Advanced Quantum Mechanic – Winter Term 2017/18

Problem Sheet 02 – due on October 25th

E. Arrigoni, R. Berger, C. Gattringer, T. Kamencek

Problem 2.1

As a first example for the quantum mechanics of a charged particle in an electromagnetic field we consider the case of a constant magnetic field (and vanishing electric field). We will revisit this problem in the lecture, but already discuss it here in a different form.

We assume the B-field in the z-direction, i.e., $\vec{B} = (0, 0, B)$. Show that

$$\vec{A} \;=\; -\frac{1}{2}\,\vec{x}\times\vec{B}\;,\quad \phi \;=\; 0\;, \tag{1}$$

are a correct choice for the vector- and scalar potentials (Are there other possible choices?). Determine $\vec{A}$ explicitly and insert this form of $\vec{A}$ in the time-independent Schrödinger equation for this problem,

$$\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\vec{A}\right)^2 \psi \;=\; E\,\psi\;, \tag{2}$$

and multiply out the square of the operator on the left hand side. Now it is convenient to switch to cylinder coordinates, where Laplace- and Nabla operator are given by (check these formulas in your vector analysis notes if you do not remember how to use them)

$$\nabla \;=\; \vec{e}_\rho\frac{\partial}{\partial\rho} + \vec{e}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \vec{e}_z\frac{\partial}{\partial z}\;,\;\; \triangle \;=\; \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}. \tag{3}$$

Use the fact that the B-field in the mixed term (the term with one Nabla operation) is proportional to $\vec{e}_\varphi$ (write also $\vec{x}$ in cylinder coordinate to see that) and simplify the mixed term. Then use the following factorization ansatz $\psi(\rho, \varphi, z) = u_m(\rho)\, e^{i\varphi m}\, e^{ikz}$ for the wave function in cylinder coordinates and determine the resulting equation for the radial functions $u_m(\rho)$.

<u>Remark:</u> With the Ansatz $u_m(\rho) = \rho^{|m|}\,\exp(-\alpha\rho^2)\,\omega(\rho)$ and the substitution $y = \rho^2$ one can transform the radial equation into the Laguerre differential equation which you might remember from the hydrogen problem. However, this is a painful battle and in the lecture we will discuss a more elegant treatment for the problem of the electron in a constant magnetic field.

Problem 2.2

In the lecture we have discussed how the vector- and scalar potentials $\vec{A}(\vec{x}, t)$ and $\phi(\vec{x}, t)$, as well as the wave function $\psi(\vec{x}, t)$ transform under a gauge transformation. In this example we study the same quantum mechanical problem for two different choices of the potentials that are related by a gauge transformation, i.e., describe the same physical fields.

We consider a particle with charge q and mass m in a constant external electric field $\vec{E} = E_0 \, \vec{e}_x$. We can restrict the problem to one spatial dimension, i.e., we consider just the x coordinate and of course the time t. In this case also the vector potential reduces to a scalar function $A(x, t)$ (which is the x-component of $\vec{A}$) and the Nabla operator is replaced by ∂ / ∂_x. Furthermore we use natural units in which $\hbar = c = 1$.

Different forms of the vector- and the scalar potentials $A(x, t)$ and $\phi(x, t)$ that are related by a gauge transformation give the same $E(x, t)$. For the given constant electric field $E(x, t) = E_0$ construct $A(x, t)$ and $\phi(x, t)$ for the following two gauge choices :

(1) $A(x, t) \neq 0$, $\phi(x, t) = 0$,

(2) $A'(x, t) = 0$, $\phi'(x, t) \neq 0$,

and find also the gauge transformation function $\Lambda(x, t)$ that connects the two choices of gauge. Ansatz: (1) $A(x, t) = c_1 t$, (2) $\phi'(x, t) = c_2 x$, $\Lambda(x, t) = c_3 \, t \, x$.

Determine the corresponding Hamiltonians: $\widehat{H}$ for the gauge choice in (1) and $\widehat{H}'$ for the choice in (2). Write down the time-dependent Schrödinger equation (TDSE) for the two Hamiltonians $\widehat{H}$ and $\widehat{H}'$.

Starting from an initial wave function $\psi(x, t = 0) = e^{ikx}$ solve the TDSE for H in the following way:

(i) Use an ansatz of the form

$$\psi(x, t) = e^{ikx} f(t) \tag{4}$$

and determine the corresponding differential equation for $f(t)$.

(ii) Write down and solve the equation for $\log f(t)$. This gives $f(t)$ and thus $\psi(x, t)$.

In the lecture we have discussed how the wave functions $\psi(x, t)$ and $\psi'(x, t)$ are related when the corresponding potentials are connected via a gauge transformation. Use this relation to determine the wave function $\psi'(x, t)$. Show that $\psi'(x, t)$ obeys the TDSE with the Hamilton operator $\widehat{H}'$.

Problem 2.3

As another repetition of material you already know from your previous QM courses we discuss the time evolution of a wave packet. This involves two key techniques: Working with improper states (plane waves) and solving the quantum mechanical initial value problem.

We consider a free particle of mass m and again study the one-dimensional problem. At $t = 0$ the initial wave function is given by

$$\psi(x, t = 0) \;=\; \psi_0(x) \;=\; A \exp\left(-\frac{1}{4d^2}x^2\right) \,. \tag{5}$$

Determine the amplitude A such that $\psi_0(x)$ is normalized correctly.

Show that the free, time-independent one-dimensional Schrödinger equation is solved by the plane waves e^{ikx} and determine the corresponding energies $E(k)$. The full time dependent solution is then given as a superposition of plane waves in the form

$$\psi(x, t) \;=\; \int_{-\infty}^{\infty} dk\, \rho(k)\, e^{-i\frac{E(k)t}{\hbar}}\, e^{ikx} \,. \tag{6}$$

Use your knowledge of Fourier transformation to determine the coefficient function $\rho(k)$ from the initial wave function $\psi_0(x)$.

As a final step insert $\rho(k)$ in (6) and solve the resulting Gaussian integral. Discuss the behaviour of $|\psi(x, t)|^2$ as a function of time.

Find the arguments that lead to the conclusion that the wave function for the same problem in three dimensions is the product of the one-dimensional wave functions for the three spatial directions.