Problem 3.1 - Übungsblatt 3

Two particles with masses $m_1 = m$ and $m_2 = 3m$ are moving in 3D space and interact through the potential

$$U(r) = \begin{cases} -V, & r \leq R \\ 0, & \text{otherwise} \end{cases} \qquad V, R > 0$$

The energy E and the angular momentum ℓ are given in the reference system of the relative coordinate, r.

1) For which values of E are there bounded orbits?

2) Are there values of E for which both bounded and unboundd orbits are possible?

3) Compute the minimum (ρ_1) and maximum (ρ_2) distances between the particles and only ρ_1 are open.

4) Determine the angle $\Delta\varphi$ by which the orbit rotates between two points in time at which the particles are at the maximum distance from each other. In which case does the orbit close: always, never, sometimes?

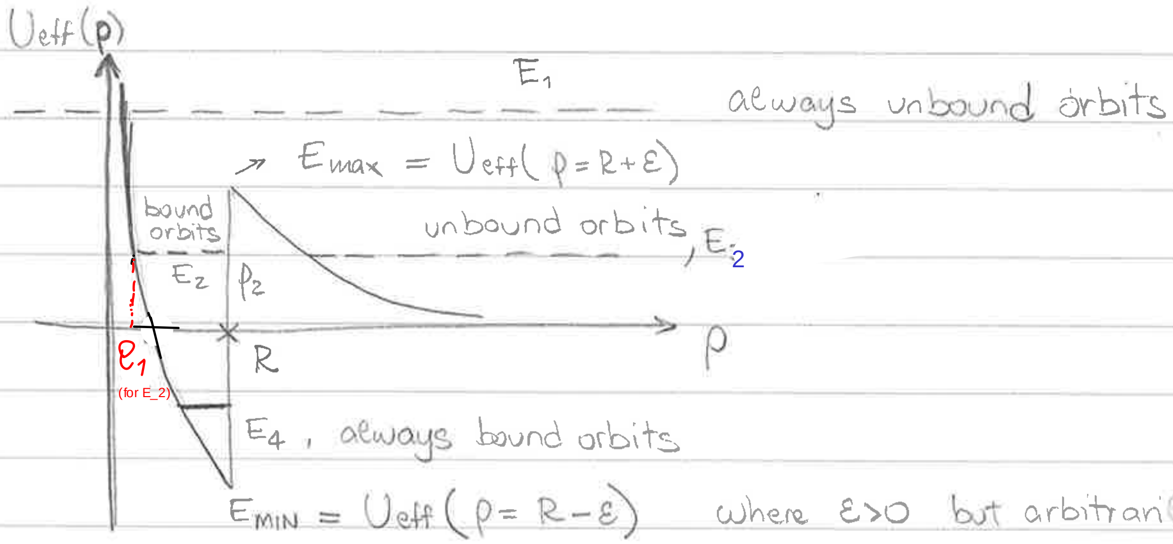
5) Assuming that a particle is at rest at the beginning and that the other is at a distance $R/2$ from the first and is moving with velocity v perpendicular to the

Connecting axis, at which values of ν are there bound orbits? Values of E and ρ are to be determined now.

Solution

Reference frame of the relative coordinate ρ

$\Rightarrow$ center of mass moves with constant velocity $\vec{v}$



where $\varepsilon > 0$ but arbitrarily small

We work in the relative coordinates, there we have to use the reduced mass μ

We compute the reduced mass μ, i.e.

$$\mu \overset{def}{=} \frac{m_1 m_2}{m_1 + m_2} \underset{\substack{m_1 = m \\ m_2 = 3m}}{=} \frac{3m^2}{4m} = \frac{3}{4} m$$

$$U(\rho) = \begin{cases} -V & \rho \leq R \\ 0 & \text{otherwise} \end{cases} = -V\,\Theta(R - \rho)$$

$$U_{eff}(\rho) = \frac{\ell^2}{2\mu\rho^2} - V \quad (\text{if } \rho \leq R) \qquad U_{eff}(\rho) = \frac{\ell^2}{2\mu\rho^2}, \; \rho > R$$

With this we can now compute the inversion point, i.e.

$$E - U_{eff}(\rho) = 0$$

For $\rho \leq R$ (*)

$$E + V - \frac{\ell^2}{2\mu\rho^2} = 0$$

$$\frac{2\mu(E+V)}{\ell^2} = \frac{1}{\rho^2} \Rightarrow \rho_1 = \frac{\ell}{[2\mu(E+V)]^{1/2}}$$

So we the condition (*) can be further restricted to being $\rho_1 = \frac{\ell}{\sqrt{2\mu(E+V)}} < R$

$$\Rightarrow E > \underbrace{\frac{\ell^2}{2\mu R^2} - V}_{E_{min} = U_{eff}(\rho = R - \varepsilon)}$$

$\forall\, \varepsilon > 0$ (arbitrarily small)

but still it has to be $E < E_{max} = \frac{\ell^2}{2\mu R^2} = U_{eff}(R+\varepsilon)$

So for bound orbits it must be

$$\boxed{\frac{\ell^2}{2\mu R^2} - V < E < \frac{\ell^2}{2\mu R^2}}$$

$E_{min} < E < E_{max}$

FOR $0 < E < E_{MAX}$ THERE CAN BE BOTH BOUND AS WELL AS UNBOUND ORBITS DEPENDING ON INITIAL CONDITIONS

FOR $E > E_{MAX}$ ONLY UNBOUND ORBITS

Computing the angle $\Delta\varphi$:

$$\Delta\varphi = 2\int_{\rho_1}^{R} \frac{d\rho}{\sqrt{\tilde{F}(\rho)}}$$

Where we use the formula:

$$\tilde{F}(\rho) = \frac{\mu^2\rho^4}{\ell^2}\left[\frac{2}{\mu}\left(E+V-\frac{\ell^2}{2\mu\rho^2}\right)\right] =$$

$$= \frac{2\mu\rho^4}{\ell^2}(E+V) - \rho^2$$

For $\rho < R$ we have

$$\tilde{F}(\rho) = C^2\rho^4 - \rho^2 \quad \text{with} \quad C^2 := \frac{2\mu(E+V)}{\ell^2} = \frac{1}{\rho_1^2}$$

from our previous calculations

$$\Delta\varphi = 2\int_{\rho_1}^{R} d\rho \frac{1}{\sqrt{\frac{\rho^4}{\rho_1^2} - \rho^2}} =$$

$$(\rho = \rho_1 x, \ d\rho = \rho_1 dx)$$

$$= 2\int_{\rho_1}^{R} \frac{d\rho}{\rho} \frac{1}{\sqrt{\left(\rho/\rho_1\right)^2 - 1}} = 2\int_{1}^{R/\rho_1} \frac{dx}{x\sqrt{x^2-1}}$$

$$= 2\cos^{-1}\left(\rho_1/R\right)$$

$$\Delta\varphi = 2\cos^{-1}\left(\rho_1/R\right)$$

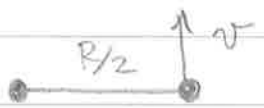
In order for the orbit to close $\Delta\varphi$ must be some rational number times 2π

i.e. $\Delta\varphi = 2\pi \cdot \frac{n}{m}$

FOR EXAMPLE FOR $\rho_1 = \frac{R}{2} \Rightarrow \Delta\alpha = \pi/6$ CLOSES

ALSO $\rho_1 = \frac{1}{\sqrt{2}} R \Rightarrow \alpha = \frac{\pi}{4}$ CLOSES.

IN GENERAL IT DOES NOT CLOSE



particle 1 (at rest)

particle 2 (moving)

The energy in the relative coordinates is given by

$$E_r = \frac{\mu}{2} v^2 - V \quad (\text{since } \rho < R)$$

(they're closer than R)

the angular momentum

$$\ell_r = \mu \frac{R}{2} v$$

$$\frac{\ell_r^2}{2\mu R^2} = \frac{\mu^2 R^2}{4} v^2 \frac{1}{2\mu R^2} = \frac{\mu v^2}{8}$$

The condition for bound orbits then reads

$$\left(E_{min} = \frac{\ell_r^2}{2\mu R^2} - V < E_r < \frac{\ell_r^2}{2\mu R^2} = E_{max} \right)$$

$$\Rightarrow \frac{\mu v^2}{8} - V < E_r < \frac{\mu v^2}{8}$$

$$E_r + V = \frac{\mu v^2}{2}$$

$$\frac{\mu v^2}{8} - V < \frac{\mu v^2}{2} - V < \frac{\mu v^2}{8}$$

$$\frac{\mu v^2}{8} < \underbrace{\frac{\mu v^2}{2} < \frac{\mu v^2}{8} + V}$$

$$\frac{\mu v^2}{2} < \frac{\mu v^2}{8} + V$$

$$\frac{\mu v^2}{2}\left(1 - \frac{1}{4}\right) < V$$

$$\frac{3\mu v^2}{8} < V \xRightarrow[\mu = \frac{3}{4}m]{} \frac{3}{8}\,\frac{3}{4} m v^2 < V \Rightarrow \frac{9}{16} m v^2 < V$$

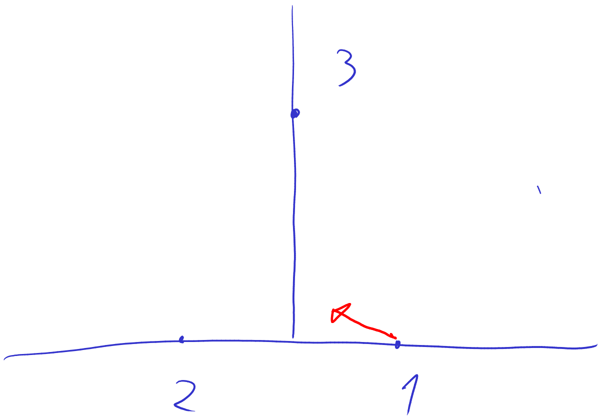
$$\boxed{v < \frac{4}{3}\sqrt{\frac{2V}{m}}}$$ Condition for bound orbits

3.2

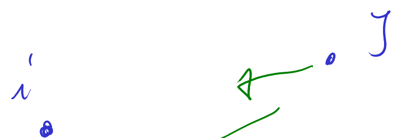
$\vec{r}_1 = (d, 0, 0)$ $\quad m_1 = m$

$\vec{r}_2 = (-d, 0, 0)$ $\quad m_2 = m$

$\vec{r}_3 = (0, 2d, 0)$ $\quad m_3 = 2m$



KRAFT VON i AUF j



$$\vec{F}_{ij} = \alpha \, |\vec{r}_{ij}|^2 \, \vec{r}_{ij}$$

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

Zeigt in Richtung i : attraktiv

$\propto \vec{r}_{ij} \rightarrow$ zentral

$\propto \vec{e}_{ij} \, f(|\vec{r}_{ij}|) \rightarrow$ ISOTROP

$$\vec{r}_{12} = 2d \, e_x$$

$$\vec{r}_{13} = d \vec{e}_x - 2d \, e_y$$

$$\vec{r}_{23} = -d e_x - 2d \, e_y$$

$$\vec{F}_1 = -\alpha d^3 \left(8 \vec{e}_x + 5 \cdot (e_x - 2 e_y) \right)$$

$$= -\alpha d^3 \left(13 e_x - 10 e_y \right)$$

Potentialenergie Zuerst berechne die Potentialenergie U_{ij} zwischen Teilchen i und j

es gilt $\vec{F}_{ij} = -\vec{\nabla}_{\vec{r}_j} U(\vec{r}_j - \vec{r}_i)$

setze $\vec{r} = \vec{r}_j - \vec{r}_i \Rightarrow \underbrace{\vec{\nabla}_{\vec{r}_j} = \vec{\nabla}_{\vec{r}}}_{\text{Kettenregel}}$

$$\vec{F}_{ij} = -\alpha r^2 \vec{r} = -\nabla U(\vec{r})$$

↑ weil im Ausdruck steht $\vec{r}_i - \vec{r}_j = -\vec{r}$

$$= -\alpha r^3 \frac{\vec{r}}{r} = -\alpha r^3 \vec{e}_r$$

$F = f(|\vec{r}|)\, \vec{e}_r$
$\rightarrow U(|\vec{r}|)$

$$-\nabla U(r) = -\frac{\partial U}{\partial r} \vec{e}_r \Rightarrow U(r) = \frac{\alpha r^4}{4} + U_0$$

$U_0 = 0$ weil $U = 0$ wenn $r = 0$

$$U_{ges} = U(|\vec{r}_1 - \vec{r}_2|) + U(|\vec{r}_1 - \vec{r}_3|) + U(|\vec{r}_2 - \vec{r}_3|) = \frac{\alpha}{4} d^4 \left(4^2 + 5^2 + 5^2\right) = \frac{\alpha}{4} d^4 \cdot 66$$

$$= \frac{33}{2} \alpha d^4$$

3.3

$$\dot{\vec{r}}_1 = u\, e_y = \dot{\vec{r}}_2 \qquad \dot{\vec{r}}_3 = -u\, e_y + 2u\, e_x$$

$$\vec{L}_1 = m\, d u\, \vec{e}_x \times \vec{e}_y = m d u\, \vec{e}_z$$

$$\vec{L}_2 = -\, m d u\, \vec{e}_z$$

$$\vec{L}_3 = 2 m d u\, (2\vec{e}_y) \times (-e_y + 2 e_x) = -8\, m d u\, \hat{e}_z$$

$$\vec{L} = -8\, m d u\, e_z$$

$$T = \frac{1}{2} m u^2 (1+1) + \frac{1}{2} 2 m u^2 (5)$$

$$= m u^2 (1+5) = 6\, m u^2$$

$$U = \frac{33}{2} \alpha d^4$$

$$E = T + U$$

3.4

SCHWERPUNKTSYSTEM

$$\vec{R} = \frac{d}{4m} m \left(\vec{e}_x - \vec{e}_x + 2 \cdot 2 \vec{e}_y \right) = d \, \vec{e}_y$$

$$\vec{r}_1^{BS} = d \left(\vec{e}_x - \vec{e}_y \right)$$

$$\vec{r}_2^{BS} = d \left(-\vec{e}_x - \vec{e}_y \right)$$

$$\vec{r}_3^{BS} = d \, \vec{e}_y$$

$$\dot{\vec{R}} = \frac{u}{4m} m \left(\vec{e}_y + \vec{e}_y - 2 \vec{e}_y + 4 \vec{e}_x \right) = u \, \vec{e}_x$$

$$\dot{\vec{r}}_1^{BS} = u \left(\vec{e}_y - \vec{e}_x \right) = \dot{\vec{r}}_2^{S}$$

$$\dot{\vec{r}}_3^{BS} = u \left(\vec{e}_x - \vec{e}_y \right)$$

$$T^{BS} = \frac{1}{2} m u^2 \left(2 + 2 + 2 \cdot 2 \right) = 4 m u^2$$

$$T_2 = \frac{1}{2} 4m \, u^2 \cdot 1 = 2 m u^2$$

$$T = T^{BS} + T_2 = 6 m u^2 \quad \checkmark$$

$$U = U^{BS}$$ DA ABSTÄNDE GLEICH BLEIBEN

$$\Rightarrow E = \underbrace{T^{BS} + U^{BS}}_{E^{BS}} + T_2$$

3.5 DREHIMPULS IM BS

$$\vec{L}_1^s = m d u \, (\vec{e}_x - \vec{e}_y) \times (\vec{e}_y - \vec{e}_x) = 0$$

$$\vec{L}_2^s = m d u \, (-\vec{e}_x - \vec{e}_y) \times (\vec{e}_y - \vec{e}_x) = m d u \, (-2 \vec{e}_z)$$

$$\vec{L}_3^s = m d u \, 2 \, \vec{e}_y \times (\vec{e}_x - \vec{e}_y) = m d u \, (-2 \vec{e}_z)$$

$$\vec{L}^s = -4 \vec{e}_z \, m d u$$

$$\vec{L}_s = 4 m \, \vec{R} \times \dot{\vec{R}} = 4 m u d \, \vec{e}_y \times \vec{e}_x = -4 e_z m u d$$

$$\vec{L}^s + \vec{L}_s = -8 \vec{e}_z m d u = \vec{L} \quad \checkmark$$

Problem 3.6 – Übungsblatt 3

A body on Earth is thrown with the initial velocity $\vec{v}_0 = (v_{0x}, 0, v_{0z})$ from the origin $\vec{r} = \vec{0}$, where $\hat{e}_x$ points towards the north and $\hat{e}_z$ towards the top.

With no apparent forces this body will move along a parabola in the xz-plane.
Determine the deviation of this body from the xz-plane at the end of the flight.

Note: the centrifugal force should be neglected.
The equations of motion therefore contain the Coriolis force only, i.e. $\vec{v}\,' = -2\vec{\omega} \times \vec{v} + \vec{g}$. (*)

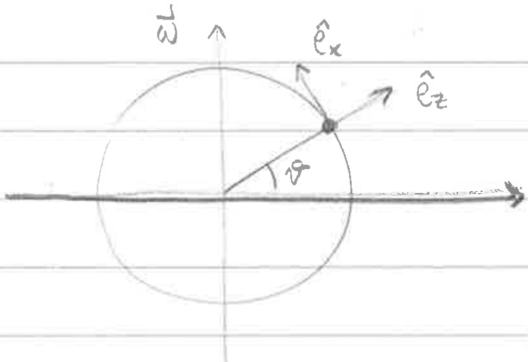
The components of the velocity in x and z direction can be determined from $\omega \to 0$ in equation (*).

Interpret qualitatively the sign of the deflection (east/west) from the point of view of an inertial reference frame.

Solution

$\vec{v}_0 = (v_{0x}, 0, v_{0z}) \qquad \vec{g} = (0, 0, -g)$

$\vec{\omega} = (\omega_x, 0, \omega_z) = \omega(\cos\vartheta, 0, \sin\vartheta)$



the equations of motion read

$$\vec{v}' = \vec{g} - 2\,\vec{\omega}\times\vec{v} \qquad (*)$$

For x, z we can take $\omega \to 0$ and solve the unperturbed motion, i.e.

$$v_x' = 0 \Rightarrow v_x(t) = v_{0x}$$
$$v_z' = -g \Rightarrow v_z(t) = v_{0z} - gt$$

Taking the y-component of $(*)$ returns

$$v_y' = -2(\vec{\omega}\times\vec{v})_y = -2(\omega_z v_x - \omega_x v_z)$$

$$\vec{\omega}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & 0 & \omega_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{pmatrix} -\omega_z v_y \\ \omega_z v_x - \omega_x v_z \\ \omega_x v_y \end{pmatrix}$$

$$v_y' = -2(\omega_z v_x(t) - \omega_x v_z(t)) =$$

$v_x(t) = v_{0x}$
$v_z(t) = v_{0z} - gt$

$$v_y' = \underbrace{-2(\omega_z v_{0x} - \omega_x v_{0z})}_{\text{constant}} - 2\omega_x g t$$

$$v_y(t) = -2(\omega_z v_{0x} - \omega_x v_{0z})t - \frac{2}{2}\omega_x g t^2$$

$\uparrow$ initial condition $v_{0y} = 0$ (see text)

$$\vec{r} = (x, y, z) \equiv \vec{r}(t)$$

$$x(t) = \int_0^t dt' \, v_x(t') = v_{0x} t$$

$$y(t) = \int_0^t dt' \, v_y(t') = -(\omega_z v_{0x} - \omega_x v_{0z})t^2 - \omega_x g \frac{t^3}{3}$$

$$z(t) = \int_0^t dt' \, v_z(t') = v_{0z} t - g\frac{t^2}{2}$$

with the initial condition $\vec{r}_0 = \vec{r}(0) = (0,0,0)$

Flight-time, T_F

$Z(t = T_F) \equiv 0$ to find where it lands

$$0 = v_{0z} T_F - \frac{g}{2} T_F^2 \Rightarrow -\frac{2v_{0z}}{g} T_F + T_F^2 = 0$$

$$T_F\left(T_F - \frac{2v_{0z}}{g}\right) = 0$$

$T_F = 0$ to be discarded

$T_F = 2v_{0z}/g$

$$y(t=T_F) = -(\omega_z v_{0x} - \omega_x v_{0z})\, T_F^2 - \frac{\omega_x g}{3} T_F^3 =$$

$$= -(\omega_z v_{0x} - \omega_x v_{0z}) \frac{4 v_{0z}^2}{g^2} - \frac{\omega_x g}{3}\left(\frac{8 v_{0z}^3}{g^3}\right) =$$

$$= \left(\frac{2 v_{0z}}{g}\right)^2 \Big[\underbrace{\frac{v_{0z}\,\omega_x}{3}}_{A \,\equiv\, \text{west}} - \underbrace{v_{0x}\,\omega_z}_{\text{east} \,\equiv\, B} \Big]$$

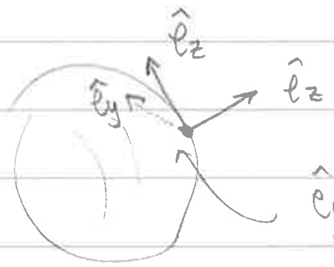
$\omega_z > 0$ for the upper emisphere

$\omega_z < 0$ for the lower emisphere

$\hat{e}_y$ is in direction "west" since

$$\hat{e}_x \times \hat{e}_z = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\hat{e}_y$$

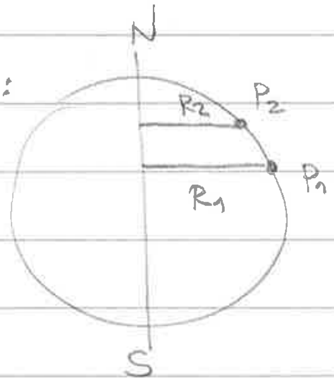
given the info we have from the text



$\hat{e}_y$ points west

Interpretation

Contribution B:



the mass starts in P_1,

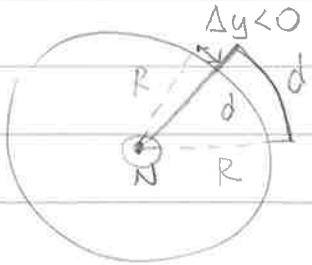
with a velocity $v_{P_1,y} = \omega R_1$

then lands in P_2 with $R_2 < R_1$

so $v_{P_2,y} = \omega R_2 < v_{P_1,y}$

Deviation is towards EAST

Contrib. A :



Earth seen from the top
(above the NORTH pole)

$\boxed{d = \omega R T_F}$

Deviation towards WEST