Advanced Quantum Mechanics WS 2024/25, Problem set 6

Final version [*]

Sind Sie auf der Suche nach einem Thema für Ihre Bachelor- oder Masterarbeit? Dann kommen Sie zur Vorstellung der Bachelor und Masterthemen am 15.1.2023 ab 14:30 Uhr im Meerscheinschlössl/Mozartgasse Graz

We use $\hbar = 1$ in these exercises

6.1 Second quantization: perturbation

Consider the hamiltonian of Ex. 5.1 for the bosonic case (for simplicity $k = 1, \cdots, \infty$, i.e. remove $k = 0$).
We add now a perturbation $\hat{V}$ which in first quantisation has the form

$$\hat{V} = h \sum_i \hat{q}_i$$

where $\hat{q}$ is an operator with matrix elements [†]

$$\langle\varphi_k|\,\hat{q}\,|\varphi_{k+1}\rangle = \langle\varphi_{k+1}|\,\hat{q}\,|\varphi_k\rangle = 1, \qquad (k = 1, \cdots, \infty)$$

and all other matrix elements of $\hat{q}$ are equal to zero.
(a) Express $\hat{V}$ in second quantisation (see lecture notes, operators in second quantisation), show that $\hat{V}$ is hermitian.
(b) Apply $\hat{V}$ to the $N = 2$ (bosonic) ground state $|G_2\rangle$, i.e. evaluate

$$\hat{V}\,|G_2\rangle\,.$$

verify that this is proportional to the first excited state $|E_2\rangle$.
(**Hint:** Before starting the calculation, consider that most of the terms in the sum in $\hat{V}$ vanish).
(c) Determine the second-order correction to the energy of $|G_2\rangle$.

6.2 Interaction in second quantization

Consider the (fermionic) hamiltonian of Ex. 5.3. We add now an interaction $\hat{V}$, which in second quantisation reads

$$\hat{V} = \frac{U}{2} \sum_{k_1,k_2=1}^{\infty} \sum_{p_1=\pm1} \sum_{p_2=\pm1} \sum_{\sigma_A,\sigma_B} c^\dagger_{k_1+p_1,\sigma_A} c^\dagger_{k_2+p_2,\sigma_B} c_{k_2,\sigma_B} c_{k_1,\sigma_A} \tag{1}$$

(a) Evaluate $\hat{V}\,|G_2\rangle$ and verify that the result is proportional to the second excited state $|S_2\rangle$

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[*] Since these exercises are extensions of the exercises of sheet 5, you'll find the corresponding solutions on the usual webpage

[†] The index i specifies to which particle $\hat{q}$ applies and is irrelevant in second quantisation, see script

(b) Evaluate the second order correction to the energy of the $N = 2$ ground state $|G_2\rangle$.
Hint: To evaluate $\hat{V}\,|G_2\rangle$ first apply the $c_{k_2,\sigma_B} c_{k_1,\sigma_A}$ term to $|G_2\rangle$ and realize that only few terms are left in the sum.

6.3 cont.: Second quantisation expression for the interaction

In first quantisation the (spin independent) interaction reads

$$\hat{V}_{FQ} = \tfrac{1}{2}\,\beta \textstyle\sum_{i\neq j}\ \hat{q}_i\ \hat{q}_j\ .$$

where $\hat{q}_i$ are the operators of Ex. 6.1 and $\beta > 0$.
Write down the corresponding second quantisation expression for both the bosonic and the fermionic case (notice that, except for spin, the expressions are basically the same. See script: interaction in second quatisation).
Find the relation between β and U from the previous exercise.

6.4 Two-level boson model, cont.

Consider the two-level boson model from the previous exercise sheet. Remember the solution

$$\alpha = \beta = 1/\sqrt{5} \qquad \gamma = -1 \qquad p = 3, \quad q = 2$$

(a) In second quantisation write down the normalized ground state $|G_N\rangle$ and first excited state $|E_N\rangle$ of $\hat{H}_0$ with N particles as well as their energies.
From here, if you wish, you can set $N = 2$ for simplicity
(b) Given a perturbation (in second quantisation)

$$\hat{V} = \lambda\ b_1^\dagger b_1 \qquad \lambda = \text{constant} \tag{2}$$

determine the first and second-order correction to the energy of $|G_N\rangle$.
Hint: First evaluate $\hat{V}\,|G_N\rangle$ and realize that it is a linear combination of $|G_N\rangle$ and $|E_N\rangle$. To evaluate the action of the annihliation operators it may be useful to apply known relations from the harmonic oscillator.

6.5* Exact many-body state

Assume that the particles in Ex. 6.2 are restricted to the $k = 1, 2$ levels ‡
(a) Determine the exact ground state energy of the Hamiltonian $\hat{H}_I \equiv \hat{H} + \hat{V}$ ($\hat{H}$ is the unperturbed Hamiltonian from 5.3) in this restricted space and for $N = 2$ particles. Assume β small but not too small. Use the resulting expression to verify the result from second-order perturbation theory.
(b) Determine the (unnormalized) exact ground state, as well as the one from first order perturbation theory and compare the two results.

‡For example, because there is an additional large energy added to the other levels.