Advanced Quantum Mechanics WS 2024/25, Problem set 1

1.1 Electric field and Gauge invariance

We consider a particle with charge q and mass m in an homogeneous, time-dependent electric field

$$\mathbf{E} = E_0 \mathbf{e}_x e^{-2\alpha t} \tag{1}$$

We can restrict the problem to one dimension, i.e. consider just the x coordinate, and use units in which $\hbar = c = 1$.

(a) Different forms of the vector ($\mathbf{A}$) and the scalar (φ) potentials related by a gauge transformation give the same $\mathbf{E}$. Find $\mathbf{A}(x,t)$ and $\varphi(x,t)$ for the following two cases (gauges. One particular solution each is sufficient):

(1) $\mathbf{A}(x,t) = \mathbf{A}_1(x,t) \neq 0, \quad \varphi(x,t) = 0$

(2) $\mathbf{A}(x,t) = 0, \quad \varphi(x,t) = \varphi_2(x,t) \neq 0.$

Find also the gauge transformation (i.e. one possible choice for the function $\chi(x,t)$ generating it) which transforms the potentials of (1) into the ones of (2). Write down the Hamiltonian H_1 for (1) and H_2 for (2).

Hint: Ansatz: (1)$\mathbf{A}_1 = f_1(t)$, (2) $\varphi_2 = f_2(t)x$, $\quad \chi = f_3(t)\ x$,

1.2 Heisenberg equations of motion

(a) Write down the Heisenberg equations of motion for the operators x and p of the charged particle discussed above for both gauges.

(b) Solve them for $v = \dot{x}$ (velocity) and p explicitly.

By comparing the results for H_1 and H_2, argue that p is not gauge invariant.

(c) Show that - different to p - the result for the velocity v is the same in both gauges, and that it describes a particle with acceleration $\frac{Eq}{m}$.

1.3 Time dependence of the wave function

(a) Write down the **time-dependent** Schrödinger equation (TDSE) for the two Hamiltonians H_1 and H_2 found in 1.1 and 1.2

(b) Starting from an initial state $\psi(x, t=0) = a\ e^{ikx}$ (a arbitrary constant) solve the TDSE for H_1 in the following way:

(i) Look for a solution in the form:

$$\psi(x,t) = e^{ikx} f(t)$$

and write down the corresponding differential equation for $f(t)$.

(ii) Write down and solve the equation for $\log f(t)$. This gives $f(t)$ and $\psi(x,t)$.

(c) Let $\psi_1(x,t)$ be the solution found above. From the lecture notes, we know that the solution $\psi_2(x,t)$ of the TDSE obtained with H_2 is given by a gauge transformation. Use $\chi(x,t)$ obtained above and write down $\psi_2(x,t)$ starting from $\psi_1(x,t)$

1.4 Charged particle in a constant magnetic field

Consider a particle with charge q and mass m in a constant magnetic field $\mathbf{B} = B\mathbf{e}_z$ (see also lecture notes). We take $\hbar = 1$

(a) Find the appropriate vector potential in the form

$$\mathbf{A} = \text{cost. } \mathbf{r} \times \mathbf{B}$$

(b) Write down the full Hamiltonian H by writing down explicitly the x, y, z components of coordinates and momenta.

(c) Now rewrite the Hamiltonian in terms of L_z (the z-component of the angular momentum), of $\mathbf{p}^2$ and of the square of the coordinates x^2 and y^2.

(d) Argue why H, p_z and L_z have common eigenstates

(e) Since this is the case, we can replace p_z and L_z by the corresponding eigenvalues. What are the possible eigenvalues of L_z? Remember that $L_z = -i\frac{\partial}{\partial\varphi}$ in cylindrical coordinates.

1.5 cont: Charged particle in a constant magnetic field

(f) After having fixed these eigenvalues, write down the Schrödinger equation in cylindrical coordinates (r, φ, z). Use the expression of $\boldsymbol{\nabla}^2$ in cylindrical coordinates (look up in the literature).

Notice that the resulting equation must contain the variable r only, since z and φ and their derivative have been eliminated by fixing p_z and L_z.

(g) Find ONE solution of the Schrödinger equation for $p_z = 0$, $L_z = 0$ (**Ansatz** $\psi = e^{-\beta\, r^2}$). What is the corresponding eigenenergy? Compare with the ground state energy of the Landau levels from the lecture notes.