to know what the defining equation of an ellipse looks like.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \overset{?}{\Leftarrow} \quad \frac{p}{r} = \epsilon\cos(\theta) + 1 \tag{90}$$

On the left, I put the equation of an ellipse centered at the origin of a cartesian coordinate system of which the axis labels are x and y. The parameters a and b are the major and the minor axes respectively. On the right, I copied equation 58 derived in section 4.1. and our task right now is to show that the left side follows from the right one. To do so, we need to express the parameters r and θ in terms of x and y. The transformations can be easily checked geometrically and are given in equation 91 [4, p. 142].

$$r = \sqrt{x^2 + y^2} \qquad cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}} \tag{91}$$

Next, let's insert this transformation into the right side of equation 90 and try to morph it into the left side.

$$\frac{p}{\sqrt{x^2 + y^2}} = \epsilon\frac{x}{\sqrt{x^2 + y^2}} + 1 \Rightarrow \sqrt{x^2 + y^2} = p - \epsilon x \Rightarrow x^2 + y^2 = p^2 - 2p\epsilon x + \epsilon^2 x^2 \tag{92}$$

We need to try and merge all the terms involving x into one expression. This can be done using the completing the square method.

[44] One might argue that the Kepler problem is a two-body problem while Bertrand's theorem just deals with one body of mass. However, as shown in chapter 4, the two-body problem is solved by mapping it onto a one-body problem and the solution in equation 58 is in fact a solution for a one-body problem. The motion for the two bodies of mass in the Kepler problem is then constructed by employing equation 49 to deduce $\vec{r}_1$ and $\vec{r}_2$ from the relative coordinates.

$$
\begin{aligned}
x^2(1-\epsilon^2) + 2p\epsilon x + y^2 = p^2 \Rightarrow \\
x^2 + x\frac{2p\epsilon}{1-\epsilon^2} + \frac{y^2}{1-\epsilon^2} = \frac{p^2}{1-\epsilon^2} \Rightarrow \\
(x + \frac{p\epsilon}{1-\epsilon^2})^2 - \frac{p^2\epsilon^2}{(1-\epsilon^2)^2} + \frac{y^2}{1-\epsilon^2} = \frac{p^2(1-\epsilon^2)}{(1-\epsilon^2)^2} \Rightarrow \\
(x + \frac{p\epsilon}{1-\epsilon^2})^2 + \frac{y^2}{1-\epsilon^2} = \frac{p^2}{(1-\epsilon^2)^2} \Rightarrow \\
\frac{(x + \frac{p\epsilon}{1-\epsilon^2})^2}{\frac{p^2}{(1-\epsilon^2)^2}} + \frac{y^2}{\frac{p^2}{1-\epsilon^2}} = 1
\end{aligned}
\tag{93}
$$

If we now rename the constants, the proof is completed [4, p. 142].

$$
\begin{aligned}
a \coloneqq \frac{p}{1-\epsilon^2} \qquad b \coloneqq \frac{p}{\sqrt{1-\epsilon^2}} \Rightarrow \\
\frac{(x+\epsilon a)^2}{a^2} + \frac{y^2}{b^2} = 1
\end{aligned}
\tag{94}
$$

The second line of equation 94 is identical to the left side of equation 90, with the only exception that the x-coordinates are moved to the left. But that doesn't mean that the shape is no longer an ellipse, it just means that it is no longer centered at the coordinate origin. Nonetheless, the orbit is still bound, showing that the statement of Bertrand's theorem with respect to the gravitational potential is indeed correct.