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Decoherence Effects in qubits

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Wintersemester 2012

Contents

Abstract

This thesis deals with a quantum mechanical description of a 2-level-system (*qubit*) interacting with an electromagnetic field to produce a NOT-operation on the qubit.

The central point of interest will not be the underlying model. The goal will be to treat the system in different ways and to compare these results. We will see that a physically reasonable behaviour can only be obtained when not treating the qubit as being isolated from the environment. A common way to handle such system-bath-interactions is the *Markovian approach*, where one assumes that the system of interest has no "memory". A very detailed description of methods used to describe such open quantum system can be found in the book by Breuer & Petruccione ([1]). For the following, the most important result from such Markovian treatment is the existence of a master equation for the density operator, which is the Lindblad equation.

The cental point of this work is a numerical solution of the Lindblad equation for a Hamiltonian describing a NOT-operation. Two different decoherence effects were considered; firstly an amplitude damping and secondly a phase damping (see sec. 7). Furthermore we have investigated the effects of decoherence and dissipation on two quantities: the population of the excited state and the changing of the relative phase between the two states. Both are of major interest since they are being used in quantum computing.

In the context of quantum information technology, effects of decoherence are modeled by *noise channels*. We will see that we can regain those quantum noise channels as defined in the book by Nielsen & Chuang [2] by a Markovian treatment.

1 Motivation

A qubit is the smallest undividable unit used in Quantum Information Theory. There it acts as the analogon to a bit in classical computing. To understand the major problems that occur in quantum computing, one has to have a basic understanding of how decoherence - and as a result of that, loss of information - occurs.

The simplest operation that can be done to a qubit is a NOT-operation. Accordingly, investigating such a fairly simple manipulation can be seen as a good introduction to being able to deal with more complex systems, either consisting of more qubits, or more complex logic operations being done to them.

In general, a qubit is an abstract mathematical object. However, there are many different practical realisations of systems with a variety of experimental approaches. Any quantum object with two distinct states could act as a qubit (atom with two energy levels, polarized photon, enery levels in molecules, ...). In the following, the two spin-states (spin-up / spin-down in the z-basis) will stand for the two different states. An introduction to the physical realization of such systems can be found in [2].

2 Mathematical Description of qubits

This section only addresses topics necessary for a basic understanding of a spin-1/2-system, a more detailed introduction to qubits and interpretations of certain concepts can be found in [1].

A qubit is a vector in a 2-dimensional Hilbert-space $\mathcal{H}_2$ describing the quantum mechanical state of a two level system:

$$|\psi\rangle = a(t)\,|\uparrow\rangle + b(t)\,|\downarrow\rangle \tag{2.1}$$

where $\{|\uparrow\rangle, |\downarrow\rangle\}$ denotes a basis spanning $\mathcal{H}_2$. Therefore the notation $\big(1{,}0\big)^T$ for the state $|\uparrow\rangle$ and $\big(0{,}1\big)^T$ for $|\downarrow\rangle$ can be used. Using this convention, eq. 2.1 could be denoted as $\big(a(t), b(t)\big)^T$. Note that the basis is time-independent, the coefficients hold the dynamics of the system.

3 Simple Model for NOT-Gate

A possible Hamiltonian for describing a spin-1/2-system could be

$$\hat{H}_0 = \omega_0\,|\uparrow\rangle\langle\uparrow|$$

where the system has energy ω_0 if the state is $|\uparrow\rangle$ and 0 otherwise. Plugging this into the Schrödinger equation, one gets[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t) = -i\omega_0 a(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}b(t) = 0$$

with the solution $a(t) = e^{-i\omega_0 t}$ and $b(t) = 0$ for the initial state $|\phi\rangle = |\uparrow\rangle$. This means that the expectation values of the state remain, only the phase between $|\uparrow\rangle$ and $|\downarrow\rangle$ oscillates in a sinusoidal way.

To actually change the state of the qubit from $|\uparrow\rangle$ to $|\downarrow\rangle$ or vice versa, energy has to be transfered from/to the system. This could be done by using electromagnetic waves. In a two-level-approximation, the Hamiltionian describing such an interaction is a modified version of the Jaynes-Cummings-Model[1]. It read as

$$\hat{H}_I(t) = f(t)\,|\uparrow\rangle\langle\downarrow| + f^*(t)\,|\downarrow\rangle\langle\uparrow|$$

with $f(t) = Qe^{-i\epsilon t}$. Now the full Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{H}_I(t)$ becomes time-dependent and reads as

1 Throughout this whole thesis, $\hbar = 1$ will be used.

$$H(t) = \begin{pmatrix} \omega_0 & Q \cdot e^{-i\epsilon t} \\ Q \cdot e^{i\epsilon t} & 0 \end{pmatrix}$$

in the $\{|\uparrow\rangle, |\downarrow\rangle\}$-basis. One obtains from from the Schrödinger-equation that

$$i\frac{\mathrm{d}}{\mathrm{d}t}a(t) = \omega_0 + f(t) \cdot b(t)$$
$$i\frac{\mathrm{d}}{\mathrm{d}t}b(t) = f^*(t) \cdot a(t)$$

with $a(t)$, $b(t)$ being the coefficients from eq. 2.1. Assuming the initial state of the system is a spin polarized in the z-direction, $|\phi(t=0)\rangle = |\uparrow\rangle$, one gets the solution[2]

$$a(t) = e^{-\frac{i}{2}(\epsilon+\omega_0)t} \cdot \big(\cos(\beta t) + i\frac{\alpha}{\beta}\sin(\beta t)\big)$$
$$b(t) = -\frac{Q \cdot i}{\beta} e^{-i\frac{\Delta}{2}t} \sin(\beta t)$$

with $\Delta = \epsilon - \omega_0$ and $\alpha = \Delta/2$, $\beta = \sqrt{\Delta^2/4 + Q^2}$. Δ is the detuning, it describes the deviation of the incoming waves frequency from the energy gap of the qubit. However, if the incoming wave is resonant with the system ($\epsilon = \omega_0$), then $\Delta = \alpha = 0$ and $\beta = Q$. Therefore the equations above become

$$a(t) = e^{-i\omega_0 t}\cos(Qt)$$
$$b(t) = -i\sin(Qt)$$

If an experimenter chooses the duration of interaction as $\tau = \pi/2Q$, the coefficients take the values $a(\tau) = 0$, $b(\tau) = -i$, which corresponds to $P(|\downarrow\rangle) = 1$ and $P(|\uparrow\rangle) = 0$. This is exactly the behaviour of a NOT-Gate.

However, the descriptions so far assume that the state of the qubit is a pure state which is well defined. Then the state can always be written in the form of eq. 2.1. If this is not true, which means that one can only give a certain probability to find the system in state $|\psi\rangle$, a formalism using density matrices has to be used.

4 Concept of Density Matrices

A density matrix in an n-dimensional Hilbert-space $\mathcal{H}_n$ is defined as[2],[1],[3]

$$\hat{\rho} := \sum_{i=1}^{n} p_i |\phi_i\rangle \langle\phi_i|$$

with $\mathcal{H}_n = span(|\phi_i\rangle)$. It has the form of an operator and contains the whole information about a physical system in $\mathcal{H}$. From that viewpoint it is similar to a state $|\phi\rangle$. However, the main difference is that one can include uncertainties about the preparation of a system: p_i in the definition above is the probability to find the system in state $|\phi_i\rangle$. For a qubit in a pure state (see eq. 2.1), the density operator reads as

$$\rho = \begin{pmatrix} aa^* & ab^* \\ a^*b & bb^* \end{pmatrix} \tag{4.1}$$

For the treatment of spin-1/2-systems, the density operator can always be written in the so called Bloch-sphere representation[2],[1]. It reads as

$$\hat{\rho} = \frac{1}{2}\left(\mathbb{1} + \vec{P} \cdot \vec{\hat{\sigma}}\right)$$

with $\vec{\hat{\sigma}}$ being the vector auf Pauli-matrices. In matrix notation this becomes

$$\rho = \frac{1}{2}\begin{pmatrix} 1 + P_z & P_x - iP_y \\ P_x + iP_y & 1 - P_z \end{pmatrix} \tag{4.2}$$

where $\vec{P} = (P_x, P_y, P_z)$ is a vector pointing at the surface of a sphere, the Bloch-sphere. This density matrix can be seen as representing a spin-1/2-system where the expectation value of the spin points in the direction $\vec{P}$.

If the system is in a pure state, then $\hat{\rho}^2 = \hat{\rho}$. A pure state can always be written in the form of eq. 2.1. It is straightforward to show that if and only if $|\vec{P}| = 1$, eq. 4.2 describes a pure state. If $|\vec{P}| = 0$, then $\hat{\rho} = \frac{1}{2}\mathbb{1}$, which describes the totally mixed state.

Given a Hamiltonian $\hat{H}$, which can be time-dependent, the dynamics of the state $\hat{\rho}$ is well defined. One can derive an equation for the density operator which is equivalent to the Schrödinger equation, the so called von-Neumann-equation[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = -i[\hat{H}(t),\hat{\rho}(t)]$$

If $\hat{H}$ is not time dependent, then

$$\hat{\rho}(t) = \hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t,t_0) = e^{-i\hat{H}(t-t_0)}\hat{\rho}(t_0)e^{i\hat{H}(t-t_0)} \tag{4.3}$$

4.1 Decoherence and Mixed States

In the last few decades, it became clear, that decoherence is a fundamental concept in the interpretation of quantum mechanics. Still, the definition is rather unclear. One of the most common is that a state is decoherent if *" interference is suppressed"*[3]. This would mean that the relative phase in a linear superposition is no longer well defined.

Let us calculate the expectation value of an operator $\hat{A}$ when the state is in the coherent superposition $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$:

$$\begin{aligned}\langle\hat{A}\rangle &:= \langle\psi|\hat{A}|\psi\rangle = (a^*\langle\uparrow| + b^*\langle\downarrow|)\hat{A}(a|\uparrow\rangle + b|\downarrow\rangle) \\ &= aa^*A_{11} + bb^*A_{22} + \underbrace{b^*aA_{21} + ba^*A_{12}}_{\text{interference term}}\end{aligned}$$

The last term acts as an interference term. In a decoherent state, it vanishes. It is obvious that it is not possible to write such a decorent state in the form of eq. 2.1

In terms of density matrices, the expression above can be written as[1]

$$\langle\hat{A}\rangle = \mathrm{Tr}(\hat{A}\hat{\rho})$$

This means that a state with no interference term is described by a density matrix where the off-diagonal elements a^*b, b^*a (see eq. 4.1) are being damped. Such a state is always a mixed state, because

$$\begin{pmatrix}\rho_{11} & 0\\ 0 & \rho_{22}\end{pmatrix} \neq \begin{pmatrix}{\rho_{11}}^2 & 0\\ 0 & {\rho_{22}}^2\end{pmatrix}$$

unless $\rho_{11} = 1, \rho_{22} = 0$ or $\rho_{11} = 0, \rho_{22} = 1$. But that won't describe a superposition of the states $|\uparrow\rangle, |\downarrow\rangle$.

In quantum computing, every process changing the systems state from a pure state to a mixed state is referred to as noise channel. Therefore, every noise channel creates decoherence[2]. However, decoherence can also arise from entanglement.

The problem so far: Let us rewrite the dynamics generated by $\hat{H}_0$:

$$\begin{aligned}\rho(t) &= \frac{1}{2}\begin{pmatrix}e^{-i\omega_0 t} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1+z_0 & x_0 - iy_0\\ x_0 + iy_0 & 1 - z_0\end{pmatrix}\begin{pmatrix}e^{i\omega_0 t} & 0\\ 0 & 1\end{pmatrix} = \\ &= \begin{pmatrix}1+z_0 & (x_0 - iy_0)e^{-i\omega_0 t}\\ (x_0 + iy_0)e^{i\omega_0 t} & 1 - z_0\end{pmatrix}\end{aligned} \tag{4.4}$$

with the spin being polarized in the direction $\vec{P} = (x_0, y_0, z_0)$ at $t = t_0$. In terms of the separate components, this means that

$$\begin{aligned}P_x(t) &= -y_0\sin(\omega_0 t) + x_0\cos(\omega_0 t)\\ P_y(t) &= y_0\cos(\omega_0 t) + x_0\sin(\omega_0 t)\\ P_z(t) &= z_0\end{aligned} \tag{4.5}$$

This solution has the property that $\rho^2(t) = \rho(t)$, therefore the system remains in a pure/mixed state if it has been pure/mixed at t_0. Note that this can also be seen from $|\vec{P}| = const.\ \forall\ t$.

From a statistical point of view, a behaviour described by the calculations so far is a fairly unrealistic one. There have been considerations, that due to random interactions of the system with the environment, every system should relax into a thermal equlibrium[4]. This would create a certain link between quantum mechanics and classical behaviour. The density matrix at thermal equlibrium however describes a mixed state.

A criteria for a system being in a mixed state, which is equivalent to $\hat{\rho}^2(t) = \hat{\rho}(t)$ is that $\mathrm{Tr}(\hat{\rho}^2) = 1$. Let us consider a unitary time evolution:

$$\mathrm{Tr}\big(\hat{\rho}^2(t)\big) = \mathrm{Tr}\big(\hat{U}(t)\hat{\rho}_0 \underbrace{\hat{U}^\dagger(t)\hat{U}(t)}_{=\mathbb{1}} \hat{\rho}_0 \hat{U}^\dagger(t)\big) = \mathrm{Tr}(\hat{\rho}_0^2) \tag{4.6}$$

since the trace of an operator is invariant under unitary transformations. Therefore, if the system has been in a pure state, it remains pure for all times. In other words: unitary time evolutions *can never create decoherence*[2]. The unitarity of time evolutions postulated in axiomatic quantum mechanics is only a correct assumption for closed systems. The whole universe for example could be seen as evolving that way. Every other system always has to be treated differently.

5 Treatment as Open System

In the last 40 years, there have been numerous investigations on the problems stated in the preceding section. It became clear, that a quantum mechanical description must never be done describing a system isolated from the environment (universe).

Therefore the density matrix used so far has to be extended. If two distinct systems (system of interest A and another system B, e.g. an "environment", the universe) are uncorrelated[3], it is possible to write the density matrix of the whole system as a product

$$\hat{\rho} = \hat{\rho}^{(A)} \otimes \hat{\rho}^{(B)}$$

If the systems are not uncorrelated, they are entangled and it is no longer possible to factorize the density matrices. However, it is always possible to make that assumption at some $t = 0$[1].

One normally is not interested in the dynamics of the whole system but in those of some reduced system (A). Therefore the *dynamical map* V(t) is defined: It is an operation mapping the reduced density matrix of system (A) at $t = 0$ to that at $t = t$ under consideration of $\hat{\rho} = \hat{\rho}^{(A)} \otimes \hat{\rho}^{(B)}$ at the initial time t=0:

$$V(t) : \hat{\rho}^{(A)}(0) \to \hat{\rho}^{(A)}(t)$$

When also making the assumption that environmental correlation times are short, memory effects can be neglected (Markov process) and it is possible to formulate the dynamics as a quantum mechanical semigroup.

[2] A more detailed discussion on the issue can be found in [3].
[3] Uncorrelated means that expectation values factorize: $\langle A \rangle = \langle A^{(A)} \rangle \cdot \langle A^{(B)} \rangle$

Lindblad has shown[5] in 1976, that the most general form of a quantum master equation can be written as[1]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = \mathcal{L}[\hat{\rho}(t)] &= -i[\hat{H}(t),\hat{\rho}(t)] + \sum_{\mu>0}\left(\hat{L}_\mu\hat{\rho}(t)\hat{L}_\mu^\dagger - \frac{1}{2}\{\hat{L}_\mu^\dagger\hat{L}_\mu, \hat{\rho}(t)\}\right) \\ &= -i[\hat{H}(t),\hat{\rho}(t)] + \mathcal{D}[\hat{\rho}(t)]\end{aligned} \tag{5.1}$$

This equation is referred to as the *Lindblad-equation.* $\mathcal{L}$ is the so called Lindblad-superoperator, $\mathcal{D}$ is the *dissipator*. It is the part that differs from the unitary evolution described by the von Neumann-equation and is responsible for *creating* decoherence and mixed states by describing interactions of the system with the environment.

Note that in the sum different *Lindblad operators* $\hat{L}_\mu$, each depending on a parameter μ, may appear. Each one of these operators may induce a quantum jump caused by interaction with the environment.

Although the subsystems are uncorrelated at $t = 0$, the generalized time evolution can create an entanglement between (A) and (B). This leads to an entropy $S > 0$ (see sec. 8.2).

6 Quantum channels

Quantum channels can be seen as a way of transferring quantum information, e.g. changing the state of qubit is a quantum channel transporting information from the system of interest to an "environment". From that point of view, dissipation described using eq. 5.1 can be seen as a sort of quantum channel transporting "noise".

The choice of the Lindblad operators $\hat{L}_\mu$ are somewhat arbitrary. However, a few variants can be seen to describe certain quantum channels.

In the theory of quantum information, three quantum channels are of interest since they have different physical origins:

- Depolarization-Channel: The phase of the bloch sphere shrinks with a certain probability.
- Amplitude damping channel: The system relaxes from the excited state into the ground state - energy is transferred from the system to the environment.
- Phase damping channel: The phase information about the superposition gets "destroyed".

6.1 Amplitude damping channel

An amplitude damping channel is one where quantum noise is produced due to loss of energy (e.g. inelastic scattering processes, see [2]).

In comparison to a closed quantum system, the Bloch sphere changes like

$$\big(P_x(t), P_y(t), P_z(t)\big) \rightarrow \big(P_x(t)\sqrt{1-\gamma}, P_y(t)\sqrt{1-\gamma}, (\gamma-1)P_z(t)-1\big)$$

where $\gamma = 1 - e^{-t/T_1}$ with a characteristic damping time T_1.

In the following we will see that such behaviour can be created using only a single Lindblad-operator, namely $\hat{L}_\mu = \sqrt{\gamma}\hat{\sigma}^-$ (see, e.g. [2]). Then the dissipator reads as

$$\mathcal{D}[\hat{\rho}(t)] = \gamma\big(\hat{\sigma}^-\hat{\rho}(t)\hat{\sigma}^+ - \frac{1}{2}\big\{\hat{\sigma}^+\hat{\sigma}^-, \hat{\rho}(t)\big\}\big) \tag{6.1}$$

where γ can be seen as a parameter describing the strength of the damping.

Plugging a density matrix of the form 4.2 into eq. 6.1 one obtains

$$\mathcal{D}[\rho(t)] = \gamma\begin{pmatrix} -(1+P_z) & -\frac{1}{2}(P_x - iP_y) \\ -\frac{1}{2}(P_x + iP_y) & 1+P_z \end{pmatrix} \tag{6.2}$$

From the Lindblad master equation one gets

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}P_z &= -\gamma(1+P_z) \\ \frac{\mathrm{d}}{\mathrm{d}t}P_x &= -\omega_0 P_y - \frac{\gamma}{2}P_x \\ \frac{\mathrm{d}}{\mathrm{d}t}P_y &= \omega_0 P_x - \frac{\gamma}{2}P_y \end{aligned}$$

which has the solution for $P_z(t) = (z_0+1)e^{-\gamma t} - 1$ for $P_z(t=0) = z_0$ (Spin up). Indeed an exponential amplitude damping occurs: $P_z(t \to \infty) = -1$, independent of z_0. Furthermore, one gets

$$\begin{aligned} P_x(t) &= e^{-\frac{\gamma}{2}t}\big(-y_0\sin(\omega_0 t) + x_0\cos(\omega_0 t)\big) \\ P_y(t) &= e^{-\frac{\gamma}{2}t}\big(y_0\cos(\omega_0 t) + x_0\sin(\omega_0 t)\big) \end{aligned} \tag{6.3}$$

which is the same behaviour as described by the time evolution in eq. 4.5 damped by a factor $e^{-\frac{\gamma}{2}t}$.

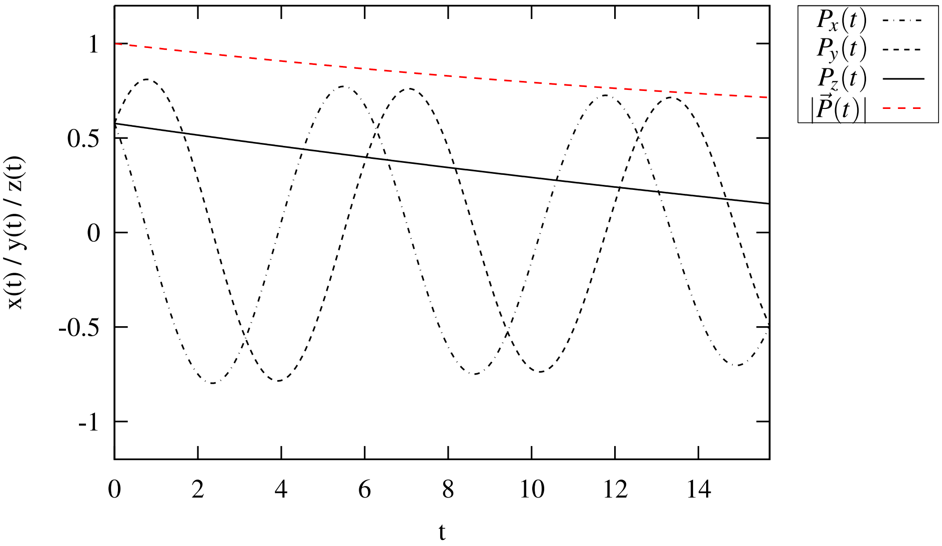

Figure 6.1: Time evolution of a spin-1/2-qubit described by a Lindblad master equation using the dissipator in eq. 6.1. The parameters are $\gamma = 0.05$ and $\omega_0 = 1$. For demonstrational purposes the polarisation at $t = 0$ has been chosen to be $1/\sqrt{3} \cdot (1, 1, 1)^T$. It can be seen that every component of the polarisation is being damped, the norm of $\vec{P}$ decreases. Therefore the system evolves into a mixed state, the qubit becomes incoherent (see eq. 4.2).

6.2 Phase damping channel

A phase damping channel leads to a loss of quantum information without the loss of energy. This might occur due to elastic scattering processes. As a result, *only* off-diagonal elements in the density operator get damped[2]

$$\rho \rightarrow \begin{pmatrix} aa^* & ab^* \cdot e^{-\lambda t} \\ a^* b \cdot e^{-\lambda t} & bb^* \end{pmatrix} \tag{6.4}$$

The Bloch sphere might change like

$$\big(P_x(t), P_y(t), P_z(t)\big) \rightarrow \big(P_x(t)\sqrt{1-\lambda}, P_y(t)\sqrt{1-\lambda}, P_z(t)\big)$$

with $\sqrt{1-\lambda} = e^{-\frac{t}{2T_2}}$.

Similar to the amplitude damping situation, we only have one Lindblad-term with $\hat{L}_\mu = \sqrt{\lambda}\hat{\sigma}_z$. The dissipator becomes

$$\mathcal{D}[\rho(t)] = \lambda \begin{pmatrix} 0 & -2(P_x - iP_y) \\ -2(P_x + iP_y) & 0 \end{pmatrix} \tag{6.5}$$

where λ is once again describing the strength of the damping.

In terms of the components of $\vec{P}$, one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}P_z = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}P_x = -\omega_0 P_y - 2\lambda P_x$$
$$\frac{\mathrm{d}}{\mathrm{d}t}P_y = \omega_0 P_x - 2\lambda P_y$$

with the solution

$$\begin{aligned} P_x(t) &= e^{-2\lambda t}\big(- x_0\sin(\omega_0 t) + x_0\cos(\omega_0 t)\big) \\ P_y(t) &= e^{-2\lambda t}\big(y_0\cos(\omega_0 t) + x_0\sin(\omega_0 t)\big) \\ P_z(t) &= z_0 \end{aligned} \tag{6.6}$$

This is actually a phase damping channel like stated above. We regain the fact that no energy is transferred by the quantum channel, since

$$\langle E\rangle = \mathrm{Tr}(\hat{H}\hat{\rho}) = \omega_0 z_0 = \mathrm{const.}$$

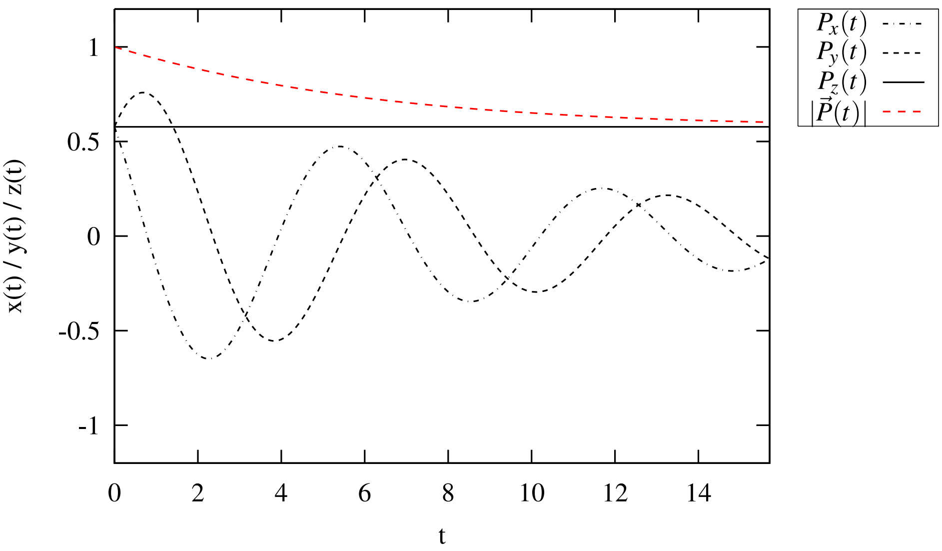

Figure 6.2: Time evolution of a qubit described by the Hamiltonian $\hat{H}_0 = \omega_0 |\uparrow\rangle\langle\uparrow|$ under consideration of a phase damping due to interaction of the system with the environment (see 6.6). The damping constant was set to $\lambda = 0.02$ and $\omega_0 = 1$. To see the effect of the phase damping, the system has been prepared in the $\vec{P} = 1/\sqrt{3}\cdot(1,1,1)^T$-state at $t = 0$. Note that the once prepared spin-up-state remains and therefore no energy is dissipating.

7 NOT-Gate in the presence of dissipation

After this preparatory work, it is now possible to get a more realistic behaviour of a NOT-Gate under full consideration of system-environment-interactions.

We will now again use the Hamiltionian

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_I(t) = \omega_0 \left|\uparrow\right\rangle \left\langle\uparrow\right| + f(t) \left|\uparrow\right\rangle \left\langle\downarrow\right| + f^*(t) \left|\downarrow\right\rangle \left\langle\uparrow\right| \tag{7.1}$$

which causes the unitary part of the Lindblad equation to be

$$-i[H,\rho] = \frac{1}{2}\begin{pmatrix} (P_x + iP_y)f(t) - (P_x - iP_y)f^*(t) & -2P_z f(t) + \omega_0(P_x - iP_y) \\ 2P_z f^*(t) - \omega_0(P_x + iP_y) & (P_x - iP_y)f^*(t) - (P_x + iP_y)f^(t) \end{pmatrix} \tag{7.2}$$

7.1 Effects of an amplitude damping channel

Adding the dissipator eq. 6.2 to the unitary part, one gets

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}P_x &= 2QP_z \sin(\epsilon t) - \omega_0 P_y - \frac{\gamma}{2}P_x \\ \frac{\mathrm{d}}{\mathrm{d}t}P_y &= -2QP_z \cos(\epsilon t) + \omega_0 P_x - \frac{\gamma}{2}P_y \\ \frac{\mathrm{d}}{\mathrm{d}t}P_z &= 2Q\big(P_y \cos(\epsilon t) + P_x \cos(\epsilon t)\big) - \gamma(1 + P_z) \end{aligned} \tag{7.3}$$

This system of differential equations can still be solved analytically by using the *interaction picture*.[4] We will however obtain a numerical approach.

The r.h.s of the differential equations are rather smooth, no singularities occur. Therefore a standard ODE-solver can be used. Here the MATLAB-algorithm `ode45`, an implementation of a Runge-Kutta-method, has been used. The numerical solution is shown in fig. 7.1.

[4] The interaction picture is defined by transforming the density matrix $\hat{\rho}(t)$ to $\hat{\rho}_I(t) := \hat{U}_I(t)\hat{\rho}_0\hat{U}_I^\dagger(t)$. $\hat{U}_I(t)$ is the time evolution operator created by the interaction part of the Hamiltonian $\hat{H}_I(t)$. A technique for handling such problems could be to solve the system for $\hat{\rho}_I(t)$ and apply the retransformation. For more details see [1].

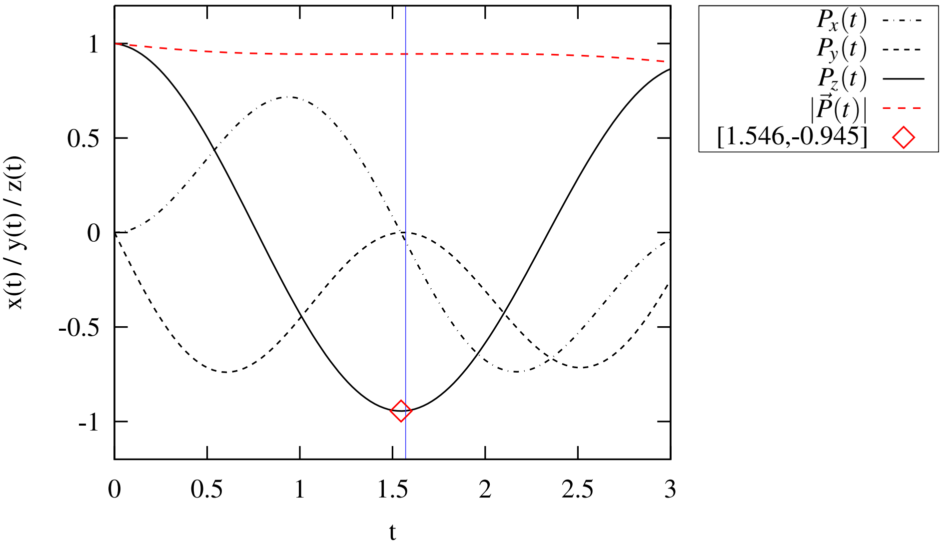

Figure 7.1: Numerical solution of the time evolution of a NOT-Gate under consideration of dissipation through an amplitude damping channel (see eq. 7.3). The parameters were set to $\omega_0 = \epsilon = 1$ (resonance), $Q = 1$, $\gamma = 0.05$. Discussion see text.

One can see that two errors occur in the quantity of interest (the value of P_z after $\tau = \pi/2Q$). An idealized NOT-Gate would work like $P_z(\tau) = -1$.
The first deviation from this desired behaviour comes from a simple damping due to energy loss. The minimum of $P_z(t)$ has a value of -0.945.
The second error comes from a shift of the minimum in time. It does not coincide with τ from the closed system approach (blue line in fig. 7.1). This has major consequences on experimental settings because the interaction time of the EM-pulse to perform the spin-flip has to be detuned by an amount that is not precisely known. For example, if one measures the value of P_z after τ, one gets $P_z(\tau) = -0.943$.

When not switching off the interacting wave after τ, the system relaxes into the totally mixed state $\vec{P} = \vec{0}$, $\rho = \frac{1}{2}\mathbb{1}$ (see fig. 7.2). This differs from the relaxation scenario in the case without interaction (sec. 6.1).

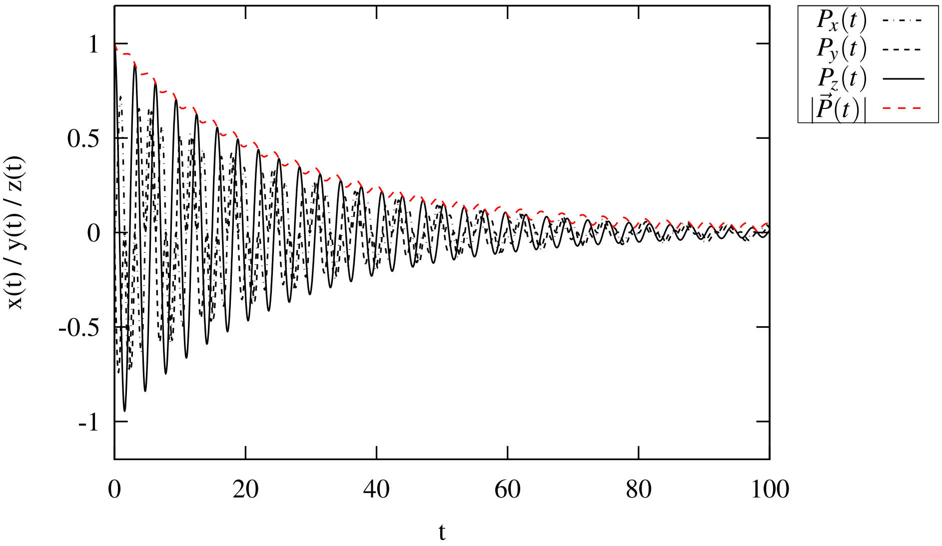

Figure 7.2: Long time evolution of the solution to eq. 7.3 with the same settings as in fig. 7.1 ($\omega_0 = \epsilon = 1, Q = 1, \gamma = 0.05$). The equilibrium is the totally mixed state with $|\vec{P}(t \to \infty)| = 0$ An interesting feature is that at certain points, the size of the bloch-sphere increases. See sec. 8.2 for a discussion on that.

7.2 Effects of a phase damping

Considering the dissipator eq. 6.5, one gets

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} P_x &= 2QP_z \sin(\epsilon t) - \omega_0 y - 2\lambda x \\
\frac{\mathrm{d}}{\mathrm{d}t} P_y &= -2QP_z \cos(\epsilon t) + \omega_0 x - 2\lambda y \\
\frac{\mathrm{d}}{\mathrm{d}t} P_z &= 2Q\big(P_y \cos(\epsilon t) + x \cos(\epsilon t)\big)
\end{aligned} \tag{7.4}$$

Again, a numerical approach was taken. The results are shown in fig. 7.3 - 7.4.

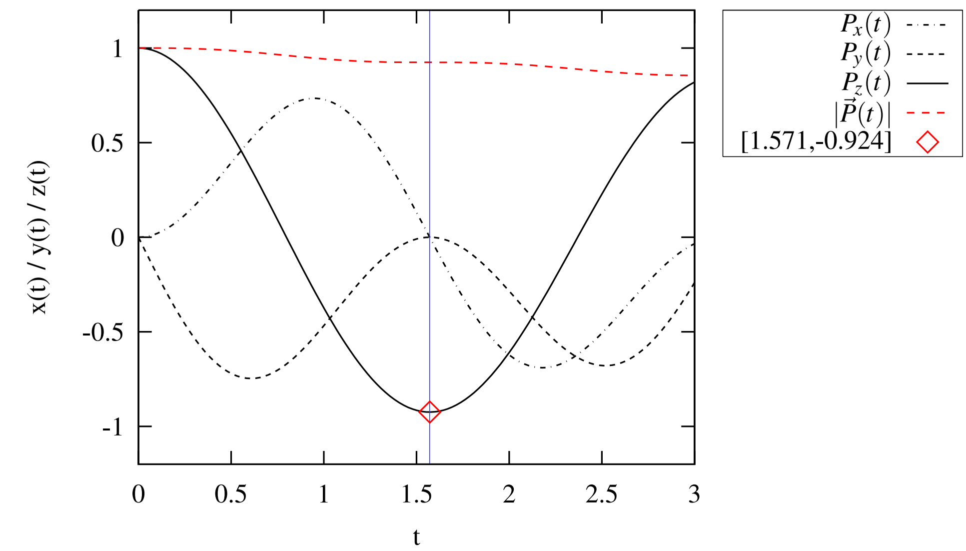

Figure 7.3: Simulation of the time evolution of a qubit under the Hamiltonian defined in eq. 7.1 and considering a noisy phase damping channel. The parameters were set to $\omega_0 = \epsilon = 1$ (resonance), $Q = 1$, $\lambda = 0.05$.

Similar to the effects of the amplitude damping channel, the value of P_z at time τ is being damped, compared to the closed system. However, the time shift is not as present as is the preceeding situation. This might be caused by the fact that P_z itself is not damped, only P_x and P_y (see eq. 7.4). A more detailed investigation would be necessary.

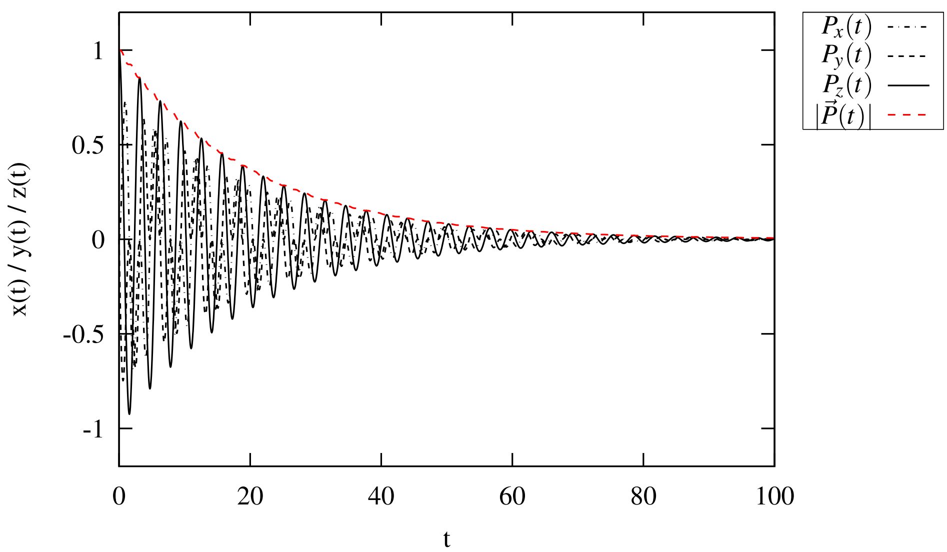

Figure 7.4: Visualization of the relaxation of the qubit shown by considering a phase damping (see eq. 7.4). Parameters are as in fig. 7.3 ($\omega_0 = \epsilon = 1$, $Q = 1$, $\lambda = 0.05$). In contrast to the amplitude damping case (fig. 7.2), the radius of the bloch-sphere (red dashed line) always decreases.

8 Results and Interpretation

When thinking of expectation values of a qubit-state, the first measureable quantity to come up with might be the population of the states $|\uparrow\rangle, |\downarrow\rangle$. Mathematically spoken, those values can be written as

$$\langle\uparrow\rangle = \mathrm{Tr}\big(\hat{\rho}\,|\uparrow\rangle\langle\uparrow|\big) = \rho_{11} = \frac{1}{2}(1 + P_z)$$
$$\langle\downarrow\rangle = \mathrm{Tr}\big(\hat{\rho}\,|\downarrow\rangle\langle\downarrow|\big) = \rho_{22} = \frac{1}{2}(1 - P_z)$$

However, another quantity of major interest is the phase.

8.1 Phase measurement using Hadamard-Gate

A Hadamard-Gate G_H[5] can be seen as first performing a rotation of the qubit around the x-axis for $\theta = 90°$ and then mirroring the bloch-sphere at the y-x-plane.

It can be denoted as

$$G_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Let us briefly interpret the effect of this operation: Assuming the states $|\alpha\rangle_1 = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|\alpha\rangle_2 = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$, after the Hadamard-operation one gets

$$G_H\,|\alpha\rangle_1 = |\uparrow\rangle$$
$$G_H\,|\alpha\rangle_2 = |\downarrow\rangle$$

Therefore the Hadamard-Gate can be used to measure the relative phase between the two qubit-states.

When acting on a density matrix written in eq. 4.2 , the Hadamard-Gate reads as

$$\rho^H = G_H \cdot \rho \cdot G_H$$

To measure the phase, one has to measure the population of the "excited state" of the tranformed matrix. In explicit form, one gets from the equation above

$$\rho^H_{11} = \frac{1}{2}\sum_{i,j=1}^{2}\rho_{ij} = \frac{1}{2}(1 + P_x)$$

[5] Normally the Hadamard-Gate is abbreviated as H. However, to avoid a confusion with the Hamiltonian, the letter G_H shall be used.

which is 0 if the relative phase is "-" and 1 if the relative phase is "+". This is in agreement with the representation of the $|x\rangle$-state in the z-basis.

The time evolutions of the population of the excited state $\langle\uparrow\rangle$ and the relative phase between $|\uparrow\rangle$ and $|\downarrow\rangle$ obtained in the preceding calculations are shown in fig. 8.1 for the amplitude damping case and fig. 8.2 for the phase damping.

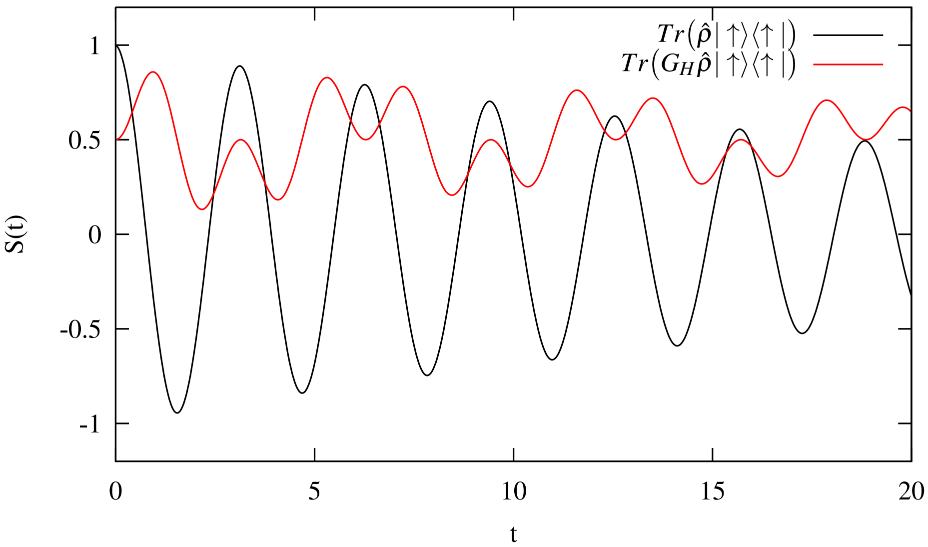

Figure 8.1: Time evolution of the expectation value of the excited state and the expectation value of the relative phase between $|\uparrow\rangle$ and $|\downarrow\rangle$ for the amplitude damping channel. Again, at t=0, the system was prepared in the spin-up-state ($P_z = 1$). Parameters are as in fig. 7.1

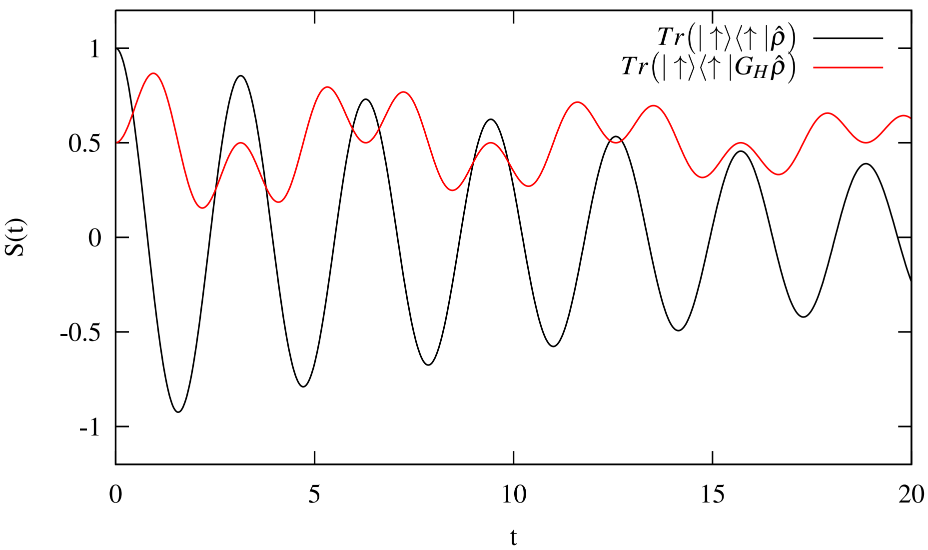

Figure 8.2: Time evolution of P_z and relative phase between $|\uparrow\rangle$ and $|\downarrow\rangle$ for the phase damping channel. Again, $P_z(t=0) = 1$, $\omega_0 = \epsilon = 1$, $Q = 1$, $\lambda = 0.05$.

8.2 Entropy change

From a statistical point of view, an entropy can be assigned to every probability distribution, the so called *Shannon-entropy*

$$S = -\sum_i p_i \cdot \ln(p_i)$$

which is a quantity to measure the information that is associated with the distribution. Von Neumann generalized the Shannon-entropy to describe information in quantum mechanical states. This leads to the *von Neumann-entropy*

$$S = -\mathrm{Tr}\big(\hat{\rho} \cdot \ln(\hat{\rho})\big)$$

In the following we will only use this definition of entropy. $S = 0$ if and only if $\hat{\rho}$ describes a pure state[2],[1].

Thinking of the 2nd law of thermodynamics, one might state that the entropy of a quantum mechanical system can only increase due to system-environment-interaction. This is true for a phase damping channel (see eq. 6.6). For $t \to \infty$ the off-diagonal elements vanish, the Bloch-sphere shrinks to $|\vec{P}| = |z_0|$. Therefore the entropy at equilibrium is well defined and finite.

However, if the entropy change induced by the amplitude damping channel is investigated, one gets a different behaviour.

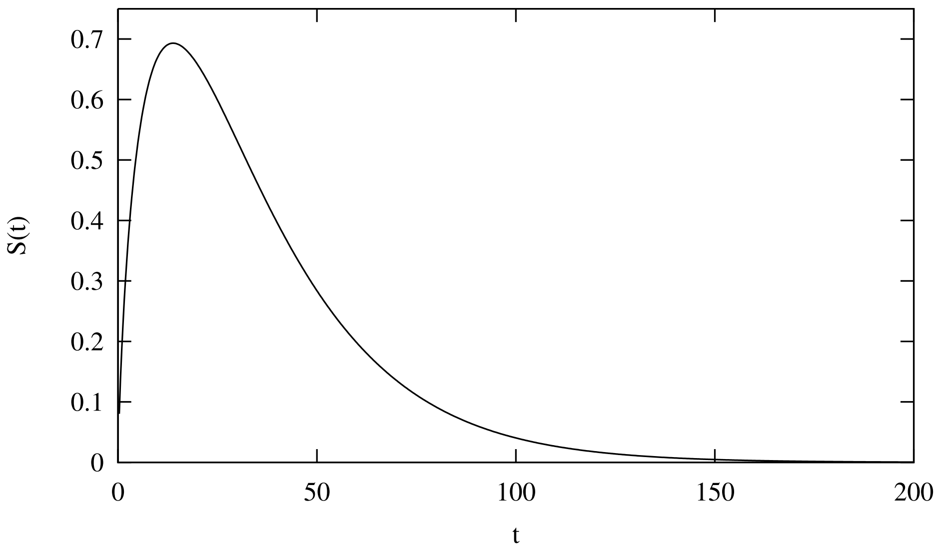

Figure 8.3: Change in the von Neumann-entropy of a qubit without an electromagnetic field, but coupled with an amplitude damping channel as discussed in sec. 6.1 (time evolution see eq. 6.3). The parameters for the plot where set to $Q = 1$, $\gamma = 0.05$. Explanation see text.

In fact, the entropy decreases after a certain point (see fig. 8.3). The reason is that due to energy dissipation, the system always relaxes into the ground state, which is a pure state. That has an entropy of $S = 0$. This can also be seen from the fact that the stable state ($t \to \infty$) is $\vec{P} = (0, 0, -1)^T$ which is pure (see eq. 6.3).

Next, we consider results for the von Neumann-entropy in the case with an electromagnetic field (see sec. 7.1 and 7.2).

Fig. 8.4 shows that for the amplitude damped case, at certain points the entropy decreases. This oscillation comes from the fact that two concurring effects are present: firstly, the amplitude damping, which pushes the system towards the ground state, and secondly the electromagnetic field, which constantly flips the spin from the ground state to the excited state and vice versa. This behaviour can also be seen from the oscillation of the radius of the Bloch-sphere (see fig. 7.2).

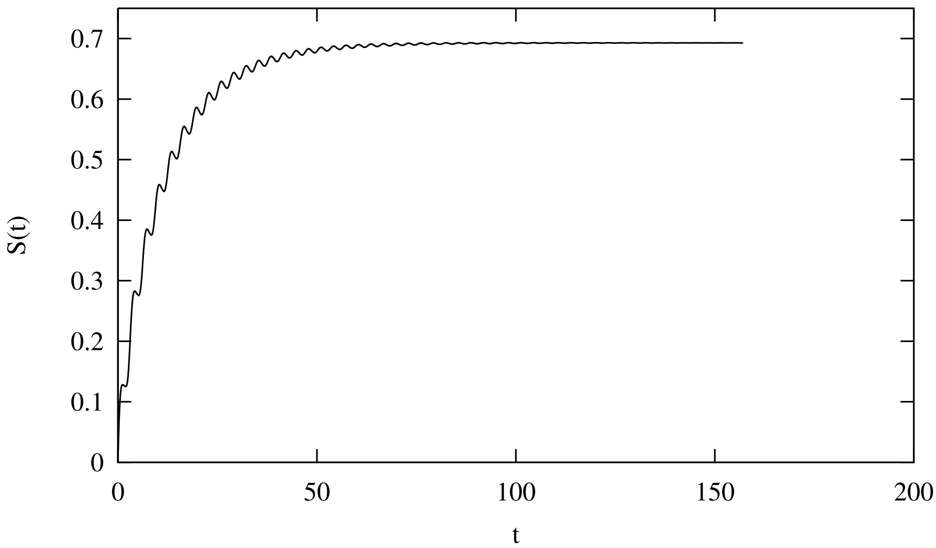

Figure 8.4: Entropy of the system vs. time for the case with electromagnetic field and amplitude channel coupling. Note that at certain points the entropy decreases. The case shown here corresponds to fig. 7.2.

However, such behaviour does not occur in the phase damping channel, the entropy always increases in the case of electromagnetic interaction.

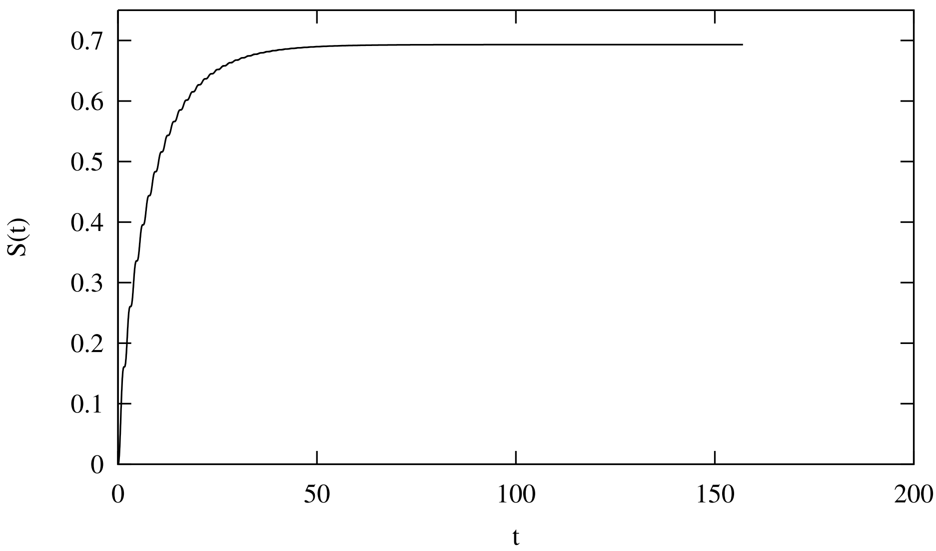

Figure 8.5: Entropy vs. time for the phase channel including an electromagnetic field. Parameters are as declared in 7.4.

One can see that for every quantum channel (phase damping and amplitude damping) the entropy has the same finite value for $t \to \infty$ due to the interaction with the electromagnetic field.

The reason is that both channels create the equilibrium state $\hat{\rho} = \frac{1}{2}\mathbb{1}$. This maximally entangled state[6] has the entropy ln(2). This is the same entropy associated with a classical bernoulli experiment with $p = q = 1/2$. Note that this equilibrium entropy does not depend on system parameters (ω_0, Q, γ) or the initial state of the system.

9 Conclusions

It was shown that it is never possible to describe a realistic behaviour of a spin-1/2-system when neglecting random environment-system-interactions, which acts as noise. If one chooses a description as an open quantum system, due to noisy quantum channels, information gets lost (phase damping and amplitude damping) and entropy increases.

When the system of interest is interacting with an electromagnetic wave, the system always relaxes to the maximally entangled state with an equlibrium entropy and a vanishing polarization. Therefore, for $t \to \infty$, all information in the system is destroyed, no matter what kind of dissipation effect (phase damping or amplitude damping) is present.

[6] Entropy can also be interpreted as a value to measure the entanglement of the system with the environment. A discussion on that can be found in [2].

References

[1] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, Oxford University Press, 2002.

[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th Anniversary Edition, Cambridge University Press, 2010.

[3] M. Schlosshauer, *Reviews of Modern Physics* **76**, 1267 (2005).

[4] R. Karplus and J. Schwinger, *Physical Review* **73**, 1020 (1948).

[5] G. Lindblad, *Communications in Mathematical Physics* **48**, 119 (1976).