Advanced Quantum Mechanics – Winter Term 2017/18

Problem Sheet 04 – due on December 12^{th}33

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Problem 4.1

During the written test it turned out, that discussing better the Gaussian integral (one of the most important integrals in theoretical physics) is clearly a good idea. More specifically you are supposed to show the following result for the 3-dimensional Gaussian integral:

$$\int_{\mathbb{R}^3} d^3r \; e^{-\frac{c^2\,\vec{r}^{\,2}}{2} - i\vec{r}\cdot\vec{q}} \; = \; \left(\frac{\sqrt{2\pi}}{c}\right)^3 \; e^{-\frac{\vec{q}^{\,2}}{2\,c^2}} \; . \tag{1}$$

We now solve the integral in three different ways which all relate to some of the questions that were asked during and after the test. All three ways use the one-dimensional integral (proven in the theory of complex functions)

$$\int_{-\infty}^{\infty} dx \, e^{-\frac{a^2x^2}{2}} \, e^{-ixb} \; = \; \frac{\sqrt{2\pi}}{a} \, e^{-\frac{b^2}{2a^2}} \; . \tag{2}$$

1. Use the fact that $\vec{r}^{\,2}$ and $\vec{r}\cdot\vec{q}$ both are sums of three terms that only depend on one of the coordinates x, y and z. Together with $d^3r = dx\,dy\,dz$ this allows you to completely factorize the integral (1) into the product of three integrals of the type (2).
2. You may also choose the z-axis for your integration over $\vec{r}$ such that it points in the direction of $\vec{q}$. In these coordinates we have $\vec{q} = q\vec{e}_z$, where $q = |\vec{q}|$. Again you can factorize the integral (1) into three integrals of the type (2). Now the integrals over x and y, do not even have a linear term in the exponent. Still, the result is again (1).
3. For the brave: You may also use spherical coordinates. Choose the z-axis for the integration parallel to $\vec{q}$, but now switch to spherical coordinates. Solve the φ-integral (trivial) and the θ-integral (as in the lecture). You will have a remaining integral over r with a factor $\sin(rq)$ in the integrand. Use $\sin(rq) = -1/r\; d/dq\; \cos(rq)$ and write the cosine as the sum of $e^{\pm iqr}/2$. For the e^{iqr} part of the integral perform the change of variables $r \to -r$ and combine the resulting integral with the e^{-iqr} part to an integral of the form (2). Solve this integral with (2). Perform the derivative d/dq and you will again find the result (1).

With different levels of complication all 3 methods lead to the same result.

Problem 4.2

In the lecture we went through a quick reminder for the properties of the angular momentum. The components $\widehat{L}_k$ of the angular momentum operator are given by

$$\widehat{L}_k \;=\; \epsilon_{klm}\,\widehat{x}_l\,\widehat{p}_m \;\mapsto\; -i\,\hbar\,\epsilon_{klm}\,x_l\,\frac{\partial}{\partial x_m}\;. \tag{3}$$

The operator $\widehat{\vec{L}}^{\,2}$ and the ladder operators $\widehat{L}_\pm$ are defined as

$$\widehat{\vec{L}}^{\,2} \;=\; \widehat{L}_1^2 + \widehat{L}_2^2 + \widehat{L}_3^2\;, \qquad \widehat{L}_\pm \;=\; \widehat{L}_1 \;\pm\; i\,\widehat{L}_2\;. \tag{4}$$

Show the following relations

$$\widehat{L}_k^\dagger \;=\; \widehat{L}_k\;, \quad \widehat{\vec{L}}^{\,2\,\dagger} \;=\; \widehat{\vec{L}}^{\,2}\;, \quad \widehat{L}_\pm^\dagger \;=\; \widehat{L}_\mp\;. \tag{5}$$

$$\left[\widehat{L}_k, \widehat{x}_l\right] \;=\; i\,\hbar\,\epsilon_{klm}\,\widehat{x}_m\;, \quad \left[\widehat{L}_k, \widehat{p}_l\right] \;=\; i\,\hbar\,\epsilon_{klm}\,\widehat{p}_m\;. \tag{6}$$

$$\left[\widehat{L}_k, \widehat{L}_l\right] \;=\; i\,\hbar\,\epsilon_{klm}\,\widehat{L}_m\;, \quad \left[\widehat{\vec{L}}^{\,2}, \widehat{L}_k\right] \;=\; 0\;, \quad \left[\widehat{L}_3, \widehat{L}_\pm\right] \;=\; \pm\hbar\widehat{L}_\pm\;. \tag{7}$$

$$\widehat{\vec{L}}^{\,2} \;=\; \widehat{L}_+\widehat{L}_- \;-\; \hbar\,\widehat{L}_3 \;+\; \widehat{L}_3^2\;, \quad \widehat{\vec{L}}^{\,2} \;=\; \widehat{L}_-\widehat{L}_+ \;+\; \hbar\,\widehat{L}_3 \;+\; \widehat{L}_3^2\;. \tag{8}$$

Problem 4.3

For the addition of angular momenta we will need to solve the following counting problem: We consider two angular momenta j_1 and j_2, where without loss of generality $j_1 \geq j_2 > 0$. The corresponding z-components are labelled by the quantum numbers $m_i \in \{-j_i, -j_i{+}1, \;......\; j_i{-}1, j_i\},\; i = 1,2$. The z-component m of the total angular momentum is given by

$$m \;=\; m_1 \;+\; m_2\;. \tag{9}$$

When m_1 and m_2 are varied independently, obviously m assumes values in the range between $-(j_1 + j_2)$ and $+(j_1 + j_2)$. However, the values of m may be degenerate.

- For each value of m determine the corresponding degeneracy.
- Compute the sum of all degeneracies and use this sum to check whether your result is plausible.