Advanced Quantum Mechanics – Winter Term 2017/18

Problem Sheet 05 – due on January 17th

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Problem 5.1

As a further example for the addition of angular momenta we consider the case of adding two angular momenta with $j_1 = j_2 = 1$. You need to determine the states $|j, m, 1, 1\rangle$ that are eigenstates of the square of the total angular momentum $(\widehat{\vec{J}})^2$ with quantum number j, of the z-component $\widehat{J}_z$ with quantum number m, as well as the squares of the individual angular momenta squared $(\widehat{\vec{J}}_1)^2$ and $(\widehat{\vec{J}}_2)^2$ with quantum numbers $j_1 = 1$ and $j_2 = 1$. These states are linear combinations of the product states $|j_1, m_1\rangle|j_2, m_2\rangle$. As in the lecture we use the abbreviation $|j, m\rangle \equiv |j, m, 1, 1\rangle$.

1. Similar to Problem 4.3 start with a list of all values the z-component $m = m_1 + m_2$ of the total spin can have for our case of $j_1 = j_2 = 1$.

2. From the degeneracy of the values of m in your list determine which multiplets $|j, m\rangle$ emerge, i.e., determine the expected values of j.

3. Determine the top state $|j_{max}, j_{max}\rangle$ for the multiplet with the largest value j_{max} of j and construct all other states $|j_{max}, m\rangle$ by applying the ladder operator $\widehat{J}_-$ as discussed in the lecture.

4. Determine the top state $|j_{max} - 1, j_{max} - 1\rangle$ of the second multiplet as the state that is orthogonal to $|j_{max}, j_{max} - 1\rangle$ of the first multiplet. Construct all other states $|j_{max} - 1, m\rangle$ in that multiplet by applying the ladder operator $\widehat{J}_-$.

5. Determine the top state $|j_{max} - 2, j_{max} - 2\rangle$ of the third multiplet as the state that is orthogonal to the states $|j_{max}, j_{max} - 2\rangle$ and $|j_{max} - 1, j_{max} - 2\rangle$ of the first and second multiplets. Construct all other states $|j_{max} - 2, m\rangle$ in that multiplet by applying the ladder operator $\widehat{J}_-$.

6. Iterate the procedure until you have expressed all multiplets for the total spin j in terms of linear combinations of the product states $|j_1, m_1\rangle|j_2, m_2\rangle$.

Check that the total number of states you constructed is correct. For all multiplets discuss the symmetry properties of the $|j, m\rangle$ under interchange of the two particles. Evaluate explicitly the eigenvalues of $(\widehat{\vec{J}})^2$ and of $\widehat{\vec{J}}_1 \cdot \widehat{\vec{J}}_2 = [(\widehat{\vec{J}})^2 - (\widehat{\vec{J}}_1)^2 - (\widehat{\vec{J}}_2)^2]/2$ for at least one member in each multiplet.

Problem 5.2

In this problem we directly calculate $\delta_0(k)$, the scattering phase shift $\delta_l(k)$ in the $l = 0$ channel for a simple spherically symmetrical potential given by

$$V(r) \;=\; \begin{cases} V_0 & \text{for } r \le a \\ 0 & \text{for } r > a \end{cases} . \tag{1}$$

In the lecture we expressed the general solution of the elastic scattering problem in the form

$$\psi(\vec{x}\,) = \sum_{l=0}^{\infty}(2l+1)\,P_l(\cos\theta)\,\frac{i^l}{kr}\,e^{i\delta_l(k)}\,\sin(kr - l\pi/2 + \delta_l(k))\;, \tag{2}$$

where $E = \hbar^2k^2/2m$. We can read off the contribution in the $l = 0$ channel:

$$\psi(\vec{x}\,)\Big|_{l=0} \;=\; \frac{1}{kr}\,e^{i\delta_0(k)}\,\sin(kr + \delta_0(k)) \;=\; A\,\frac{1}{r}\,\sin(kr + \delta_0(k))\;, \tag{3}$$

where the r-independent factors were collected in the constant A.

We now directly solve the specific problem for the potential (1) and read off $\delta_0(k)$: As in the lecture we use the ansatz $\psi(\vec{x}\,) = R_l(r)\,Y_{lm}(\theta,\varphi)$, where $Y_{lm}(\theta,\varphi)$ are the spherical harmonics (for $l = 0$: $Y_{00}(\theta,\varphi) = 1/\sqrt{4\pi}$). $R_l(r)$ is a solution of the radial Schrödinger equation,

$$\left(-\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{1}{r^2}l(l+1)\right] + V(r)\right)R_l(r) \;=\; E\,R_l(r)\;. \tag{4}$$

Use the ansatz $R_l(r) = w_l(r)/r$ for the radial equation at $l = 0$ and show

$$\left[\frac{d^2}{dr^2} + k^2 - U(r)\right]w_0(r) \;=\; 0\;, \tag{5}$$

where $U(r) = V(r)2m/\hbar^2$. Show that for $k^2 < U_0 \equiv 2mV_0/\hbar^2$ one finds $w_0(r) = A\,\sin(kr + \delta_0(k))$ and $w_0(r) = A'\,\sinh(r\sqrt{U_0 - k^2})$ as the solutions for $r > a$ and $r \le a$. Here A, A' and $\delta_0(k)$ are integration constants. What happened to the second integration constant for the solution for $r \le a$?

Putting together the full solution $R_0(r)Y_{00}(\theta,\varphi)$ for $r > a$, one finds that this indeed has the form of (3) and that the integration constant $\delta_0(k)$ is the scattering phase shift in the $l = 0$ channel.

The last step is to determine the integration constants (up to an overall factor) by using the continuity of $w_0(r)$ and its derivative at $r = a$. Show that this gives rise to the equation

$$\frac{1}{k}\,\tan(ka + \delta_0(k)) \;=\; \frac{1}{\sqrt{U_0 - k^2}}\,\tanh(a\sqrt{U_0 - k^2})\;. \tag{6}$$

Determine the scattering phase shift for the limits $a \to 0$ and $V_0 \to 0$.

We finally are also interested in the limit of $k \to 0$. For this limit we parameterize the scattering phase shift as $\delta_0(k) \sim a_0\,k$, where a_0 is the so-called scattering length. Determine the scattering length a_0.