Advanced Quantum Mechanics WS 2026/27, Problem set 3 [1]

We use $\hbar = 1$ in these exercises

3.1 Addition of angular momentum

Two interacting particles $i = 1, 2$ with spin $s_1 = 1$ and $s_2 = 2$, respectively are described by the hamiltonian

$$\hat{H} = \lambda \, \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \qquad \text{(scalar product)} \, ,$$

where $\hat{\mathbf{S}}_i$ is the spin operator of particle i, and λ a positive constant. The spatial part of the wavefunction is neglected.

(a) Let $\hat{\mathbf{J}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$ be the total spin What are the possible (eigen)values of $\hat{\mathbf{J}}^2$ and of the associated quantum number j?

(b) Determine the eigenvalues of $\hat{H}$ as well as their degeneracies (consider what was discussed in class: scalar product and $\hat{\mathbf{J}}^2$ have common eigenstates). How many eigenstates are there in total?

(c) [2] The basis states of the two-spins system can be expressed, on the one hand, as tensor products

$$|m_1\rangle \, |m_2\rangle \ . \tag{1}$$

Or, as discussed in the lecture, as $|j, m\rangle$, i.e. eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$. Express the states $|j = 3, m = 3\rangle$ and $|j = 3, m = -3\rangle$ in terms of the product states (1).

(d) Verify that the total number of basis states in the product state (1) and in the $|j, m\rangle$ representation (cf. (b)) is the same.

3.2 Cont. [2]

Unless explicitly requested, it is not necessary to normalize the states.

(a) Express the state $|j = 3, m = 2\rangle$ in terms of the product states (1). For this use the recipe given in the lecture, i.e. apply $\hat{J}^- = \hat{S}_1^- + \hat{S}_2^-$.

(b) Express the state $|j = 2, m = 2\rangle$ in terms of the product states (1). This is the state orthogonal to $|j = 3, m = 2\rangle$ in the $m = 2$ subspace. Normalize this state.

(c) Evaluate the expectation value of $\hat{S}_1^z$ in the state $|j = 2, m = 2\rangle$.

(d) What is the probability that a measurement of $\hat{S}_2^z$ in this state gives $+1$?

3.3 Cont. [2]

(a) At time $t = 0$ the two-particle system is prepared in the state $|+1\rangle \, |+1\rangle$ and evolves according to $\hat{H}$. After how long time does the system return to the same initial state state (apart for a constant)?

(b) Express the state $|j = 3, m = 1\rangle$ in terms of the states (1) (Hint: apply again $\hat{J}^-$, you don't need to normalize this state).

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[1] Final version

[2] For this exercise you should **not** use the table of Clebsch-Gordan coefficients

3.4 Clebsch-Gordan coefficients

For this exercise you should use the table of Clebsch-Gordan coefficients.
(a) Express the state $|j = 1, m = 0\rangle$ in terms of the product states (1).
(b) Express the product state $|m_1 = 0\rangle\,|m_2 = 1\rangle$ in terms of total angular momentum states $|j, m\rangle$.
(c) Evaluate the expectation value of $\hat{S}_2^z$ in the state $|j = 1, m = 0\rangle$. What is the probability that a measure of $\hat{S}_1^z$ in this state gives 0 ?

3.5* Optional

Two (distingushable) spin-$\frac{1}{2}$ particles are bound in a $\ell = 1$ orbital state. Their hamiltonian is

$$\hat{H} = 2B\ \hat{\mathbf{L}} \cdot (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2) + 2A\ \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$$

where $\hat{\mathbf{S}}_i$ are the (vector) operators for the spins of the two particles $i = 1, 2$ and $\hat{\mathbf{L}}$ is the orbital angular momentum operator. A and B are constants.
(a) Determine the eigenvalues of $\hat{H}$ and their degeneracies (for generic A, B).
(b) Identify the ground state(s) for $B > A > 0$. Argue why the expectation value of (all components of) $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}_1$ in this ground state is zero.
Hint.: to solve this problem one should first combine the two spins into a state of given total spin quantum number $s = 0$ or $s = 1$. Then combine each of these states with the orbital angular momentum. Generalize the discussion about the scalar product carried out in the lecture notes.
For the last expectation value you don't need to do any calculation: just think what the properties of rotations are!