Introduction to theoretical physics

Sheet 1

Exercise 1:

Be $\phi(\mathbf{r})$, $\mathbf{v}(\mathbf{r})$, $\mathbf{w}(\mathbf{r})$, $\mathbf{A}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$ continuously differentiable scalar- and vector-fields ("bold-face" $\mathbf{r}, \mathbf{v}, \mathbf{w}$ denotes vectors). Show that:

i) $\nabla \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\nabla \times \mathbf{v}) - \mathbf{v} \cdot (\nabla \times \mathbf{w})$

ii) $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

iii) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$

iv) $\nabla \times (\nabla \times \mathbf{w}) = \nabla(\nabla \cdot \mathbf{w}) - \nabla^2 \mathbf{w}$

Hint: you can use one of the following equivalent methods: (**(a)** ist safer, **(b)** ist faster, best of all if you compare both methods)

a) Use the component notation, i.e. $\mathbf{v} \to v_i$ and the Einstein sum convention. The "Nabla-operator" $\nabla \to \partial_i = \frac{\partial}{\partial x_i}$ can be treated as a vector. The vector product, for example, can be expressed in terms of the completely antisimmetric tensor ε_{ijk} as $(\nabla \times \mathbf{v})_i = \varepsilon_{ijk}\partial_j v_k$. (Indices occurring twice, here j und k, are summed over by convention). The identity $\varepsilon_{ijk}\varepsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$ can be used whenever needed.

b) (Nabla-calculus): The "Nabla-operator" is a linear differential operator, which can be formally treated as a vector (i.e., the usual rules of vector algebra can be applied), provided one specifies with a special notation the vactor- and scalar fields on which the Nabla operator acts as a differential operator. For example, this can be done in the following way:

$$\nabla \times (\phi \mathbf{v}) \;=\; \nabla \times \overset{\downarrow}{\phi}\, \mathbf{v} + \nabla \times \phi\, \overset{\downarrow}{\mathbf{v}} \;=\; \nabla \times \mathbf{v}\, \overset{\downarrow}{\phi} + \phi \nabla \times \overset{\downarrow}{\mathbf{v}} \;=\; -\mathbf{v} \times (\nabla\phi) + \phi \nabla \times \mathbf{v}$$

The arrow on top of a certain quantity means that the Nabla operator applies to this quantity only. After such a transformation one can adopt the usual expression of vector algebra.

Exercise 2:

Evaluate or prove the following expressions ($\mathbf{r}$ ist der position vector, restrict to the case $\mathbf{r} \neq \mathbf{0}$).

a) $\nabla|\mathbf{r}|$

b) $\nabla \cdot \mathbf{r}$

c) $\nabla\left(\frac{1}{|\mathbf{r}|}\right)$

d) $\nabla \times \mathbf{r}$

e) $\nabla \cdot (f(|\mathbf{r}|)\mathbf{r})$

f) $\nabla \times \left(f(|\mathbf{r}|)\frac{\mathbf{r}}{|\mathbf{r}|}\right)$

g) $\nabla \cdot [\mathbf{a}(\mathbf{r} \cdot \mathbf{a})] = |\mathbf{a}|^2$ ($\mathbf{a}$ is a constant vector):

h) $\nabla \cdot (0, 0, (x^2 + y^2)z) = x^2 + y^2$ ((x,y,z) cartesian coordinates).

Exercise 3*:
Consider two point charges q_1 and $-q_2$ (q_1, $q_2 > 0$, $\frac{q_1}{q_2} = \alpha < 1$), which are locate at a distance $2d$ from each other.
Determine the electrostatic potential $\phi(\mathbf{r})$ of this charge configuration (which vanishes for $|\mathbf{r}| \to \infty$). Determine the shape of the equipotential surface $\phi(\mathbf{r}) = 0$
Determine the **E**-field.

Exercise 4:
(a) Determine the charge distribution producing the potential $\Phi(\mathbf{r}) = qe^{-\alpha r}$
(Hint: use the Laplace operator in spherical coordinates).
Determine the corresponding **E**-field.
(b) Determine the potential and the **E**-field (everywhere except around the origin $\mathbf{r} = 0$) of the charge distribution $\rho(\mathbf{r}) = \mathbf{p} \cdot \nabla\delta(\mathbf{r})$, with $\mathbf{p}$ a constant vector .
(Hint: $\int f(\mathbf{r})\partial_i\delta(\mathbf{r}) \to -\int \delta(\mathbf{r})\partial_i f(\mathbf{r})$ (partial integration)).