

Advanced Quantum Mechanics WS 2026/27, Problem set 2

Preliminary version ¹

We use $\hbar = 1$ in these exercises

2.1 Tensor product and angular momentum

Consider the following two states of an electron:

$$|\psi_i\rangle = |+\rangle \otimes |\varphi_i\rangle \quad \text{where } i = 1, 2$$

Here, $|\varphi_i\rangle$ describes the spatial part and $|+\rangle$ the spin part.

$|+\rangle$ corresponds to a (normalized) spin state polarized in the $+x$ direction.

The spatial parts of the two states $i = 1, 2$ are expressed in terms of angular momentum states $|l, m\rangle$ (the radial part is omitted) as

$$|\varphi_1\rangle = |l, m\rangle \quad |\varphi_2\rangle = \alpha |l, l\rangle + \beta |l, l-1\rangle,$$

where α and β are real constants, $l \geq 1$ is an integer, and $m = -l, \dots, l$.

Consider now the spin-orbit coupling Hamiltonian

$$H = \lambda \mathbf{S} \cdot \mathbf{L},$$

where λ is real, S^n and L^n ($n = x, y, z$) are the components ² of the spin and orbital angular momentum operators, respectively. “ $\cdot$ ” means, as usual, scalar product.

- (a) What conditions must α and β fulfil in order for $|\psi_2\rangle$ to be normalized?
- (b) Prove that $\langle +x | S^n | +x \rangle = \frac{1}{2}$ for $n = x$ and 0 for $n = y, z$.
- (c) For normalized states, determine the expectation values

$$\langle \psi_1 | H | \psi_1 \rangle \quad \text{and} \quad \langle \psi_2 | H | \psi_2 \rangle$$

Hints, also for the next part:

- (i) use (b)
- (ii) Remember that L^x can be written in terms of $L^\pm$.
- (iii) For (c), evaluate $L^\pm |\varphi_i\rangle$ and then multiply with $\langle \varphi_i |$ from the left.
- (iv) $L^\pm |l, m\rangle = \sqrt{l(l+1) - m(m \pm 1)} |l, m \pm 1\rangle$

2.2 Continuation of 2.1

Now fix $l = 2, m = 1$.

- (a) Evaluate the states

$$|\psi_3\rangle = c_3 S^+ L^- |\psi_1\rangle \quad |\psi_4\rangle = c_4 S^z L^z |\psi_1\rangle$$

Write the resulting states in normalized form. You need not determine c_3 and c_4 separately.

Furthermore, for each of the three states $|\psi_1\rangle, |\psi_3\rangle, |\psi_4\rangle$

- (b) Determine the probability that a measurement of S^z gives $+\frac{1}{2}$.
- (c) Determine the probability that a measurement of L^z gives $+1$.

Hint: Prove and use $|\langle +z | +x \rangle|^2 = \frac{1}{2}$. ³

¹See the information about the preliminary version in the material web site.

²As mentioned in class, we use superscripts to indicate components of vectors.

³ $|+z\rangle$ is a (normalized) spin state polarized in the $+z$ direction.

2.3 Two particles in a harmonic oscillator

Consider two identical, noninteracting fermionic particles, each of mass $M = 1$, in a one-dimensional harmonic oscillator of frequency $\omega > 0$. Spin degrees of freedom are neglected. (We omit the constant zero-point energy $\omega/2$):

$$H = H_1 + H_2, \quad H_i = \omega a_i^\dagger a_i, \quad i = 1, 2.$$

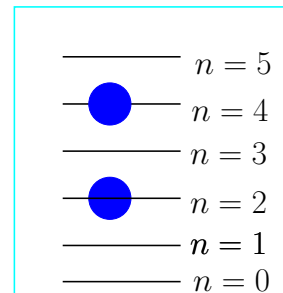
Here $a_i^\dagger$ and a_i raise and lower the oscillator quantum number of particle i . The normalized single-particle states satisfy

$$H_i |n_i\rangle = \omega n_i |n_i\rangle, \quad n_i = 0, 1, 2, \dots$$

An eigenbasis of the two-fermion system is formed by the properly antisymmetrized tensor products of these states.

(a) Write down the normalized ground state $|G\rangle$ in this representation. Determine its energy and degeneracy (the number of linearly independent states with the same energy).

Hint: Think of the particles as occupying oscillator levels. The Pauli principle permits at most one fermion per level. The total energy is the sum of the two single-particle energies. Obtain the corresponding two-particle state by antisymmetrizing the tensor product, as described in the lecture notes.



A highly excited state

- (b) Repeat (a) for the first excited level, with state $|E\rangle$.
- (c) Repeat (a) for the second excited level, giving an orthonormal basis of states $|S_i\rangle$.
- (d) Repeat (a)–(c) for identical bosons with no spin degrees of freedom.

2.4* Optional: perturbation theory

For the fermionic system in Exercise 2.3, consider the perturbation

$$\hat{W} = \alpha \hat{x}_1 \hat{x}_2 \tag{1}$$

where α is real and $|\alpha| \ll M\omega^2$. Here α is the perturbation strength, unrelated to the coefficient in Exercise 2.1.

- (a) Evaluate the first-order correction to the ground-state energy.
- (b) Evaluate the first-order correction to the energies of the (degenerate!) second excited states $|S_1\rangle, |S_2\rangle$.