Problem 2.3 — Übungsblatt 2

Two bodies having masses m_1 and m_2 are moving under the infeluence of mutual gravitational attraction and in an additional constant gravitational field in the $-z$-direction.

- Determine the equation of motion in the center of gravity and the relative coordinates.
- In addition, calculate the total energy as function of the coordinates of the centre of gravity and the relative coordinate, and verify that this quantity can be divided into a centre of gravity part and a relative part.

Force from 2 -> 1,
(sorry, notation opposite with respect to lecture notes)

Solution

Equations of motion (EOMs) for the two masses

$$(\mathrm{I}) \qquad m_1 \vec{r}_1'' = -m_1 \vec{g} + \vec{F}_{12}$$

$$(\mathrm{II}) \qquad m_2 \vec{r}_2'' = -m_2 \vec{g} + \vec{F}_{21}$$

$$\vec{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

Newton's 3rd law : $\vec{F}_{21} = -\vec{F}_{12}$

•) Center of gravity coordinate $\vec{R}$

$$M = m_1 + m_2 \qquad \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}$$

We take eq.s (I) and (II) and add them, i.e.

$$m_1 \vec{r}_1'' + m_2 \vec{r}_2'' = M \vec{R}'' = -M\vec{g}$$

So that the effect of $\vec{F}_{21}$ balances $\vec{F}_{12}$

$$M\vec{R}'' = -M\vec{g} \Rightarrow \vec{R}'' = -\vec{g}$$

Such an EOM describes the accelerated (uniformly) motion of a mass M.

••) relative coordinate

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

We subtract $\frac{1}{m_2}(\mathrm{II})$ from $\frac{1}{m_1}(\mathrm{I})$ and we get

$$-\frac{1}{m_2} m_2 \vec{r}_2{}'' + \frac{1}{m_1} \vec{r}_1{}'' = -\vec{g} + \vec{g} + \frac{\vec{F}_{12}}{m_1} - \frac{\vec{F}_{21}}{m_2}$$

$$\vec{r}\,'' = \vec{F}_{12}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \vec{F}_{12}\left(\frac{m_1+m_2}{m_1 m_2}\right) = \frac{M \vec{F}_{12}}{m_1 m_2}$$

$$\frac{m_1 m_2}{M}\vec{r}\,'' = \vec{F}_{12} \Rightarrow \mu \vec{r}\,'' = \vec{F}_{12}$$

$\uparrow$ reduced mass, $\mu := \frac{m_1 m_2}{M}$

Also $\vec{F}_{12} = -\gamma \frac{m_1 m_2}{r^3}\,\vec{r}$ is the gravitational force between the two masses.

•) Now the total energy

From $\vec{R} = \frac{1}{M}(m_1 \vec{r}_1 + m_2 \vec{r}_2)$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$

we can write $\begin{cases} \vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r} \end{cases}$

$$E_{tot} = \frac{1}{2} m_1 (\vec{r}_1{}')^2 + \frac{1}{2} m_2 (\vec{r}_2{}')^2 + \underbrace{(m_1 z_1 + m_2 z_2) g}_{\substack{\text{gravitational} \\ \text{potential} \\ (\text{heights} = z_i\text{'s})}} - \underbrace{\frac{m_1 m_2 \gamma}{r}}_{\substack{\text{potential} \\ 1 \leftrightarrow 2}}$$

$$(\vec{r}_1{}')^2 = \left(\vec{R}' + \frac{m_2}{M}\vec{r}\,'\right)^2 = (\vec{R}')^2 + \left(\frac{m_2}{M}\right)^2 (\vec{r}\,')^2 + \frac{2 m_2}{M}\vec{R}'\vec{r}\,'$$

$$(\vec{r}_2{}')^2 = \left(\vec{R}' - \frac{m_1}{M}\vec{r}\,'\right)^2 = (\vec{R}')^2 + \left(\frac{m_1}{M}\right)^2 (\vec{r}\,')^2 - \frac{2 m_1}{M}\vec{R}'\vec{r}\,'$$

In principle also in skript

$$\frac{1}{2}m_1(\vec{r}_1')^2 + \frac{1}{2}m_2(\vec{r}_2')^2 =$$

$$= \frac{1}{2}(m_1+m_2)(\vec{R}')^2 + \frac{1}{2M^2}(\vec{r}')^2(m_1 m_2^2 + m_1^2 m_2) + \left(\begin{matrix}\text{mixed terms}\\ \text{cancel out}\end{matrix}\right)$$

$$= \frac{1}{2}M(\vec{R}')^2 + \frac{1}{2M^2}m_1 m_2 \cancel{(m_1+m_2)}(\vec{r}')^2 =$$

$\uparrow$ $M \equiv m_1 + m_2$

$\mu \equiv m_1 m_2 / M$

$$= \frac{1}{2}M(\vec{R}')^2 + \frac{1}{2}\mu(\vec{r}')^2$$

So the total energy reads

$$E_{tot} = \frac{1}{2}M(\vec{R}')^2 + \frac{1}{2}\mu(\vec{r}')^2 + (m_1 z_1 + m_2 z_2)g - \gamma\frac{m_1 m_2}{r}$$

given the definition

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$$

we see that $(\vec{R})_z = R_z = \dfrac{m_1 z_1 + m_2 z_2}{M}$

$$E_{tot} = \frac{1}{2}M(\vec{R}')^2 + \frac{1}{2}\mu(\vec{r}')^2 + MR_z g - \gamma\frac{m_1 m_2}{r}$$

$$= \underbrace{\frac{1}{2}M(\vec{R}')^2 + MR_z g}_{\text{Kin.+ Pot. energies of the center of mass}} + \underbrace{\frac{1}{2}\mu(\vec{r}')^2 - \gamma\frac{m_1 m_2}{r}}_{\text{Kin. + Pot. energies of the relative coordinate.}}$$