Crossover from Luttinger- to Fermi-Liquid Behavior in Strongly Anisotropic Systems in Large Dimensions

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We consider the low-energy region of an array of Luttinger liquids coupled by a weak interchain hopping. The leading-logarithmic divergences can be resummed to all orders, within a self-consistent perturbative expansion in the hopping, in the large-dimension limit. The anomalous exponent scales to zero below the one-particle crossover temperature. As a consequence, coherent quasiparticles with finite weight appear along the whole Fermi surface. Extending the expansion self-consistently to all orders turns out to be crucial to restore the correct Fermi-liquid behavior.

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One-dimensional metals are characterized by the failure of the quasiparticle concept, signaled by the vanishing of the single-particle spectral weight at the Fermi surface (FS). This produces a number of additional anomalous single- and two-particle properties, as the absence of a step in the momentum distribution, spin-charge separation, the vanishing of the local density of states \( \rho(\omega) \) as a power-law-like \( \rho(\omega) \propto \omega^\alpha \), as well as nontrivial power-law behavior of several correlation functions. All these anomalous properties (except for spin-charge separation) are controlled by the single exponent \( \alpha \). Metals showing these properties are called Luttinger liquids (LL) [1]. On the other hand, most isotropic two- and higher-dimensional metals show Fermi-liquid (FL) properties, their low-energy regime can be described by well-defined quasiparticles, and the exponents of the correlation functions coincide with those inferred from dimensional analysis.

We thus encounter two distinct universality classes when going from one to higher dimensions, and it is certainly a fascinating issue to study the crossover region between them. Castellani, Di Castro, and Metzner [2] have approached this problem by carrying out an analytic continuation from one dimension \( D = 1 \) to \( D = 1 + \epsilon \). They have shown that the FL regime is always recovered for \( D > 1 \), except for broken-symmetry states.

An alternative way is to approach the crossover for the anisotropic case by starting from a \( (D - 1) \)-dimensional array of uncoupled LL and switching on a small single-particle hopping \( t_\perp /\sqrt{D} \) between them [3]. This also represents, in principle, the experimental situation, for example, of the organic quasi-one-dimensional conductors. The crucial question is: does the system go over to a FL for arbitrarily small \( t_\perp \) and sufficiently low temperatures? One expects that, at least for temperatures \( T \) greater than a certain crossover temperature \( T_{1D} \), the system still shows LL properties. In fact, it has been shown (see, e.g., Ref. [4]), that the region of validity of the LL extends down to \( T_{1D} = t_{\text{eff}} = t_{\perp}^{1/(1 - \alpha)} \) [5–7]. Therefore, the interesting, nontrivial region is the one for which the characteristic energy scale of the system \( T_{\parallel} \) is smaller than \( t_{\text{eff}} \). Only in this region, can a possible FL behavior occur.

Since \( t_{\perp} \) is small, the natural idea is to carry out a perturbative calculation in \( t_{\perp} \). Many results have been obtained by restricting to the lowest order or to the first few orders in \( t_{\perp} \) [4,8,9]. An expansion in powers of \( t_{\perp} \) has been introduced in Ref. [4] and systematized diagrammatically by the author [10]. This was done by extending a method introduced by Metzner to expand around the atomic limit of the Hubbard model [11]. The lowest order in \( t_{\perp} \) for the (inverse) self-energy has been studied by Wen [8] and by Boies et al. [4]. Within their approximation (below referred to as LO), the quasiparticle structure is recovered for arbitrarily small \( t_{\perp} \) on most of the FS. However, the anomalous exponent \( \alpha \) still dominates the low-energy behavior (\( k/t_{\text{eff}} \ll 1 \) [5]) behavior for small \( c_\perp \) since the self-energy behaves as \( k^{1 - \alpha} \), i.e., its asymptotic behavior is still LL like. As a physical consequence, close to the regions for which \( c_\perp \) vanishes, the spectral function is incoherent and the quasiparticles are thus poorly defined (see Fig. 2). In particular, the quasiparticle weight (or residue) \( Z \) vanishes as \( Z \propto (t_{\perp} c_\perp)^{1/(1 - \alpha)} \). However, the LO approximation is not sufficient to make definite conclusions not even for small \( t_{\perp} \). The reason is that the perturbation \( t_{\perp} \) is relevant, i.e., each order in \( t_{\perp} \) carries a strongly divergent power of the energy \( E \gg t_{\perp}^{-1} \). Any perturbative expansion restricted to lowest order is thus uncontrolled at low energies \( E_{\parallel} \ll t_{\text{eff}} \). This is the reason why theoretical results are still contradictory about the nature of the ground state in this energy region. Since, as discussed above, this is precisely the relevant region for a possible FL behavior, it is worthwhile investigating it in a controlled way.

In this Letter, we analyze for the first time unambiguously the region \( E_{\parallel} \ll t_{\text{eff}} \) by extending the calculation of the self-energy to infinite order in \( t_{\perp} \). Specifically, we sum self-consistently the infinite series of loop diagrams of Fig. 1a. The sum of these diagrams with fully dressed hopping (cf. Fig. 1b) gives the exact result in the (anisotropic) large-\( D \) limit. Drastic changes are obtained with respect to the LO result, which corresponds to taking just the first term on the right-hand side in Fig. 1a [8,10]. Our main result (Fig. 2) is that the anomalous exponent \( \alpha \) scales to zero and the asymptotic behavior of the correlation...
functions is dominated by mean-field-like exponents and thus it becomes FL like. In particular, our result restores coherent quasiparticles with a finite weight $Z$ around the regions where $c_{\perp} = 0$. This striking outcome points out the importance of extending the expansion self-consistently to all order in the cumulants. Stopping the series at any finite order would still get an anomalous behavior and a vanishing $Z$.

We consider a hypercubic array of LL with total dimension $D$, characterized by an exponent $\alpha$, and with a hopping $t_{\perp}/\sqrt{D}$ coupling them [3]. Here, we take the spinless case since it allows for crucial simplifications in the calculation. Since we are interested in the effects and in the fate of the anomalous exponent $\alpha$, we believe that spin-charge separation should not play a role.

Since it is not possible to sum all diagrams in the expansion, we want to select a workable subset of diagrams according to some physical limit in order to avoid an arbitrary choice. The $D \to \infty$ limit allows us to restrict ourselves to the diagrams indicated in Fig. 1a. Our $D \to \infty$ procedure is different from the standard dynamical mean-field theory [12] in several aspects. First, and most important, in our case the system is strongly anisotropic, since the hopping in one (in the $||$ direction is not rescaled by the $\sqrt{D}$ factor and is much larger than in the other $D - 1$ ($\perp$) directions. Our system thus represents a 1D chain embedded in an effective self-consistent medium. Since this captures the one dimensionality and the anomalous exponent of the initial system, it is particularly suited to study the problem of the crossover from one to higher dimensions. Notice that the system has more degrees of freedom than the standard $D \to \infty$ single-impurity problem. As a consequence, the self-energy is local with respect to the $\perp$ coordinates but has a nontrivial dependence on the $||$ ones [5]. The second difference is merely procedural: instead of solving the dynamical mean-field equations, we sum all the diagrams contributing to the self-energy in $D \to \infty$. The internal lines of these diagrams must be evaluated self-consistently (see Fig. 1). Third, since we are interested in the asymptotic behavior, we keep only the leading singularities in each diagram, as we will discuss below.

As anticipated, each power of $t_{\perp}$ carries a divergent power of the energy $\propto E_{||}^{1-\alpha}$ [4,10]. A crucial point in our approach is to consider an expansion in the dressed hopping $T_{\perp}$ instead of the bare one $t_{\perp}$ (see below and Fig. 1). Since the scaling behavior of $T_{\perp}$ turns out to be $T_{\perp} \propto E_{||}^{1-\alpha}$ this cancels exactly the power-law divergences met in the $t_{\perp}$ expansion. However, logarithmic divergences, arising from the integration of the cumulants still occur, as we shall see below.

Further ingredients of the perturbation theory are the bare (i.e., $t_{\perp} = 0$) $\perp$-local $m + 1$-particle cumulants of the LL $|G(0)\rangle \langle 0|_{0\cdots m_{\perp}}$. For these, we take the form valid for $T = 0$ and large distances [13]. It will prove convenient to work in real (i.e., $x$) space, in contrast to usual perturbation theory. As discussed above, in the $D \to \infty$ limit, the inverse self-energy $\Gamma(x_0)$ is $\perp$ local, and is obtained as the sum of the loop diagrams in Fig. 1a as

$$\Gamma(x_0) = \langle 0|_{0\cdots m_{\perp}} \frac{1}{\left[ \prod_{k=1}^{m} d^2 y_k \right]} \int_{1|0}^{\infty} \prod_{k=1}^{m} d^2 x_k T_{\perp} (-x_k, 0) \times G(0)|_{0\cdots m_{\perp}} \{ y_0 + x_0, \ldots, y_m + x_m \ | y_0, \ldots, y_m \},$$

where $1|0$ indicates the region $|x_1| > |x_2| > \cdots > |x_m|$ to which the integral can be restricted by exploiting the symmetry under exchange of the coordinates $1, \ldots, m$. In Eq. (1), $T_{\perp} (x, x_0 = 0)$ is the dressed hopping written in real space. As anticipated, we are interested in the dominant low-energy behavior (here, $|x_0|/t_{\perp} \gg 1$) of correlation functions and thus we can restrict to the leading logarithmic divergences in the loop integrals (1)

FIG. 1. Diagrams contributing to the inverse self-energy $\Gamma(k)$ (gray circle) in the $D = \infty$ limit within an expansion in the dressed hopping $T_{\perp}$ (a). (b) Dressed hopping (dashed line) and its diagrammatic expression in terms of the bare hopping $t_{\perp}$ (full line). (c) Dyson equation for the full Green's function $\tilde{G}(k)$ (cf. Ref. [10]).

FIG. 2. Quasiparticle weight $Z$ (a) as a function of the off-chain kinetic energy $c_{\perp}$ (in units of $t_{\perp}$) for the LL exponent $\alpha = 1/3$. Spectral function $A(\omega )$ (b) for $k_{\parallel} = 0$ and $c_{\perp} = 0.1$, and imaginary part of the self-energy $\text{Im}\Gamma^{-1}$ (c) as a function of $\omega$. Our $D = \infty$ results (solid lines) are compared with the LO result (dashed line). In (a) we also plot for comparison the result obtained including just the first two diagrams on the right-hand side of Fig. 1a (dotted line).
are interested in the leading contribution to the fully dressed inverse self-energy (1) in (4). We Within our self-consistent calculation [15] one must use the relative coordinates $x_i$ through $l_i = \alpha \log(|x_i|_{\text{eff}})$ [13]. We can thus first carry out the integration over the “center-of-mass” coordinate $y_{m+1}$ in (2) by simply considering the effect on the bare cumulant. The leading-logarithmic contributions to this integral can be evaluated in a lengthy but straightforward way [13].

We now need to evaluate the dressed hopping $T_{\perp}$ to insert in (2). This can be obtained in $k$ space in terms of $\Gamma(k)$ [10] by solving the Dyson-like equation of Fig. 1b [5]

$$T_{\perp}(k, c_l) = t_{\perp} c_l / (1 - \Gamma(k) r_{\perp} c_l).$$  (4)

Within our self-consistent calculation [15] one must use the fully dressed inverse self-energy (1) in (4). We are interested in the leading contribution to $T_{\perp}(x, 0)$ at large $|x|$. This comes from two types of singularities in momentum space: (i) from the shifted poles of (4) at $c_{\perp} \neq 0, |k| \neq 0$ and (ii) from the $c_{\perp} = 0, |k| = 0$ power-law singularity of $\Gamma(k)$. Taking, to start with, the LO form for $\Gamma$, namely, $\Gamma(k) = G_{0}^0(k) \sim |k|^{-\alpha - 1}$, the shifted-pole contribution (i) turns out to behave like $|x|^{2a - 4}$, while (ii) yields the strongest contribution $\propto |x|^{-\alpha - 3}$. This is due to the fact that the shifted-pole singularity (i) is smeared out after integration over the $\perp$ momenta.

We now evaluate $T_{\perp}$ in terms of the inverse self-energy $\Gamma$, so that it can be inserted in (2). From (3) with $m = 0$, $\Gamma(x_0) = G_{0}^0(x_0) f_0(l_0)$. By taking the Fourier transform (FT) of this expression into momentum space, inserting in (4), and transforming $T_{\perp}$ again into real space, we obtain, within our leading-logarithmic scheme [5],

$$T_{\perp}(x_0, 0) = \frac{i \alpha}{\pi} |x|^{\alpha - 3} e^{i r \arg x} \left[ f_0(l_0) + f_{\perp}^0(l_0) \right],$$  (5)

where $f_0(l_0) = 1/\mathcal{F}_0(l_0)$, and $f_{\perp}^0(l_0) = \frac{d}{dl_0} f_0(l_0)$. Thanks to the $|x|^{\alpha - 3}$ term in (5), the FT of $T_{\perp}$ now scales as $|k|^{3 - \alpha}$, as already anticipated. Therefore, the diagrams written in terms of the dressed propagator $T_{\perp}$ all scale in the same way for low energies and thus only logarithmic singularities are left [16].

Using Eqs. (2), (3), and (5), and integrating over $y_{m+1}$ and $x_{m+1}$, yields a recursive self-consistent equation [13] for the renormalization functions $F_m$, which we have solved together with the self-consistency condition $F_0(l) = 1/F_0(l_0)$, by expanding in powers of the variable $l$ up to 40th order and analyzed the series by means of several types of differential Padé approximants. The result of this analysis yields for the large-positive-$l$ behavior of the renormalization function $F(l) \sim e^{cl}$, where the exponent $c$ turns out to be equal to $l$ within a great degree of accuracy (about 0.01%).

Introducing this result in the expression for the inverse self-energy $\{\text{Eq. (3) with } m = 0\}$ yields $\Gamma(x) = G_{0}^{0}(x) / (x|x|_{\text{eff}})^{\alpha} \sim 1/|k|$, i.e., the anomalous exponent $\alpha$ exactly cancels out. The same things happen in momentum space. Here, one obtains $\Gamma(k) = G_{0}^{0}(k) \times F_0(\alpha \log |x|_{\text{eff}} / |k|) \sim (i \omega - rk)^{-1} [\text{since } G_{0}^{0}(k) \propto |k|^{\alpha} / (i \omega - r k)]$. This can be interpreted as the LL exponent $\alpha$ renormalizing to zero for energies smaller than $t_{\text{eff}}$.

The Green’s function of the coupled array of LL is given by the Dyson-like equation [10] $G(k) = [\Gamma(k)^{-1} - i t_{\perp} c_{\perp}^{-1}]^{-1}$ (cf. Fig. 1c). The main important consequence of our result is that the low-energy behavior of correlation functions, as, say, the Green’s function, is now FL like. Specifically, by an homogeneous rescaling of all energies $k \rightarrow s k$, and $c_{\perp} \rightarrow s c_{\perp}$, $G(k) \propto \alpha^{-1}$ for $s \ll 1$, i.e., the anomalous dimension vanishes. This result reflects onto the spectral properties in the region close to $c_{\perp} = 0$. The quasiparticle weight $Z(c_{\perp})$ on the FS $[k_{\perp}(c_{\perp})]$ is given by $Z(c_{\perp})^{-1} = \frac{d}{dc_{\perp}} 1/\mathcal{G}[k_{\perp}(c_{\perp}), i \omega]_{i \omega = 0}$, and is plotted in Fig. 2a for $\alpha = 1/3$ and compared with two results obtained within the LO approximation (dashed line) and by taking in addition the first loop diagrams of Fig. 1a (dotted line), respectively. For small $c_{\perp}$, the LO result would give a $Z$ vanishing as $Z(c_{\perp}) \propto t_{\text{eff}}^{2} c_{\perp}^{2} / (1 - \alpha)$, thus yielding poorly defined quasiparticles around $c_{\perp} = 0$. Inclusion of the first loop gives a vanishing $Z$ too. On the contrary, our result yields a finite $Z$ for $c_{\perp} \rightarrow 0$ as can also be seen from the figure. This also reflects in the spectral function plotted in Fig. 2b. Our result shows a coherent quasiparticle peak in contrast to the
rather broad and shallow maximum produced by the LO approximation. To understand this behavior, we have plotted the scattering rate $\langle \text{Im}\Gamma^{-1} \rangle$ for $k_{\|} = 0$ in Fig. 2c. As one can see, while for the LO approximation $\text{Im}\Gamma^{-1}$ vanishes quite slowly (as $w^{1-a}$), in our result $\text{Im}\Gamma^{-1}$ vanishes faster than linearly, as required for well-defined quasiparticles. However, one should mention that our calculation, restricted to the leading divergences, yields only reliable results for $\text{Im}\Gamma^{-1}$ close to the FS points with $c_{\perp} = 0$. In other FS points, we cannot reliably state whether $\text{Im}\Gamma^{-1}$ vanishes fast enough or not. Arguments similar to the one of ordinary perturbation theory [17] cannot be extended to the present case, due to the momentum dependence of the vertices in the $t_{\perp}$ expansion.

In summary, our result shows that a quasi-one-dimensional metal with small perpendicular hopping $t_{\perp}$ crosses over from a LL behavior with anomalous exponent $\alpha$ for energies larger than $t_{\text{eff}}$ and flows to a FL fixed point with mean-field-like exponents for smaller energies in large dimensions. This is due to the LL exponent $\alpha$ renormalizing to zero for $\mathcal{E}_t \ll t_{\text{eff}}$. As a consequence, well-defined quasiparticles are recovered along the whole FS. The result contrasts with the LO approximation for which the anomalous exponent is still present in the asymptotic behavior of the self-energy below $t_{\text{eff}}$ and the spectrum is incoherent for small $c_{\perp}$.

In principle, we cannot say whether our result is valid also for $D = 2$ or $D = 3$. However, it turns out that the non-$\perp$-local dressed hopping $\mathcal{T}_{\perp}(x, x_{\perp} \neq 0)$ vanishes faster than the $\perp$-local one $\mathcal{T}_{\perp}(x, 0)$ for large $|x|$. Non-$\perp$-local contributions are thus irrelevant and the present result could be thus extended to finite dimensions. This is valid for values of $\alpha$ smaller than a critical value $\alpha_{2p}$, for which a transition to a two-particle regime takes place [10]. In addition, it may possibly exist a critical dimension below which non-$\perp$-local diagrams may enhance the contribution from the shifted $|\mathbf{k}|$, $c_{\perp} \neq 0$ poles, and give rise to nesting or superconducting instabilities at selected regions of the FS. One additional important conclusion is that it is crucial to include an infinite number of diagrams in the $t_{\perp}$ expansion, in order to obtain reliable results.

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[3] We rescale $t_{\perp}$ by a factor $D^{-1/2}$, in order to obtain a nontrivial $D \to \infty$ limit, as pointed out in Ref. [12].


[5] We give here our conventions: (i) The Fermi velocity $v_F$ and energy $E_F$ are set equal to 1. (ii) The momenta of the right- and left-moving fermions in the isolated LL are measured from the respective Fermi momenta. (iii) $\mathcal{E}_t$ is the largest on-chain energy scale between the external frequency $\omega$, the momentum $k_{\|}$, and the temperature $T$. (iv) The coordinates ($x_\|, \tau, r$), and $\mathbf{k} = (k_{\|}, \omega, r)$. Integrals over these coordinates implicitly include a sum over $r$. (v) $t_{\perp} c_{\perp} = 2t_{\perp} \sum_{j=1}^{D-1} \cos(k_{\perp}/\sqrt{D})$ is the kinetic energy in the $\perp$ direction. (vi) Finally, we call $\perp$ local a quantity which is local in the $x_\perp$ coordinates or, equivalently, independent of the $k_{\perp}$ momenta.

[6] The region of LL behavior may extend down to quite low temperatures. For example, for typical parameters of the Bechgaard salts (see Ref. [18]) as $T_L \sim 100$ K, $T_s \sim 1000$ K, and, as it is believed $\alpha > 1/2$, this region extends down to less than 10 K.


[14] Our procedure is a generalization of the renormalization group (RG) extended to the case of the expansion of Fig. 1. It can be shown that this corresponds to taking all orders in $\log(t_{\text{eff}}/\mathcal{E}_t) = O(1)$, while considering $\alpha$ small, which is justified *a posteriori*, since $\alpha$ turns out to be an irrelevant parameter in the RG sense. This procedure is similar to the “leading-logarithm” approach, which has been used by many authors in order to sum the paquet series for the electron gas. See, for example, I. T. Diatlov, V. V. Sudakov, and K. A. Ter-Martirosian, Sov. Phys. JETP 5, 631 (1957); B. Roulet, J. Gavoret, and P. Nozières, Phys. Rev. 178, 1072 (1969); Y. A. Bychkov, L. P. Gor’kov, and I. E. Dzyaloshinskii, Sov. Phys. JETP 23, 489 (1966).

[15] In contrast to the usual logarithmic paquet expansion [14], the Green’s function is renormalized already at one loop in our calculation and thus its self-consistency has to be fully taken into account.

[16] Actually, as we will show below, the exponent $\alpha$ renormalizes to zero for $\mathcal{E}_t \ll t_{\text{eff}}$, and thus $\mathcal{T}_{\perp}$ will eventually scale as $|\mathbf{k}|$. However, since this happens for the renormalized cumulants (3) too, this conclusion remains unchanged.
