Advanced Quantum Mechanics WS 2024/25, Problem set 7 §

7.1 Scattering within the first order Born approximation

Particles of mass m and momentum $\mathbf{k}$ are scattered by a potential

$$V(r) = \beta \; e^{-r^2/a^2} \qquad r \equiv |\mathbf{x}| \; .$$

Within the first-order Born approximation:
(a) Determine the scattering amplitude $f(\mathbf{k}, \mathbf{k}_{out})$. You don't need to carry out the final integral over r. Carry out the transformation $r = a \; x$ in the integral.
We consider now the case of small scattering angles θ.
(b) Argue why in this case one can replace $\sin y \to y$ in the integral (y is the argument of the sin).
Be more specific about the condition: θ must be much smaller than which quantity (it must be obviously dimensionless).
The same approximation can be alternatively made if I have small energies. What does it mean "small" in this case. Do I have restrictions on θ for the approximation to be valid in this case?
(c) Determine the differential cross section $d\sigma/d\Omega$ for small θ.
(d) Be ρ the particle density. Determine how many particle per unit time are scattered through an angle smaller than θ.
Hint: $\int_0^\infty x^n e^{-x^2} \; dx \equiv I_n$ is given.

7.2 One-dimensional delta potential †

(a) Consider a particle of mass m scattered by a one-dimensional potential

$$V(x) = u \; \delta(x)$$

Write down the Lippmann-Schwinger equation for the wave function.
Write down the equation for the exact wave function in the point $x = 0$, and solve the Lippmann-Schwinger equation exactly.
Determine the reflection amplitude r, i.e. the coefficient of the backscattered wave (i..e. of e^{-ikx} for $x < 0$, the incoming wave being normalized as e^{ikx}).
From the above results determine the range of validity of the first-order Born approximation. Is it valid for arbitrarily low energies?

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§ final version

† Ex. 7.2 and 7.3 are "star" exercises in the sense that they bring extra marks for the exercise class. However, exercise 7.2 (complete) and 7.3 (just (a)) may appear as exam topics **starting from the second exam.**

7.3 Quantized electric and magnetic field †

Consider a quantum mechanical state (call it $|n\rangle$) with n photons with wave vector $\mathbf{q} = q\mathbf{e}_z$ in a volume Ω and polarised in the $\mathbf{e}_x$ direction, i.e. we take $\mathbf{u}_{\mathbf{q},1} = \mathbf{e}_x$ (notation as in lecture notes).

(a) Write down this state with the proper normalisation in second quantisation, i.e. with the help of the photon vacuum state $|0\rangle$, and the creation operators for these photons $a^\dagger_{\mathbf{q},1}$.

(b) Show that the expectation value $\langle n| \hat{\mathbf{E}}(\mathbf{r}) |n\rangle$ of the electric field operator in this state is zero.

(c) Consider a state $|\psi\rangle$, which is a linear combination of two such states with different n:

$$|\psi\rangle \equiv \frac{1}{\sqrt{2}} \left(|n\rangle + i\ \alpha\, |n+1\rangle\right) \qquad |\alpha| = 1\ .$$

Determine the expectation value $\langle\psi| \hat{\mathbf{E}}(\mathbf{r}=0) |\psi\rangle$ of the electric field (evaluated in $\mathbf{r} = 0$) in this state.

(d) Determine the expectation value of the ($\mathbf{r} = 0$) magnetic field $\langle\psi| \hat{\mathbf{B}}(\mathbf{r}=0) |\psi\rangle$.