Advanced Quantum Mechanic – Winter Term 2017/18

Problem Sheet 03 – due on November 15th

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Problem 3.1

The Hamiltonian of the 1-dimensional harmonic oscillator is given by

$$\widehat{H}^{(0)} \;=\; -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \;+\; \frac{m\omega^2}{2}\,x^2 \,. \tag{1}$$

In your first quantum mechanics class you computed its eigenfunctions which we denote as $\psi_n(x)$ and the eigenvalues $E_n = \hbar\omega(n+1/2),\; n \in \mathbb{N}_0$. Now we couple a constant electric field E and consider

$$\widehat{H}^{(q)} \;=\; \widehat{H}^{(0)} \;+\; q\,\phi(x) \,. \tag{2}$$

Find a suitable scalar potential $\phi(x)$ for the constant E field and solve the eigenvalue problem for $\widehat{H}^{(q)}$. The problem can be solved by rewriting the combined potential of oscillator and electrostatic part into a new form (take some inspiration from our discussion of the Landau levels in the lecture). Then one can immediately read off the eigenvalues and eigenfunctions.

Let us now consider the problem of the harmonic oscillator in an electric field as a toy model for the Stark effect (shift of spectral lines in an electric field): Can we observe a shift of spectral lines when we switch on E? If your answer is no, then try to construct a space-dependent field $E(x)$ where one can observe a shift of spectral lines. If your answer is yes, then try to construct a space-dependent field $E(x)$ where no shift occurs ($E(x) \equiv 0$ does not count).

Problem 3.2

In the Born approximation the scattering amplitude is given by

$$f_k(\theta,\varphi) \;=\; -\,\frac{m}{2\pi\hbar^2}\int d^3x'\,V(\vec{x}\,')\,e^{-i\vec{x}'\cdot\vec{q}_k(\theta,\varphi)} \,. \tag{3}$$

The incoming particles move in the z-direction with a wave vector $\vec{k} = k\vec{e}_z$, with energy $E = \hbar^2k^2/2m$. For this setting the vector $\vec{q}_k(\theta,\varphi)$ has the form $\vec{q}_k(\theta,\varphi) = k(\vec{e}_r - \vec{e}_z)$, with $\vec{e}_r = (\sin\theta\,\cos\varphi, \sin\theta\,\sin\varphi, \cos\theta)$.

Compute the differential cross-section $d\sigma/d\Omega = |f_k(\theta,\varphi)|^2$ for the following spherically symmetrical potential

$$V(\vec{x}\,') \;=\; \begin{cases} V_0 & \text{for } |\vec{x}\,'| \le d_0 \\ \\ 0 & \text{for } |\vec{x}\,'| > d_0 \end{cases} \tag{4}$$

<u>Hint 1:</u> Introduce spherical coordinates $\vec{x}\,' = r'(\sin\theta' \, \cos\varphi', \sin\theta' \, \sin\varphi', \cos\theta')$ for the integration over $\vec{x}\,'$ and choose the z' axis parallel to $\vec{q}_k(\theta,\varphi)$.

<u>Hint 2:</u> Show that $|\vec{q}_k(\theta,\varphi)| = 2k\sin(\theta/2)$ and express the result for $d\sigma/d\Omega$ as a function of $\sin(\theta/2)$ (the cross section depends only on the scattering angle θ, as is the case for any spherically symmetrical potential).

Perform the combined limit $d_0 \to 0$ and $V_0 \to \infty$ with fixed $\alpha = V_0(d_0)^3$.

<u>**Problem 3.3**</u>

In the lecture we have discussed the Aharanov Bohm effect. On the right hand side of the double slit screen we derived the solution for the squared absolute value of the wave function as

$$|\psi_B(\vec{x})|^2 \;=\; |\,\psi_0^{(1)}(\vec{x})\, e^{\frac{ie}{\hbar c}\phi_B} \;+\; \psi_0^{(2)}(\vec{x})\,|^2 \;, \tag{5}$$

where $\psi_0^{(1)}(\vec{x})$ and $\psi_0^{(2)}(\vec{x})$ are the wave functions for the particles going through slit 1 and 2 in the absence of a magnetic field, and ϕ_B is the magnetic flux in the coil between the slits when the B-field is switched on.

We consider the situation where the screen with the two slits coincides with the x-y plane. The two infinite slits are parallel to the x-axis at $y = \pm d$. Use the cylindrical waves

$$\begin{aligned} \psi_0^{(1)}(y,z) &= c\,\frac{e^{ikr^{(1)}}}{\sqrt{r^{(1)}}} \;,\quad r^{(1)} = \sqrt{(y-d)^2 + z^2}\;, \\ \psi_0^{(2)}(y,z) &= c\,\frac{e^{ikr^{(2)}}}{\sqrt{r^{(2)}}} \;,\quad r^{(2)} = \sqrt{(y+d)^2 + z^2}\;, \end{aligned} \tag{6}$$

for the $B = 0$ solutions on the right hand side of the double slit screen ($z > 0$). Determine the positions in space where we have constructive interference in the full solution (5) and find the location of the intensity maxima. Discuss how they depend on ϕ_B.