Advanced Quantum Mechanics WS 2024/25, Problem set 3 [1]

We use $\hbar = 1$ in these exercises

3.1 Addition of angular momentum

Two interacting particles $i = 1, 2$ with spin $s_1 = 1$ and $s_2 = 2$, respectively are described by the hamiltonian

$$\hat{H} = \lambda\, \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \qquad \text{(scalar product)}\ ,$$

where $\hat{\mathbf{S}}_i$ is the spin operator of particle i, and λ a positive constant. The spatial part of the wavefunction is neglected.
(a) Let $\hat{\mathbf{J}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$ be the total spin What are the possible (eigen)values of $\hat{\mathbf{J}}^2$ and of the associated quantum number j?
(b) Determine the eigenvalues of $\hat{H}$ as well as their degeneracies (consider what was discussed in class: scalar product and $\hat{\mathbf{J}}^2$ have common eigenstates). How many eigenstates are there in total?
(c) [2] The basis states of the two-spins system can be expressed, on the one hand, as tensor products

$$|m_1\rangle\, |m_2\rangle\ . \tag{1}$$

Or, as discussed in the lecture, as $|j, m\rangle$, i.e. eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$.
Express the states $|j = 3, m = 3\rangle$ and $|j = 3, m = -3\rangle$ in terms of the product states (1).
(d) Verify that the total number of basis states in the product state (1) and in the $|j, m\rangle$ representation (cf. (b)) is the same.

3.2 Cont. [2]

Unless explicitly requested, it is not necessary to normalize the states.
(a) Express the state $|j = 3, m = 2\rangle$ in terms of the product states (1). For this use the recipe given in the lecture, i.e. apply $\hat{J}^- = \hat{S}_1^- + \hat{S}_2^-$.
(b) Express the state $|j = 2, m = 2\rangle$ in terms of the product states (1). This is the state orthogonal to $|j = 3, m = 2\rangle$ in the $m = 2$ subspace. Normalize this state.
(c) Evaluate the expectation value of $\hat{S}_{1z}$ in the state $|j = 2, m = 2\rangle$.
(d) What is the probability that a measurement of $\hat{S}_{2z}$ in this state gives $+1$?

3.3 cont.: Clebsch-Gordan coefficients

For this exercise you should use the table of Clebsch-Gordan coefficients.
(a) Express the state $|j = 1, m = 0\rangle$ in terms of the product states (1).
(b) Express the product state $|m_1 = 0\rangle\, |m_2 = 1\rangle$ in terms of total angular momentum states $|j, m\rangle$.

[1] Final version

[2] For this exercise you should **not** use the table of Clebsch-Gordan coefficients

(c) Evaluate the expectation value of $\hat{S}_2^z$ in the state $|j=1, m=0\rangle$. What is the probability that a measure of $\hat{S}_1^z$ in this state gives 0 ?

3.4 Wigner-Eckart's theorem

Consider a hydrogen atom in a state $|n, \ell, m\rangle$, where $n = 7$ and ℓ, m are the usual angular momentum quantum numbers.
For which values of (ℓ, m) are the following matrix elements nonzero?
[Indicate explicitly the pairs (ℓ, m), example: $(2,1),(3,0),(5,-2)$].
(a) $\langle n, \ell, m|\, \hat{z}\, |n, 2, -2\rangle$
(b) $\langle n, \ell, m|\, \hat{y}\, |n, 3, -2\rangle$
(c) $\langle n, \ell, m|\, \hat{y} - i\, \hat{x}\, |n, 4, 0\rangle$
(d) $\langle n, \ell, m|\, (\hat{x}^2 + \hat{y}^2 + \hat{z}^2 + 2\, \hat{\mathbf{r}} \cdot \hat{\mathbf{p}})\, |n, 3, -1\rangle$

3.5 cont. [3]

Given the matrix element

$$\langle n, 1, 1|\, \hat{z}\, |n, 2, 1\rangle = \alpha$$

determine the following matrix elements in terms of α:
(a) $\langle n, 1, m|\, \hat{z}\, |n, 2, 0\rangle$
(b) $\langle n, 1, 1|\, \hat{y}\, |n, 2, m\rangle$
(c) $\langle n, 1, 0|\, \hat{x} - i\, \hat{y}\, |n, 2, m\rangle$

3.6* Optional

Two (distingushable) spin-$\frac{1}{2}$ particles are bound in a $\ell = 1$ orbital state. Their hamiltonian is

$$\hat{H} = 2B\, \hat{\mathbf{L}} \cdot (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2) + 2A\, \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$$

where $\hat{\mathbf{S}}_i$ are the (vector) operators for the spins of the two particles $i = 1, 2$ and $\hat{\mathbf{L}}$ is the orbital angular momentum operator. A and B are constants.
(a) Determine the eigenvalues of $\hat{H}$ and their degeneracies (for generic A, B).
(b) Identify the ground state(s) for $B > A > 0$. Argue why the expectation value of (all components of) $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}_1$ in this ground state is zero.
Hint.: to solve this problem one should first combine the two spins into a state of given total spin quantum number $s = 0$ or $s = 1$. Then combine each of these states with the orbital angular momentum. Generalize the discussion about the scalar product carried out in the lecture notes. For the last question use Wigner-Eckart theorem

[3] We recommend you to use a Clebsch-Gordan table here