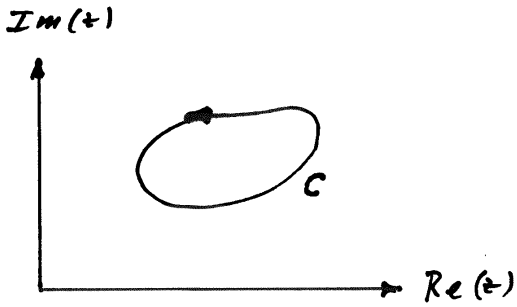
Appendix 1:
The calculation of integrals using Cauchy's formula

<u>Lit.:</u> S. Flügge, *Mathematische Methoden der Physik I*, Springer-Verlag, Heidelberg, 1979, S. 1ff.

The basis of all following statements is Cauchy's formula

$$\oint_C dz\, f(z) = 0\,. \qquad (1)$$

f is a complex function of the complex argument z, and the integration is performed on the closed loop C (in anti-clockwise direction):



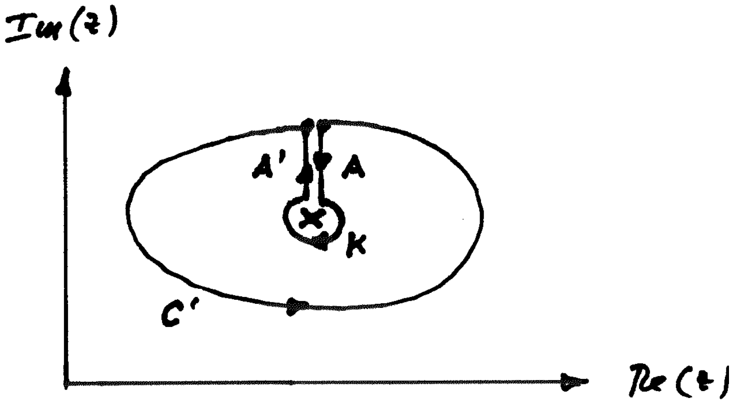
The above simple form of Cauchy's formula is only valid if the function f has no sigularities within C and on the boundary of C.

What happens if $f(z)$ does have such a singularity (a pole of n^{th} order) within C?
In this case, the integrand can be written as

$$f(z) = \frac{F(z)}{(z - z_0)^n} \qquad (n = 1, 2, \ldots)$$

where $F(z)$ means an analytical function.

A possibility to attack the problem is to deform the closed path C according to the following diagram:



C is now a sum of four elements:

$$C = C' + A + K + A' \,.$$

Obviously, the contributions to the path integral along A and A' cancel, and one gets (note the contrary directions of C' and K)

$$\int_{C'} dz \, \frac{F(z)}{(z-z_0)^n} = -\int_K dz \, \frac{F(z)}{(z-z_0)^n} \,.$$

Defining polar coordinates for the circle K with center at z_0 and radius r, namely

$$z = z_0 + r\mathrm{e}^{i\varphi} \qquad \text{and} \qquad dz = ir\mathrm{e}^{i\varphi}\, d\varphi \,,$$

one obtains

$$\int_K dz \, \frac{F(z)}{(z-z_0)^n} = -ir \int_{\varphi=0}^{2\pi} d\varphi \, \mathrm{e}^{i\varphi} \frac{F(z)}{r^n \mathrm{e}^{in\varphi}} \,.$$

Due to the regularity of $F(z)$ at $z = z_0$, this function can be Taylor-expanded as

$$F(z) = \sum_{k=0}^{\infty} \frac{F^{(k)}(z_0)}{k!} (z-z_0)^k = \sum_{k=0}^{\infty} \frac{F^{(k)}(z_0)}{k!} \, r^k \, \mathrm{e}^{ik\varphi} \,.$$

A combination of the last three equations leads to

$$\int_{C'} dz \frac{F(z)}{(z-z_0)^n} = i \sum_{k=0}^{\infty} \frac{F^{(k)}(z_0)}{k!} \, r^{k-n+1} \underbrace{\int_0^{2\pi} d\varphi \, \mathrm{e}^{i(k-n+1)\varphi}}_{=2\pi\,\delta_{k,n-1}} = 2\pi i \frac{F^{(n-1)}(z_0)}{(n-1)!} \,.$$

We call this Cauchy's formula or Cauchy's residua statement, because

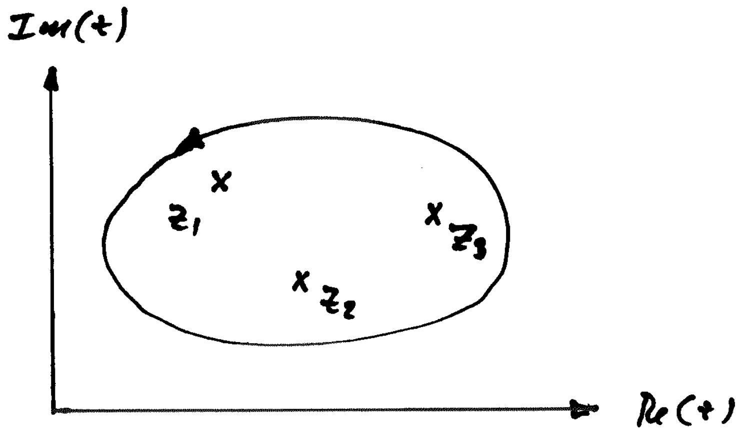
$$Res_f(z_0) \equiv \frac{F^{(n-1)}(z_0)}{(n-1)!}$$

means the residuum of the function $f(z)$ for the pole of n^{th} order at z_0.

According to the last diagram, Eq. (1) can be generalized as follows: the integral of $f(z)$ with respect to a closed loop C on the z plane (direction = anti-clockwise) is given by

$$\oint_C dz\, f(z) = 2\pi i \sum_j Res_f(z_j)\,, \tag{2}$$

what means that the integral is only determined by the sum of the residua at the poles at z_1, z_2, ...:

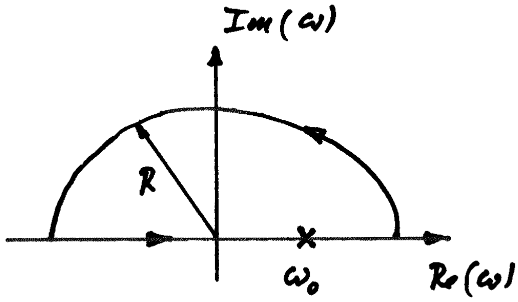


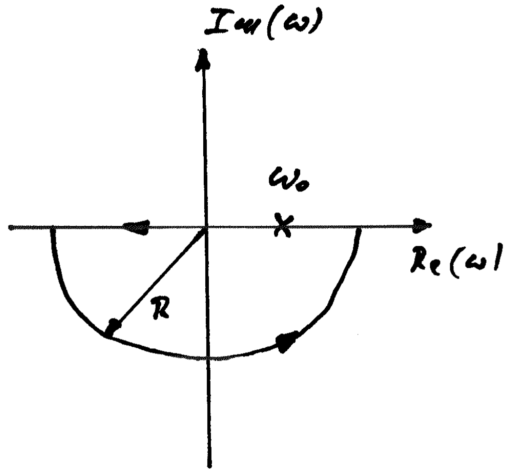
In Solid State Physics, formula (2) is frequently used for the evaluation of integrals like

$$I(t) = \int_{-\infty}^{+\infty} d\omega\, \frac{F(\omega)\, e^{i\omega t}}{(\omega - \omega_0)^n} \qquad n = 1, 2, \ldots$$

where ω_0 means a real number, i.e., the singularity lies on the real ω axis.

In such a situation, the best way to choose an integration path on the complex ω plane is one of the following ones:





The integration from $\omega = -\infty$ to $+\infty$ can be realized by drawing a semicircle of radius R either on the upper or the lower half plane (UHP or LHP), including the limit $R \to \infty$. By doing so, one gets in case of an integration over the UHP[1]

$$\int_C d\omega\, f(\omega) = \int_{-\infty}^{+\infty} d\omega\, f(\omega) + \int_{UHC} d\omega\, f(\omega) = 2\pi i\, Res_f(\omega_0)$$

and over the LHP

$$\int_C d\omega\, f(\omega) = -\int_{-\infty}^{+\infty} d\omega\, f(\omega) + \int_{LHC} d\omega\, f(\omega) = 2\pi i\, Res_f(\omega_0)\,.$$

One further obtains

$$\int_{-\infty}^{+\infty} d\omega\, f(\omega) = -\int_{UHC} d\omega\, f(\omega) + 2\pi i\, Res_f(\omega_0)$$

or

$$\int_{-\infty}^{+\infty} d\omega\, f(\omega) = \int_{LHC} d\omega\, f(\omega) - 2\pi i\, Res_f(\omega_0)\,.$$

Now, what's about the semicircle integrals concerning the function to be integrated?

$$f(\omega) = \frac{F(\omega)\, \mathrm{e}^{i\omega t}}{(\omega - \omega_0)^n}$$

[1]UHC means "upper half-circle", LHC means "lower half-circle".

By using the transformation $\omega = R\,\mathrm{e}^{i\varphi}$, one gets for the integral over the UHC

$$iR\int_0^{\pi} d\varphi\,\frac{F(R\mathrm{e}^{i\varphi})}{(R\mathrm{e}^{i\varphi}-\omega_0)^n}\,\mathrm{e}^{itR\cos\varphi}\,\mathrm{e}^{-tR\sin\varphi}$$

or for the integral over the LHC

$$iR\int_0^{\pi} d\varphi\,\frac{F(-R\mathrm{e}^{i\varphi})}{(-R\mathrm{e}^{i\varphi}-\omega_0)^n}\,\mathrm{e}^{-itR\cos\varphi}\,\mathrm{e}^{+tR\sin\varphi}\,.$$

What concerns the limits of these integrals for $R \to \infty$, one yields for an integration over the upper (lower) half-plane

$$\lim_{R\to\infty}\int_0^{\pi} d\varphi \ldots \frac{\mathrm{e}^{-tR\sin\varphi}}{R^{n-1}} \qquad \text{or} \qquad \lim_{R\to\infty}\int_0^{\pi} d\varphi \ldots \frac{\mathrm{e}^{+tR\sin\varphi}}{R^{n-1}}$$

The consequences of this behavior are as follows:

- In case of a pole of first order (n=1):
  - For $t > 0$: the integral over the half circle is zero (∞) if the integration is performed over the UHP (LHP): For $t > 0$, the integration has to be done over the UHP.
  - For $t < 0$: the integral over the half circle is zero (∞) if the integration is performed over the LHP (UHP): For $t < 0$, the integration has to be done over the LHP.

  This rule called Jordan's lemma is of great importance for practical calculations.

- In case of a pole of higher order ($n > 1$):
  In that case, the integral goes to zero for any t, without taking into account over which half plane the integration is performed.

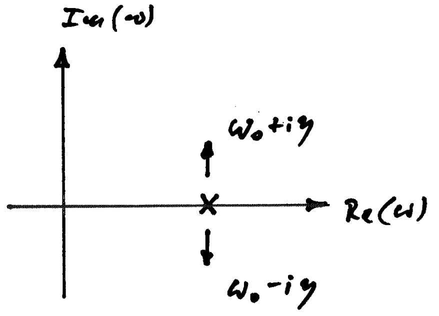
If the above rules are obeyed, all "half-plane integrals" disappear and one gets for $t > 0$

$$\int_{-\infty}^{+\infty} d\omega\,\frac{F(\omega)\mathrm{e}^{i\omega t}}{(\omega-\omega_0)^n} = +2\pi i Res\left[\frac{F(\omega)\mathrm{e}^{i\omega t}}{(\omega-\omega_0)^n}\right](\omega_0)$$

and for $t < 0$

$$\int_{-\infty}^{+\infty} d\omega\,\frac{F(\omega)\mathrm{e}^{i\omega t}}{(\omega-\omega_0)^n} = -2\pi i Res\left[\frac{F(\omega)\mathrm{e}^{i\omega t}}{(\omega-\omega_0)^n}\right](\omega_0)$$

A last problem is still open: it is *technically* disadvantageous if the singularity lies exactly on the real ω axis. For this reason, this pole is shifted into the UHP (LHP) by the factor +(-) $i\eta$ (with η as a real positive number $<< 1$):



After the integration, the limit $\eta \to 0$ has to be performed:

$$I(t) = \lim_{\eta \to 0} \int_{-\infty}^{+\infty} d\omega \, \frac{F(\omega)\, \mathrm{e}^{i\omega t}}{(\omega - \omega_0 \mp i\eta)^n} \, .$$

Finally, the application of Cauchy's formula is demonstrated in connection to the integral representation of the Heaviside step function (see Sec. 1.2.1 on the non-interacting Green's function):

$$\Theta(\tau) = -\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \quad \frac{\mathrm{e}^{-i\omega\tau}}{\omega + i\eta} \, .$$

Obviously, the singularity lies at $\omega = -i\eta$, i.e., within the LHP. For this pole is of first order, Jordan's Lemma has to be obeyed.

- Evaluation of the integral for $\tau > 0$: the integration has to be performed over the LHP, and the residuum of the integrand

  $$f(\omega) = -\frac{1}{2\pi i} \quad \frac{\mathrm{e}^{-i\omega\tau}}{\omega + i\eta}$$

  reads

  $$\lim_{\eta \to 0} \left[-\frac{1}{2\pi i} \quad \mathrm{e}^{-\eta\tau} \right] = -\frac{1}{2\pi i} \, .$$

  Consequently, the integral has the value $\Theta(\tau > 0) = 1$.

- Evaluation of the integral for $\tau < 0$: the integration has to be performed over the UHP, and in this region,there is no singularity at all. Consequently, the integral has the value $\Theta(\tau < 0) = 0$.