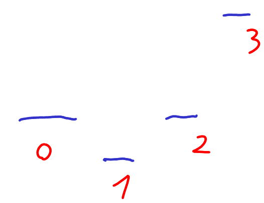
5.1

$$\varepsilon_n = \alpha (k-1)^2 :$$

k	0	1	2	3
$\frac{\varepsilon_n}{\alpha}$	1	0	1	4

a) $$H = \sum_{n=0}^{\infty} \alpha (n-1)^2 \, b_n^+ b_n$$



b) $$|G_N\rangle = \frac{1}{\sqrt{N!}} \left(b_1^+\right)^N |0\rangle$$

$\varepsilon = 0$ deg $= 1$

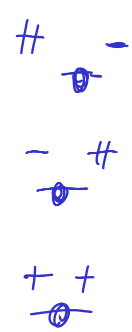
Notice, if it is asked to write a state in second quantisation this is the answer. Not $|0, N, 0, 0, 0\rangle$ this is the same state, yes, but it's in the occupation number representation

c) $$|E_{N,1}\rangle = \frac{1}{\sqrt{(N-1)!}} \, b_0^+ \left(b_1^+\right)^{N-1} |0\rangle$$

$$|E_{N,2}\rangle = \frac{1}{\sqrt{(N-1)!}} \, b_2^+ \left(b_1^+\right)^{N-1} |0\rangle$$

$\varepsilon = \alpha$ deg $: 2$

d) $$|S_N\rangle = \frac{1}{\sqrt{(N-2)!}} \left(b_1^+\right)^{N-2} \times \begin{cases} \frac{1}{\sqrt{2}} \left(b_0^+\right)^2 |0\rangle \\ \frac{1}{\sqrt{2}} \left(b_2^+\right)^2 |0\rangle \\ b_0^+ b_2^+ |0\rangle \end{cases}$$



$\varepsilon = 2\alpha$ deg $= 3$

5.3

(a)

$$\hat{H} = \sum_{n=1}^{\infty} \sum_{\sigma} \alpha (n-1)^2 \, c^{\dagger}_{n\sigma} c_{n\sigma}$$

IMPORTANT TO INDICATE

(b)

$|G_{1\sigma}\rangle = c^{\dagger}_{1\sigma}|0\rangle$ $\quad \sigma = \pm 1$ $\quad E_n = 0$ $\quad deg = 2$

$|G_2\rangle = c^{\dagger}_{1\uparrow} c^{\dagger}_{1\downarrow}|0\rangle$ $\quad E_n = 0$ $\quad deg = 1$

$|G_{3\sigma}\rangle = c^{\dagger}_{1\uparrow} c^{\dagger}_{1\downarrow} c^{\dagger}_{2\sigma}|0\rangle$ $\quad \sigma = \pm 1$ $\quad E_n = \alpha$ $\quad deg = 2$

(c)

$|E_{1\sigma}\rangle = c^{\dagger}_{2\sigma}|0\rangle$ $\quad \sigma = \pm 1$ $\quad E_n = \alpha$ $\quad deg = 2$

$|E_{2\sigma\sigma'}\rangle = c^{\dagger}_{1\sigma} c^{\dagger}_{2\sigma'}|0\rangle$ $\quad \sigma = \pm 1$ $\quad \sigma' = \pm 1$ $\quad E_n = \alpha$ $\quad deg = 4$

$|E_{3\sigma}\rangle = c^{\dagger}_{1\sigma} c^{\dagger}_{2\uparrow} c^{\dagger}_{2\downarrow}|0\rangle$ $\quad \sigma = \pm 1$ $\quad E_n = 2\alpha$ $\quad deg = 2$

d

$|S_{1\sigma}\rangle = c^\dagger_{3\sigma}|0\rangle$ $\quad \sigma = \pm 1 \quad \varepsilon_n = 4\alpha \quad \text{deg} = 2$

$|S_2\rangle = c^\dagger_{2\uparrow}c^\dagger_{2\downarrow}|0\rangle$ $\quad \varepsilon_n = 2\alpha \quad \text{deg} = 1$

$|S_3\rangle = \; = c^\dagger_{1\uparrow}c^\dagger_{1\downarrow}c^\dagger_{3\sigma}|0\rangle$ $\quad \sigma = \pm 1 \quad \varepsilon_n = 4\alpha \quad \text{deg} = 2$

5.2

$$H = 2h_1^\dagger h_1 + 2\left(h_1^\dagger h_2 + h_2^\dagger h_1\right) - h_2^\dagger h_2$$

$$h_1 = \alpha\left(2d_A + \gamma d_B\right) \qquad h_2 = \beta\left(d_A + 2d_B\right)$$

a)

$$[h_1, h_1^\dagger] = \alpha^2\Big(4[d_A, d_A^\dagger] + \gamma^2[d_B, d_B^\dagger] + \cancel{2\gamma^2[d_A, d_B^\dagger]} + \cancel{2\gamma^2[d_B, d_A^\dagger]}\Big)$$

$$= \alpha^2(4+\gamma^2)$$

$$[h_2, h_2^\dagger] = \beta^2\left([d_A, d_A^\dagger] + 4[d_B, d_B^\dagger]\right)$$

$$= 5\beta^2$$

$$[h_1, h_2^\dagger] = \alpha\beta\left(2[d_A, d_A^\dagger] + 2\gamma[d_B, d_B^\dagger]\right)$$

$$= 2\alpha\beta(1+\gamma) \overset{!}{=} 0 \quad \Rightarrow \gamma = -1$$

$$[h_2, h_1^\dagger] = \alpha\beta(2+2\gamma) \quad \Rightarrow \gamma = -1$$

$$\Rightarrow \alpha = \beta = \frac{1}{\sqrt{5}}$$

(b)

$$H = \frac{2}{5}\left(4 d_A^\dagger d_A + d_B^\dagger d_B - 2 d_A^\dagger d_B - 2 d_B^\dagger d_A\right)$$

$$+ \frac{2}{5}\left(2 d_A^\dagger d_A - 2 d_B^\dagger d_B + 4 d_A^\dagger d_B - d_B^\dagger d_A + h.c.\right)$$

$$- \frac{1}{5}\left(d_A^\dagger d_A + 4 d_B^\dagger d_B + 2 d_A^\dagger d_B + 2 d_B^\dagger d_A\right)$$

$$= \frac{1}{5}\Big((8+8-1) d_A^\dagger d_A + (2-8-4) d_B^\dagger d_B + (-4+6-2)\left(d_A^\dagger d_B + d_B^\dagger d_A\right)\Big)$$

$$= 3 d_A^\dagger d_A - 2 d_B^\dagger d_B$$

$$p = 3 \qquad q = 2$$

(c)

$$|G_2\rangle = \frac{1}{\sqrt{2}}\left(d_B^\dagger\right)^2 |0\rangle \qquad E_m = -4 \quad deg = 1$$

$$|E_2\rangle = d_A^\dagger d_B^\dagger |0\rangle \qquad E_n = 1 \quad deg = 1$$

5.4*

(i) $N = (2!\,3!)^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{3}}$

$$\Delta E = 2\omega_2 + 3\omega_4 = c(2k_2 + 3k_4)$$

$$= c\,\frac{\pi}{L}(4+12) = 16\,c\,\frac{\pi}{L}$$

(ii) $|E\rangle = b_1^\dagger|0\rangle = |1, \dots\rangle$

$\Delta E_E = c\,\omega_1 = c\,\dfrac{\pi}{L}$ DEG $= 1$

(iii) $|S_A\rangle = \dfrac{1}{\sqrt{2}}(b_1^\dagger)^2|0\rangle = |2, \dots\rangle$

$|S_B\rangle = b_2^\dagger|0\rangle = |0, 1, \dots\rangle$

$\Delta E_S = c\,2\,\dfrac{\pi}{L}$ DEG $= 2$

third exc states $|T_A\rangle = \dfrac{1}{\sqrt{\ }}(b_1^\dagger)^3|0\rangle = |3, \dots\rangle$

$|T_B\rangle = b_1^\dagger\, b_2^\dagger|0\rangle = |1, 1, \dots\rangle$

$|T_C\rangle = b_3^\dagger|0\rangle = |0, 0, 1, \dots\rangle$

DEG $= 3$

$\Delta E_T = 3c\,\dfrac{\pi}{L}$