Advanced Quantum Mechanics WS 2024/25, Problem set 5

Final version (minor changes in red)

We use $\hbar = 1$ in these exercises

5.1 Bosons in second quantization

Consider bosonic particles whose first quantisation Hamiltonian $\hat{H}_{FQ}$ has eingenstates (levels) $|\varphi_k\rangle$ and eigenenergies $\varepsilon_k = \alpha \cdot (k-1)^2$, with $k = 0, 1, 2, \cdots, \infty$ and α a positive constant.
(a) Let $b_k^\dagger, b_k$ be the creation and destruction operators for a particle in the level $|\varphi_k\rangle$.
(a) Write down the corresponding Hamiltonian $\hat{H}$ in second quantisation
For a given (generic) value of the total number of particles N ($N \geq 2$)
(b) write down the normalized ground state $|G_N\rangle$ of $\hat{H}$ in second quantisation and determine its eigenenergy and degeneracy.
(c) same as (b) for the first excited state(s) $|E_N\rangle$.
(d) Same as (b) for the second excited state(s) $|S_N\rangle$.

Hint. (valid also for following exercises): To help visualize the states and to determine their eigenenergies you can use the same level diagrams that you have used for previous exercises (for bosons just pick one value of N in order to understand how it works). Also in this case the particles are noninteracting, therefore the total (eigen)energy of a many-particle state is simply the sum of the energies of each particle.

5.2 Two-level boson model

We consider bosonic particles on a two-level system described by creation and destruction operators $b_k^\dagger$ and b_k associated to the two levels $k = 1, 2$. The second quantisation Hamiltonian for the system reads (in appropriate units)

$$\hat{H}_0 = 2\, b_1^\dagger b_1 + 2 \left(b_1^\dagger b_2 + b_2^\dagger b_1 \right) - b_2^\dagger b_2 \tag{1}$$

Solve the hamiltonian in the following way:
Introduce two new creation ($d_A^\dagger$, $d_B^\dagger$) and destruction (d_A, d_B) operators

related to the b_k by the transformations

$$b_1 = \alpha\,(2d_A + \gamma\, d_B) \qquad b_2 = \beta\,(d_A + 2d_B) \qquad \alpha, \beta, \gamma = \text{real constants} \quad (2)$$

and their hermitian conjugates.
(a) Given that the d_x ($x = A, B$) obey the correct commutation relations

$$[d_x, d_y^\dagger] = \delta_{x,y} \quad (3)$$

determine for which values of the constants α, β,γ the b_k, $b_k^\dagger$ obey correct commutation relations as well (one solution is sufficient).
(b) Show that in terms of the d_x the Hamiltonian becomes

$$\hat{H}_0 = p\ d_A^\dagger d_A - q\ d_B^\dagger d_B \qquad p, q > 0 \quad (4)$$

and determine the values of p and q.
(c) Write down the normalized ground state $|G_2\rangle$ with $N = 2$ particles, as well as the corresponding first excited state $|E_2\rangle$ in second quantisation (use the appropriate creation operators!) and determine their eigenenergies and degeneracies.
(If you did not solve (b) you can do (c) by assuming unknown values of $p, q > 0$)

5.3 Electrons in second quantisation

Consider the system of Ex. 5.1 for the case of electrons with spin and for $k = 1, 2, \cdots, \infty$.
Let $c_{k,\sigma}^\dagger, c_{k,\sigma}$ be the annihilation and creation operators associated with the orbital level $|\varphi_k\rangle$ with spin index $\sigma = \pm 1$ (or $\sigma = \uparrow, \downarrow$).
(a) write down the Hamiltonian $\hat{H}$ in second quantisation
(b) write down the normalized ground state(s) $|G_N\rangle$ of $\hat{H}$ with $N = 1, 2, 3$ particles in second quantisation and determine their eigenenergies and degeneracies.
(c) same as (b) for the first excited state(s) $|E_N\rangle$.
(d) same as (b) for the second excited states $|S_N\rangle$.

5.4* Quantum vibrating string

Consider the quantum vibrating string discussed in class. Its Hamiltonian is

$$\hat{H} = \sum_{n=1}^{\infty} \omega_n \left(b_n^\dagger b_n + \frac{1}{2} \right) \qquad \omega_n = c k_n \qquad k_n = \frac{\pi n}{L} \,.$$

See lecture notes for the various parameters. Its ground state $|0\rangle$ is given by all oscillators (modes) being in their ground level:*

$$|0\rangle \equiv |N_1 = 0, N_2 = 0, N_3 = 0, \cdots\rangle$$

The energy of the ground state is infinite $E_0 = \sum_{n=1}^{\infty} \frac{\omega_n}{2}$. However, excitation energies, i.e. differences between energies of excited states and E_0 are typically finite. Therefore, below we assume energies in the form $E_i = E_0 + \Delta E_i$ and we are only interested in ΔE_i.
Excited states of the vibrating string can be written in two different ways, on the other hand in the form $|N_1, N_2, N_3, \cdots\rangle$ or by starting from the ground state and applying the ladder operators. For example:

$$|0, 2, 0, 3, \cdots\rangle = \mathcal{N} \left(b_2^\dagger \right)^2 \left(b_4^\dagger \right)^3 |0\rangle \tag{5}$$

is a state with the $n = 2$ mode in its second excited level and the $n = 4$ mode in its third excited level.
(i) Determine the normalisation constant $\mathcal{N}$ and the eigenenergy of the state (5)
(ii) Write down the (normalized) first excited state(s) of $\hat{H}$ in both forms (similar to the left and right hand side of (5)) and determine its eigenenergy and degeneracy.
(iii) Same as (ii) for the second and third excited state(s).

*in our convention, $\cdots$ means that all the remaining modes have their $N_i = 0$.