Introduction to Theoretical Physics

Problem Set 6

Task 14: Solution to the wave equation with given initial conditions

Consider the following fields at the time $t = 0$ in vacuum, free of charge and current ($\mathbf{r} = (x, y, z)$)

$$\mathbf{E}(\mathbf{r}, t = 0) = E_0(\mathbf{e}_x + a\ \mathbf{e}_y + b\ \mathbf{e}_z)e^{-\frac{z^2}{2\alpha} - d\ y^2} \qquad \mathbf{B}(\mathbf{r}, t = 0) = 0\ . \tag{1}$$

a) Give the requirements for the constants a, b and d in case there are any.

From now on assume $a = b = d = 0$ (from Task b onwards!)

b) Calculate the Fourier components of the electric field (only the k_z-dependent components) $\mathbf{E}(k_z, t = 0)$ and the corresponding time derivative $\frac{\partial}{\partial t}\mathbf{E}(k_z, t)|_{t=0}$ at the time $t = 0$. (Hint: The time derivative of $\mathbf{E}$ can be calculated from $\mathbf{B}$ using one of Maxwell's equations).

c) Give the general solution to the wave equation for those Fourier components.

d) Give the particular solution for the Fourier components, using the initial conditions obtained in subtask (b)

e) Determine $\mathbf{E}(\mathbf{r}, t)$ by reverse transformation for an arbitrary point in time t.

f) Determine $\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t)$ using Maxwell's equations.

g) Calculate $\mathbf{B}(\mathbf{r}, t)$.

Hint:

$$e^{-\frac{(z-q)^2}{2\alpha}} = \frac{1}{2\pi}\int dk\ e^{ik(z-q)}\ \sqrt{2\pi\alpha}\ e^{-\frac{\alpha}{2}k^2}$$

Task 15:

In order to show that the vacuum equations also permit solutions to the wave equation where $\mathbf{E}$ and $\mathbf{B}$ are parallel, assume now a superposition of two opposing circularly polarized plane waves (E_0 is real-valued):

$$\mathbf{E} = (\mathbf{E}_+ + \mathbf{E}_-)\ , \quad \mathbf{E}_\pm = E_0(\mathbf{e}_x \cos(kz \mp ckt) - \mathbf{e}_y \sin(kz \mp ckt)).$$

Write down $\mathbf{E}_+$, $\mathbf{E}_-$ and $\mathbf{E}$ in complex notation.

Determine $\mathbf{E}$ in real notation.

Determine the corresponding fields $\mathbf{B}_+$ and $\mathbf{B}_-$ as well as the total $\mathbf{B}$ field, using Maxwell's equations (complex notation!).

Show that $\mathbf{E}$ and $\mathbf{B}$ are parallel to each other.