Advanced Quantum Mechanics WS 2024/25, Problem set 0

Review of bachelor level quantum mechanics.
We have here only star exercises, i.e. they gives extra marks. I still strongly advice to get familiar with these topics.

We use $\hbar = 1$ in these exercises

0.1* Harmonic oscillator

Consider an harmonic oscillator described by the hamiltonian

$$H = \epsilon \; a^\dagger \; a$$

where $a^\dagger$, a are the creation and annihilation operators (Erzeugung- und Vernichtungsoperatoren). Notice that $\hbar = 1$ and the $\frac{1}{2}$ term has been omitted for simplicity. Also the $\hat{}$ on operators as on $\hat{H}$ will be often neglected in this course.
The $|n\rangle$ with $n = 0, 1, 2, \cdots$ are the eigenstates of H with eigenvalues $\epsilon \; n$.
Consider now the operator

$$X \equiv \frac{a + a^\dagger}{2}$$

(a) Evaluate the expectation value $\langle 1| \, X \, |1\rangle$
Consider the state

$$|\psi\rangle = \alpha \left(|0\rangle + 2 \;\; |1\rangle \right) \qquad\qquad \alpha > 0$$

(b) Determine the normalisation constant α
(c) Determine $\langle \psi| \, X \, |\psi\rangle$

0.2* cont. Harmonic oscillator: Schrödinger time dependence

(a) Determine the time evolution $|\psi(t)\rangle$ of the state $|\psi\rangle$ (in the Schrödinger picture).
(b) Determine the expectation value $\langle \psi(t)| \, X \, |\psi(t)\rangle$

0.3* cont. Harmonic oscillator: Heisenberg time dependence

(a) Solve the equation of motion for the operator $X^H(t)$, i.e. X in the Heisenberg picture.
Hints: First write down the Heisenberg equation of motion for the operator $a^H(t)$ and solve it. Notice that both X and H are time independent in the Schrödinger picture.
Then use the fact that $a^{\dagger\ H}(t) = \left(a^H(t)\right)^\dagger$. Finally use linearity: $X^H(t) = \frac{a^H(t)+a^{\dagger\ H}(t)}{2}$.
(b) Determine the expectation value $\langle\psi| X^H(t) |\psi\rangle$ and compare with the result of the previous exercise.
Important information for the future: the index ^{H} to specify the Heisenberg picture is often omitted in advanced quantum mechanics and in the literature. Generally, it is understood that the parentheses "(t)" in an operator indicate the Heisenberg time evolution, unless otherwise specified. So for example X indicates an operator in the Schrödinger picture, while $X(t)$ is its counterpart in the Heisenberg picture.

0.4* Angular momentum matrix elements

Consider angular momentum states $|l, m\rangle$, i.e. eigenstates of the operators $\mathbf{L}^2$ and L_z with the known properties:

$$\mathbf{L}^2\ |l,m\rangle = l(l+1)\ |l,m\rangle \qquad \mathbf{L}_z\ |l,m\rangle = m\ |l,m\rangle$$

where $\mathbf{L} = (L_x, L_y, L_z)$ is the angular momentum operator.
Remember the effect of the ladder operators $L_\pm = L_x \pm iL_y$ on those states

$$L_\pm\ |l,m\rangle = \sqrt{l(l+1) - m(m \pm 1)}\ |l, m \pm 1\rangle$$

Evaluate the following states and expectation values: *

$$L_x\ |2,0\rangle \qquad \langle 2,0|\, L_x \,|2,0\rangle \qquad L_y\ |3,1\rangle$$

$$L_x^2\ |1,0\rangle \qquad \langle 1,0|\, L_x^2 \,|1,0\rangle$$

*By $|1,0\rangle$ we mean a state with $l = 1, m = 0$,etc.