Summary Sec. 1.3:
The interacting Green’s function
Feynman diagrams

From Sec. 1.2.1: Remember the most important properties of the one-particle Green’s function for non-interacting electrons:

\[ G^0_{\alpha\beta}(r, t; r', t') = \delta_{\alpha,\beta} G^0(r - r', t - t'). \]

That means:

1. As long as the interaction between the electrons is spin-independent, \( G^0 \) is diagonal in spin space.

2. As long as no relativistic effects concerning the interaction are taken into account, \( G^0 \) is temporally homogenous.

3. As long as the electrons are not subjected to any local (external) potential, \( G^0 \) is spatially homogeneous.

If the above conditions are fulfilled, the corresponding properties are also valid for a system of interacting electrons:

\[ G_{\alpha\beta}(r, t; r', t') = \delta_{\alpha,\beta} G(r - r', t - t'). \]

Perturbation expansion of \( G \), based on the non-interacting functions \( G^0 \)? What does that mean?

\[ G_{\alpha\beta} = \sum_{\nu=0}^{\infty} f_{\nu}(\cdots G^0_{\alpha\beta} \cdots \hat{V} \cdots) \]

\( \nu \) means the order of the term with respect of the interaction operator \( \hat{V} \) (“how often appears \( \hat{V} \) in the term?”).
The electron-electron interaction operator:
\[ \hat{V} = V(r - r'; t - t') = \frac{e^2}{|r - r'|} \delta(t - t') \]

with
\[ V(r - r') = \frac{1}{\Omega} \sum_k V(k) e^{ik \cdot (r - r')} \quad \text{and} \quad V(k) = \frac{4\pi e^2}{k^2}. \]

What’s about the divergence of \( V(k) \) for \( k = 0 \)?

\( V(0) \) refers to the spatial mean value of the potential energy of the electrons. Quantummechanically, such a constant energy value can always be shifted to zero without any change of the physical content of the Schrödinger equation.

Therefore, for all following formulas, one can write per definition:
\[ V(k = 0) = 0 \]

**Feynman diagrams in \((k, \omega)\) space:**

- R. Feynman (1918-1988), Nobel prize 1965, invented the diagram technique for his work in quantum electrodynamics; meanwhile, it has become extremely useful in many other fields in physics.

- Elements of Feynman diagrams:
  - **Rule 1:** Draw all topographically different, connected diagrams which consist von \( \nu \) interaction lines and \( 2\nu + 1 \) particle lines (propagator lines) (\( \nu \) means the order of the diagram).
    Each graph has to be entered and left by *one and only one* particle line.
• Each particle line means a non-interacting Green’s function which is named by a wave vector, a frequency, and two spin indices denoting the spin orientation of the electron at the beginning and the end of the propagator.

• At each vertex, momentum and energy conservation has to be fulfilled.

• In case of a spin-independent interaction (as, e.g., a Coulomb interaction), no spin-flip must happen at the vertices.

Mathematical evaluation:

• Each particle line is replaced by a non-interacting Green’s function $G_{\alpha\beta}(k, \omega)$, each interaction line by a Fourier coefficient of the bare Coulomb potential $V(q)$.

• Summations over all internal spin indices and integrations over all internal wave vectors.

• Each diagram of order $\nu$ has to be multiplied by the factor

$$\left( \frac{i}{\hbar} \right)^{\nu} (2\pi)^{-4\nu} (-1)^F$$

with $F$ as the number of closed particle loops in the diagram.

• Each particle line looking like

![Diagram](image)

has to be replaced by $e^{i\omega\eta} G^0(k, \omega)$.

• After all integrations, the limit $\eta \to 0$ has to be performed.
All Feynman diagrams of zeroth, first, and second order, and some examples of Feynman diagrams of third order:
Evaluation of the graphs of zeroth and first order:

\[(k, \omega) \rightarrow k \text{ etc.}\]

Zeroth-order term (a):

\[G_{\alpha\beta}^a(k) = \delta_{\alpha\beta}G^0(k)\]

First-order term (b):

\[G_{\alpha\beta}^b(k) = \frac{i}{\hbar} \frac{1}{(2\pi)^4} (-1)^{0} V(0) \left[G^0(k)\right]^2 \sum_{\gamma} \sum_{\sigma} \delta_{\alpha\gamma} \delta_{\gamma\beta} \delta_{\sigma\sigma} \int d^4 k_1 e^{i\omega_1 \eta} G^0(k_1) = 0.\]

**Tadpole terms** do not contribute to the Green’s function!

First-order term (c):

\[G_{\alpha\beta}^c(k) = \frac{i}{\hbar} \frac{1}{(2\pi)^4} \left[G^0(k)\right]^2 \sum_{\gamma} \sum_{\sigma} \delta_{\alpha\gamma} \delta_{\gamma\sigma} \delta_{\sigma\beta} \int d^4 q V(q) e^{i(\omega-\omega_1)\eta} G^0(k-q)\]

\[= \delta_{\alpha,\beta} \frac{i}{\hbar} \frac{1}{(2\pi)^4} \left[G^0(k, \omega)\right]^2 \int_{\Omega} d^3 q V(q) \int d\omega_1 e^{-i\omega_1 \eta} G^0(k-q, \omega-\omega_1).\]

(1)
Selfenergy and proper selfenergy:
→ central terms in electron theory

If we look on the table of Feynman diagrams, we see that the first-order term $G^{c}_{\alpha \beta}$ has corresponding higher-order terms leading to an infinite sum of graphs as

\[
(t:) = \frac{\Sigma(M)}{t} = \sum_{n=0}^{\infty} \frac{\Sigma^{(n)}(t:)}{t^n}
\]

The same type of interaction ("Wechselwirkungs-Motiv" in German) appears zero-times, once, twice, threetimes etc. Graphically, this can be written as

$\Sigma(M)$ is called the selfenergy insertion of the interaction type $M$ into the electron particle line.

Please note: $\Sigma(M)$ is not "the" electron selfenergy but only the $M^{th}$ part of it. Actually, the type of interaction shown in the above figures is the simplest approximation to the selfenergy:

\[
\Sigma = \Sigma(t:) + \text{"further terms"}
\]
The definition of a so-called proper selfenergy term can also easily be explained in a graphic way:

Such a definition of proper selfenergy terms enables the representation of the infinite sum of diagrams in a recursive way:

Mathematically, such a Dyson equation means an integral equation in the \( \{r; t\} \) space and an algebraic equation in the \( \{k; \omega\} \) space: the corresponding translation into a mathematical formula is extraordinarily simple [once more: \( k \) means \((k, \omega)\)]:

\[
G(k) = G^0(k) + G(k) \Sigma^{pr}(k) G^0(k).
\]

This equation can easily be solved by writing

\[
G(k) = \frac{G^0(k)}{1 - \Sigma^{pr}(k) G^0(k)} = \frac{1}{[G^0(k)]^{-1} - \Sigma^{pr}(k)},
\]

or, again diagramatically:
How does the formula for the simplest approximation to $\Sigma^{pr}$ look like? This can be got out by remembering that the first-order Green’s function term (c) can be written in a twofold way: firstly, according to Eq. (1) of this summary, as

$$G^c(k, \omega) = G^0(k, \omega) \times \left\{ \frac{i}{\hbar (2\pi)^4} \int_{\Omega} d^3q \ V(q) \int d\omega_1 e^{-i\omega_1 \eta} G^0(k - q, \omega - \omega_1) \right\} G^0(k, \omega).$$

and secondly, directly from the corresponding diagram, as

$$G^c(k, \omega) = G^0(k, \omega) \Sigma^{pr(c)}(k, \omega) G^0(k, \omega).$$

The result:

$$\Sigma^{pr(c)}(k, \omega) = \frac{i}{\hbar (2\pi)^4} \int_{\Omega} d^3q \ V(q) \int d\omega_1 e^{-i\omega_1 \eta} G^0(k - q, \omega - \omega_1).$$

(2)

- How can this equation be further evaluated (especially what concerns the integration over $\omega_1$)?
  → see the following appendix.

- What is the physical content of this equation?
  → see Sec. 1.4.