2.1

a) $\langle \psi_2 | \psi_2 \rangle = \langle +x | +x \rangle$ $\langle \varphi_2 | \varphi_2 \rangle = \langle \varphi_2 | \varphi_2 \rangle$

$\langle \varphi_2 | \varphi_2 \rangle = \alpha^2 + \beta^2 \overset{!}{=} 1$

b) $S^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $|+x\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$

$= \langle +x | S^x | +x \rangle = \frac{1}{2}$

$S^z = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \Rightarrow \frac{1}{2}(1, 1)\, S^z \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$

$S^y = \frac{i}{2}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \Rightarrow \frac{1}{2}(1, 1)\, S^y \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$

ALTERNATIVE WAY

$|+x\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right)$

$S^x = \frac{1}{2}\left(|+z\rangle\langle -z| + |-z\rangle\langle +z|\right)$

$S^x |+x\rangle = \frac{1}{2} \frac{1}{\sqrt{2}}\left(|-z\rangle\langle +z| + |+z\rangle\langle -z|\right)$

$= \frac{1}{2} |x\rangle$ (actually we knew this by definition)

$\langle +x | S^x | +x \rangle = \frac{1}{2}$

$S^z = \frac{1}{2}\left(|+z\rangle\langle +z| - |-z\rangle\langle -z|\right)$

$$\langle +x | S^z | +x \rangle = \ldots \text{ I leave it to you } \ldots = 0$$

$$S^y = \frac{i}{2}\left(|-z\rangle\langle +z| - |+z\rangle\langle -z| \right)$$

$$\langle +x | S^y | +x \rangle = \ldots \quad = 0$$

We also need $\quad L^{\pm} = L^x \pm i L^y \Rightarrow L^x = \dfrac{L^+ + L^-}{2}$

c) $\langle \psi_i | H | \psi_i \rangle = \lambda \sum_\alpha \langle x | S^\alpha | x \rangle \langle \varphi_i | L^\alpha | \varphi_i \rangle$

↑ only $\alpha = x$ contributes

$$= \lambda \, \frac{1}{2} \langle \varphi_i | L^x | \varphi_i \rangle$$

$$\langle \psi_1 | H | \psi_1 \rangle = \lambda \, \frac{1}{2} \langle l, m | L^x | l, m \rangle$$
$$= \lambda \, \frac{1}{4} \langle l, m | L^+ + L^- | l, m \rangle = 0$$

since $\quad L^{\pm} | l, m \rangle \propto | l, m \pm 1 \rangle$

$$\langle \Psi_2 | H | \Psi_2 \rangle = \lambda \sum_\alpha \langle x | S^\alpha | x \rangle \langle \varphi_2 | L^\alpha | \varphi_2 \rangle$$

$\propto$ only $\alpha = x$

$$\langle \Psi_2 | H | \Psi_2 \rangle = \lambda \frac{1}{2} \langle \varphi_2 | L^x | \varphi_2 \rangle$$

$$L^x | \varphi_2 \rangle = \frac{1}{2} (L^+ + L^-) \left(\beta | l, l-1 \rangle + \alpha | l, l \rangle \right)$$

$$L^+ | l, l-1 \rangle = \sqrt{l(l+1) - l(l-1)} \, | l, l \rangle = \sqrt{2l} \, | l, l \rangle$$

$$L^- | l, l \rangle = \sqrt{2l} \, | l, l-1 \rangle$$

$$L^+ | l, l \rangle = 0 \qquad L^- | l, l-1 \rangle \propto | l, l-2 \rangle$$

$$\Rightarrow L^x | \varphi_2 \rangle = \frac{1}{2} \sqrt{2l} \left(\beta | l, l \rangle + \alpha | l, l-1 \rangle + \underbrace{c}_{\text{irrelevant}} | l, l-2 \rangle \right)$$

$$\langle \varphi_2 | L^x | \varphi_2 \rangle = \frac{1}{2} \sqrt{2l} \, 2 \alpha \beta = \sqrt{2l} \, \alpha \beta$$

$$\langle \Psi_2 | H | \Psi_2 \rangle = \lambda \sqrt{\frac{l}{2}} \, \alpha \beta$$

THE CONSTANT IS IRRELEVANT

2, 2

$$|\psi_3\rangle \propto S^+ L^- |+x\rangle |2,1\rangle$$
$$= S^+ |+x\rangle \otimes L^- |2,1\rangle$$

$$|+x\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right)$$

$$S^+ |+x\rangle = \frac{1}{\sqrt{2}}\left(\underbrace{S^+|+z\rangle}_{=0} + \underbrace{S^+|-z\rangle}_{|+z\rangle}\right)$$

$$L^- |2,1\rangle \propto |2,0\rangle$$

YOU DON'T NEED THE $\sqrt{\cdots}$ FACTORS

$$\Rightarrow \boxed{|\psi_3\rangle = |+z\rangle |2,0\rangle}$$

NORMALIZED

$$S^z |+x\rangle = \frac{1}{2}\frac{1}{\sqrt{2}}\left(|+z\rangle - |-z\rangle\right)$$

$$L^z |2,1\rangle = |2,1\rangle$$

$$\Rightarrow |\psi_4\rangle = C\left(|+z\rangle - |-z\rangle\right)|2,1\rangle$$

$$C = \frac{1}{\sqrt{2}}$$ SO IS NORMALIZED

IN GENERAL (SEE SCRIPT)

$$P\left(|\psi\rangle, S_z=+\tfrac{1}{2}\right) = \frac{\left|\left(|+z\rangle\langle+z|\otimes I\right)|\psi\rangle\right|^2}{\langle\psi|\psi\rangle}$$

FOR

$$|\psi_1\rangle \Rightarrow \left|\left(|+z\rangle\underbrace{\langle+z|+x\rangle}_{\frac{1}{\sqrt{2}}}\right)|2,1\rangle\right|^2 = \frac{1}{2}$$

$$\left(\text{PROOF}: |+x\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right) \Rightarrow \langle+z|+x\rangle = \frac{1}{\sqrt{2}}\right)$$

FOR

$$|\psi_3\rangle = |+z\rangle|2,0\rangle$$

$$P\left(S_z=+\tfrac{1}{2}\right) = 1$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle - |-z\rangle\right)|2,1\rangle$$

$$P\left(S_z=+\tfrac{1}{2}\right) = \left|\left(|+z\rangle\langle+z|\ \frac{1}{\sqrt{2}}\left(|+z\rangle - |-z\rangle\right)\right)|2,1\rangle\right|^2$$

$$= \frac{1}{2}$$

$$P(L_z = +1)$$

$$|\psi_1\rangle = |+x\rangle\,|2,1\rangle \quad \Rightarrow \quad P = 1$$

$$|\psi_3\rangle = |+z\rangle\,|2,0\rangle$$

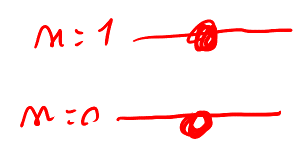
$$\Rightarrow \quad P = 0$$

$$|\psi_4\rangle = |\text{SOME SPIN STATE}\rangle \otimes |2,1\rangle$$

$$\Rightarrow \quad P = 1$$

2.3

$$|G\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right)$$



Eigenenergy: two possible methods to determine it

(a) (SIMPLER)

Since particles are noninteracting, simply add the energy of each article

$$E_G = \varepsilon_{n=0} + \varepsilon_{n=1} = w\cdot 0 + w\cdot 1 = w$$

NON DEGENERATE!
PARTICLES INDISTINGUISHABLE

(b) (more complicated but systematic, not necessary here)

Check $H|G\rangle = E_G|G\rangle$

$$H = H_1 \otimes I + I \otimes H_2$$

H_1, H_2 are harmonic oscillator Hamiltonians

it's better to do one piece at a time (also cf. Chap. 7 of lecture notes)

$$H|0\rangle|1\rangle = (H_1|0\rangle)\otimes|1\rangle + |0\rangle\otimes(H_2|1\rangle)$$

$$= (0 + W)\,|0\rangle\otimes|1\rangle = W\,|0\rangle|1\rangle$$

$$H|1\rangle|0\rangle = W\,|1\rangle|0\rangle$$

notice, normalization is not important in this case

$$\Rightarrow H\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right) = W\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right)$$

$$\Rightarrow H|G\rangle = W|G\rangle \qquad \Rightarrow E_G = W$$

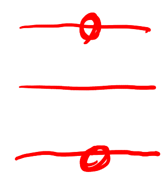
For <u>Eigenenergies</u> it is not convenient to evaluate

$$\langle G|H|G\rangle$$

(b)

For the next two states, I'll use method (a)

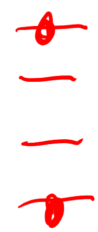
$$|E\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|2\rangle - |2\rangle|0\rangle\right)$$



$$E_E = 2W \qquad \text{NON DEG.}$$

(c)

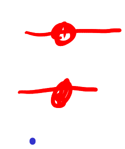
$$|S_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|3\rangle - |3\rangle|0\rangle\right)$$



$$E_{S_1} = 3W$$

DEG = 2

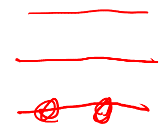
$$|S_2\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle|2\rangle - |2\rangle|1\rangle\right)$$



$$E_{S_2} = 3W$$

(d) BOSONS

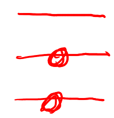
$$|G\rangle = |0\rangle|0\rangle$$



$E_G = 0$

NON DEG.

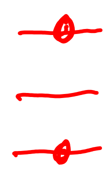
$$|E\rangle = \left(|0\rangle|1\rangle + |1\rangle|0\rangle\right)\frac{1}{\sqrt{2}}$$



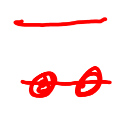
$E_E = \omega$

NON DEG.

$$|S_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |2\rangle\right)$$



$$|S_2\rangle = |1\rangle|1\rangle$$



$E = 2\omega$ DEG $= 2$