Advanced Quantum Mechanics WS 2024/25, Problem set 2 [1]

We use $\hbar = 1$ in these exercises

2.1 Tensor product and angular momentum

Consider the following two states of an electron:

$$|\psi_i\rangle = |{+x}\rangle \otimes |\varphi_i\rangle \qquad \text{where} \quad i = 1, 2$$

Here, $|\varphi_i\rangle$ describes the spatial (wave function) and $|{+x}\rangle$ the spin part.
$|{+x}\rangle$ corresponds to a (normalized) spin state polarized in the $+x$ direction.
The spatial parts of the two states $i = 1, 2$ are expressed in terms of angular momentum states $|l, m\rangle$ (the radial part is omitted) as

$$|\varphi_1\rangle = |l, m\rangle \qquad |\varphi_2\rangle = \alpha\,|l, l\rangle + \beta\,|l, l-1\rangle \ ,$$

where α and β are real constants.
Consider now the spin-orbit coupling Hamiltonian

$$H = \lambda\, \mathbf{S} \cdot \mathbf{L} \ ,$$

where S^n $(n = 1, 2, 3)$ are the components [2] of the spin and L^n of the orbital angular momentum operator. "·" means, as usual, scalar product.
(a) What conditions must α and β fullfill in order for $|\psi_2\rangle$ to be normalized?
(b) Prove that $\langle +x|\, S^n\, |{+x}\rangle = \frac{1}{2}$ for $n = x$ and 0 for $n = y, z$.
(c) Determine the expectation values

$$\langle\psi_1|\, H\, |\psi_1\rangle \quad \text{and} \quad \langle\psi_2|\, H\, |\psi_2\rangle$$

Hints, also for the next part:
(i) use (b)
(ii) Remember that L^x can be written in terms of $L^{\pm}$.
(iii) For (c), evaluate $L^{\pm}\,|\varphi_i\rangle$ and then multiply with $\langle\varphi_i|$ from the left.
(iv) $\qquad L^{\pm}\,|l, m\rangle = \sqrt{l(l+1) - m(m \pm 1)}\,|l, m \pm 1\rangle$

2.2 cont.

Now fix $l = 2, m = 1$.
(a) Evaluate the states

$$|\psi_3\rangle = c_3\; S^+ L^-\,|\psi_1\rangle \qquad |\psi_4\rangle = c_4\; S^z L^z\,|\psi_1\rangle$$

with constants c_3, c_4 such that the states are normalized (you don't need to determine the constants in front of the states you evaluate. It saves work)
Furthermore, for each of the three states $|\psi_1\rangle, |\psi_3\rangle, |\psi_4\rangle$

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[1] Final version.
[2] As mentioned in class, we use superscripts to indicate components of vectors.

(b) Determine the probability that a measure of S^z gives $+\frac{1}{2}$.
(c) Determine the probability that a measure of L^z gives $+1$.
Hint: It is useful to use $\langle +z | +x \rangle = \frac{1}{\sqrt{2}}$. Prove this. [3]

2.3 Two particles without spin in an harmonic oscillator

Consider two indistinguishable fermionic particles in a one-dimensional harmonic oscillator with eigenfrequency ω described by the Hamiltonian (The particles don't have spin and the $\frac{1}{2}$ in H was omitted):

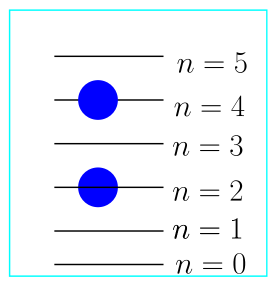
$$H = H_1 + H_2 \qquad H_i = \omega \; a_i^\dagger a_i$$

where $a_i^\dagger$, a_i are the creation and annihilation operators for particle i (i=1,2). Denote by $|n_i\rangle$ the single-particle levels, i.e.

$$H_i \, |n_i\rangle = \omega \; n_i \, |n_i\rangle$$

As we know from the lecture, the eigenstates of this two-particle system are the properly antisymmetrized tensor products of the $|n_i\rangle$.
(a) Write down in this representation the normalized ground state $|G\rangle$ (the state with the lowest energy) of the system. Determine its (eigen-)energy and its degeneracy (i.e. how many independent states there are with the same energy).

Hint: You can imagine the two particles as occupying the levels n of the harmonic oscillator (see figure), whereby Pauli principle has to be obeyed. The total (eigen) energy is simply the sum of the energies of each particle. The two-particle state is then obtained by antisymmetrizing the tensor product state as described in the lecture notes.



A highly excited state

(b) Same for the first excited state $|E\rangle$.
(c) Same for the second excited state(s) $|S_i\rangle$.
(d) repeat (a),(b) and (c) for bosonic particles (here no Pauli principle!).

2.4* Optional

Consider the fermionic system of the previous exercise. Consider the perturbation

$$\hat{W} = \alpha \; \hat{x}_1 \; \hat{x}_2 \; , \tag{1}$$

(a) Evaluate the first-order correction to the energy of the ground state $|G\rangle$.
(b) Evaluate the first-order correction to the energies of the (degenerate!) second excited states $|S_1\rangle$, $|S_2\rangle$.

[3] $|+z\rangle$ is a (normalized) spin state polarized in the $+z$ direction.