Chapter 3

Renormalization Group

3.1 Introduction

We discuss the basic ideas of the renormalization group (RNG) approach in terms of the 1d Ising model

$$H = -J \sum_{i=1}^{N} s_i s_{i+1} - h' \sum_i s_i \ , \tag{3.1}$$

as every step can be performed analytically. We assume $N = 2^L$, which allows us to thin out every other spin repeatedly. In addition we assume periodic boundary conditions (pbc), i.e. $s_{N+1} = s_1$. For the partition function we need

$$\tilde{H} := -\beta H := K \sum_i s_i s_{i+1} + h \sum_i s_i + NC \ , \tag{3.2}$$

where we have added a constant energy NC, as it will become relevant in the renormalization scheme. The pdf for a spin configuration $\{s\}$ then reads

$$\rho_N = \prod_{i=1}^{N} e^{K s_i s_{i+1} + \frac{h}{2}(s_i + s_{i+1}) + C} \tag{3.3}$$

For later use we have written the B-field term slightly differently. Next we want to determine the reduced pdf for the even sites only, which is obtained by the trace over the spins on the odd sites

$$\rho_{\text{even}} = \Big(\prod_i^{\text{odd}} \sum_{s_i = \pm 1} \Big) \rho_N \tag{3.4}$$

$$= \Big(\prod_i^{\text{even}} e^{h s_i} e^{2C} \Big) \Big(\prod_i^{\text{odd}} \sum_{s_i = \pm 1} \Big) e^{K(s_{i-1} s_i + s_i s_{i+1}) + h s_i} \ . \tag{3.5}$$

This marginal pdf is required, if we want to calculate the spin-spin correlation for spins on even sites only. In particular, it allows to compute the spin-spin correlation

$$\langle s_{2i} s_{2(i+1)} \rangle \, . \tag{3.6}$$

If we repeat the marginalization once more, we obtain a marginal pdf that allows to compute

$$\langle s_{4i} s_{4(i+1)} \rangle \, . \tag{3.7}$$

Hence, the repeated marginalization of half of the sites allows to determine how spin spin-correlations behave on different length scales. Now we want to really compute equation (3.4) [previous page] .

$$\rho_{\text{even}} = \left(\prod_i^{\text{even}} e^{h s_i + 2C} \right) \left(\prod_i^{\text{odd}} \sum_{s_i = \pm 1} e^{s_i \left(K(s_{i-1} + s_{i+1}) + h \right)} \right) \tag{3.8}$$

$$= \left(\prod_i^{\text{even}} e^{h s_i + 2C} \right) \left(\prod_i^{\text{odd}} 2 \ \cosh \left(K(s_{i-1} + s_{i+1}) + h \right) \right) \tag{3.9}$$

$$= \prod_i^{\text{even}} e^{h s_i + 2C + \ln(2)} \cosh \left(K(s_i + s_{i+2}) + h \right) \tag{3.10}$$

$$= \prod_i^{N/2} e^{\frac{h}{2}(s_{2i} + s_{2(i+1)}) + 2C + \ln(2)} \cosh \left(K(s_{2i} + s_{2(i+1)}) + h \right) \, . \tag{3.11}$$

Next we define $s_i^{(b)} = s_{bi}$ and find in particular for $b = 2$

$$\rho_{\text{even}} = \left(\prod_i^{N/2} e^{\frac{h}{2}(s_i^{(2)} + s_{i+1}^{(2)}) + 2C + \ln(2)} \cosh \left(K(s_i^{(2)} + s_{i+1}^{(2)}) + h \right) \right) \, . \tag{3.12}$$

Finally, we want to express the marginal density formally identically to equation (3.3)

$$\rho_{\text{even}} := \rho_{N/2} = \prod_{i=1}^{N/2} e^{K' s_i^{(2)} s_{i+1}^{(2)} + \frac{h'}{2}(s_i^{(2)} + s_{i+1}^{(2)}) + C'} \, , \tag{3.13}$$

which is possible, since each factor

$$e^{K' s_i^{(2)} s_{i+1}^{(2)} + \frac{h'}{2}(s_i^{(2)} + s_{i+1}^{(2)}) + C'} = e^{\frac{h}{2}(s_i^{(2)} + s_{i+1}^{(2)}) + 2C + \ln(2)} \cosh \left(K(s_i^{(2)} + s_{i+1}^{(2)}) + h \right)$$

only depends on $s_i^{(2)} + s_{i+1}^{(2)}$, and $s_i^{(2)} s_{i+1}^{(2)}$, for which 3 different values are possible each, which defines the 3 parameters K', h', C'. The corresponding conditions are

$$s_i^{(2)} = s_{i+1}^{(2)} = 1: \qquad e^{K'+h'+C'} = e^{h+2C+\ln(2)} \cosh(h+2K) \tag{3.14a}$$

$$s_i^{(2)} = s_{i+1}^{(2)} = -1: \qquad e^{K'-h'+C'} = e^{-h+2C+\ln(2)} \cosh(h-2K) \tag{3.14b}$$

$$s_i^{(2)} = -s_{i+1}^{(2)}: \qquad e^{-K'+C'} = e^{2C+\ln(2)} \cosh(h) \tag{3.14c}$$

From equation (3.14c) we obgtain

$$e^{C'} = e^{K'+2C+\ln(2)} \cosh(h) \tag{3.15}$$

Insertion in the first two equations yields

$$e^{2K'+h'} \cosh(h) = e^{h} \cosh(h+2K) \tag{3.16a}$$

$$e^{2K'-h'} \cosh(h) = e^{-h} \cosh(h-2K) \; . \tag{3.16b}$$

Multiplication of these equations yields

$$e^{4K'} = \frac{\cosh(h+2K)\cosh(h-2K)}{\cosh^2(h)} \; , \tag{3.17}$$

and division of these equations yields

$$e^{2h'} = e^{2h} \frac{\cosh(h+2K)}{\cosh(h-2K)} \; . \tag{3.18}$$

equation (3.15) can be written as

$$e^{4C'} = e^{4K'} e^{8C+4\ln(2)} \cosh^4(h) \tag{3.19}$$

Along with equation (3.17) this yields

$$e^{4C'} = \cosh(h+2K)\cosh(h-2K) e^{8C+4\ln(2)} \cosh^2(h) \tag{3.20}$$

The equations 3.17, 3.18, and 3.20 uniquely define the values of h', K', C', which we now denote as $h^{(2)}, K^{(2)}, C^{(2)}$. The key finding is that the reduced density matrix is formally identical to the original one with modified parameters and due to the translational invariance it is actually the same for the even and odd sites. Therefore we denote it simply by $\rho^{(2)}$. Now we can repeat this procedure and obtain $\rho^{(3)}$, which is the reduced density if only every fourth site is retained. The corresponding parameters are $h^{(3)}, K^{(3)}, C^{(3)}$ are related to the parameters $h^{(2)}, K^{(2)}, C^{(2)}$ of the previous iteration via equation (3.17) , equation (3.18) , and equation (3.20) . In general we obtain the iteration scheme

$$e^{4K^{(n+1)}} = \frac{\cosh\big(h^{n)} + 2K^{(n)}\big)\cosh\big(h^{n)} - 2K^{(n)}\big)}{\cosh^2(h^{n)})} \tag{3.21a}$$

$$e^{2h^{(n+1)}} = e^{2h^{(n)}} \frac{\cosh\big(h^{(n)} + 2K^{(n)}\big)}{\cosh(h^{(n)} - 2K^{(n)})} \tag{3.21b}$$

$$e^{4C^{(n+1)}} = \cosh\big(h^{(n)} + 2K^{(n)}\big)\cosh\big(h^{(n)} - 2K^{(n)}\big)e^{8C^{(n)}+4\ln(2)}\cosh^2(h^{(n)}) \; . \tag{3.21c}$$

The iteration starts with $K^{(1)} = K, h^{(1)} = h$, and $C^{(1)} = C$. For a first discussion, we consider the case $h = 0$, i.e. no external magnetic field. Then the first iteration yields for h

$$e^{2h^{(2)}} = \frac{\cosh\big(2K^{(1)}\big)}{\cosh(2K^{(1)})} = 1 \; , \tag{3.22a}$$

i.e. $h^{(2)} = 0$. Hence, $h^{(n)} = 0$ for all iteration steps. For the parameter K we then obtain the recursion relation

$$e^{4K^{(n+1)}} = \cosh^2(2K^{(n)}) = \frac{1}{4}\left(e^{2K^{(n)}} + e^{-2K^{(n)}}\right)^2 \tag{3.23}$$

$$e^{-4K^{(n+1)}} = \frac{4}{\big(e^{2K^{(n)}} + e^{-2K^{(n)}}\big)^2} = \frac{4e^{-4K^{(n)}}}{\big(1 + e^{-4K^{(n)}}\big)^2} \tag{3.24}$$

We introduce the definition $x^{(n)} = e^{-4K^{(n)}}$ for which we obtain

$$x^{(n+1)} = f(x^{(n)})$$
$$f(x) := \frac{4x}{\big(1 + x\big)^2}$$

The figure illustrates that if we start the recursion with any value $0 < x^{(1)} \leq 1$, i.e. $0 \leq K < \infty$, which corresponds to the parameter of the original physical system, the iteration ends at the fixed point $x^{(\infty)} = 1$. This means that the physical feature (e.g. spin-spin correlation) for very long distances is equivalent to that of an Ising model with $x = 1$, which corresponds to $K = 0$. This fixed point is called *high temperature fixed point*, as $x = 1$, or rather $K = 0$ is also obtained for $T \to \infty$ ($\beta \to 0$). The 1d Ising model considered at very long length scales looks like an infinite temperature or non-interacting solution, which means it is disordered (no long range order). This fixed point is stable or attractive.

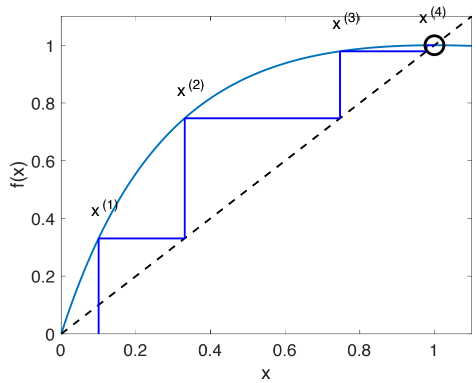

Figure 3.1: **Recursion relation for x in the 1d Ising model. Starting with an arbitrary $x^{(1)} \in (0,1]$ the fixed point is $x^{(\infty)} = 1$. Only for $x^{(1)} = 0$ the other fixed point $x^{(\infty)} = 0$ is relevant.**

The other fixed point $x^* = 0$ is the non-trivial *critical fixed point*. It corresponds to $K = \infty$, i.e. $T = 0$. In this case, starting with the physical parameter $x = 0$, i.e. $T = 0$, the system stays at $x = 0$ and has therefore long-range order. Here it is the trivial case, without thermal fluctuations, there is long range order, since there are no quantum fluctuations, in contrast to the case of the spin-1/2 Heisenberg model.
Next we study the energy parameter C for the case $h = 0$. Then we have

$$C^{(n+1)} = 2C^{(n)} + \frac{1}{2}\ln\Big(\cosh\big(2K^{(n)}\big)\Big) + \ln(2)\ . \tag{3.25}$$

As the recursion for $K^{(n)}$ was independent of that for $C^{(n)}$, we can insert the previous result for $K^{(n)}$. Hence for $n \to \infty$ (very long length scale),we have $K^{(n)} \to 0$ and we find

$$\begin{aligned} e^{4C^{(n+1)}} &= e^{4(2C^{(n)}+\ln(2))} \\ C^{(n+1)} &= 2C^{(n)} + \ln(2)\ . \end{aligned}$$

The factor 2 is obvious due to the decimation of the number of spins by the factor 2. The recursion relation for the case $h = 0$ in equation (3.23) [previous page] can also be written as

$$K^{(n+1)} = g(K^n) \tag{3.26}$$

$$g(K) = (\frac{1}{2}\ln\Big(\cosh(2K)\Big)\ , \tag{3.27}$$

wich for large $K^{(n)}$ becomes

$$
\begin{aligned}
K^{(n+1)} &= \frac{1}{2}\ln\left(\frac{e^{2K^{(n)}}(1+e^{-4K^{(n)}})}{2}\right) \\
&= K^{(n)} - \frac{\ln(2)}{2} + \ln\left(1+e^{-4K^{(n)}}\right) .
\end{aligned}
$$

Hence we have for large $K^{(n)}$

$$
K^{(n+1)} \approx K^{(n)} - \frac{\ln(2)}{2} . \tag{3.28}
$$

If this relation applies, then

$$
\begin{aligned}
K^{(n)} &\approx K^{(n-1)} - \frac{\ln(2)}{2} \\
&\approx K^{(n-2)} - \frac{\ln(2)}{2} - \frac{\ln(2)}{2} \\
&\dots \\
K^{(n)} &\approx K^{(0)} - n \cdot \frac{\ln(2)}{2} .
\end{aligned} \tag{3.29}
$$

The renormalization group approach can also be used to study how the correlation length depends on β or rather K. If there is no long range order, the correlation length decreases exponentially as

$$
\langle s_0 s_m \rangle \propto e^{m/\xi(K)} .
$$

The correlation length will depend on $\beta J = K$. In each renormalization step, the unit cell increases by a factor of 2. In the original system we have

$$
\langle s_0 s_{2m} \rangle_{K^{(0)}} \propto e^{(2m)/\xi(K^{(0)})} .
$$

The same correlation function is equal to that of the renormalized system as follows

$$
\langle s_0 s_{2m} \rangle_{K^{(0)}} = \langle s_0^{(2)} s_m^{(2)} \rangle_{K^{(1)}} \propto e^{m/\xi(K^{(1)})} .
$$

Comparing the exponentials yields

$$
\frac{\xi(K^{(0)})}{2} = \xi(K^{(1)}) = \xi(g(K^{(0)})) .
$$

Repeating the renormalization m times we obtain the relation

$$
\xi(K^{(0)}) = 2^m \xi(g^{(m)}(K^{(0)})) . \tag{3.30}
$$

The left hand site is the quantity we are interested in for low temperature or rather $K \gg 1$. In this case, according to equation (3.29) , after m RNG steps we have

$$g^{(m)}(K^{(0)}) = K^{(m)} \approx K^{(0)} - \frac{m}{2}\ln(2) \ .$$

Remember we are interested in $K \gg 1$. Each iteration reduces K by $\ln(2)/2 \approx 0.35$. Then if we choose m sufficiently large, eventually, we will reach $K^{(m^*)} = g^{(m^*)}(K^{(0)}) \sim O(1)$. The required number of steps is

$$\begin{aligned} K^{(0)} - \frac{m^*}{2}\ln(2) = \mathcal{O}(1) \\ m^* = \frac{2\big(K^{(0)} - \mathcal{O}(1)\big)}{\ln(2)} \\ m^* \approx \frac{2K^{(0)}}{\ln(2)} \ . \end{aligned}$$

Along with equation (3.30) [previous page] we obtain

$$\xi(K) = 2^{m^*}\xi(\underbrace{g^{(m^*)}(K)}_{=\mathcal{O}(1)}) \ .$$

Now $g(K = \mathcal{O}(1))$ is some unimportant constant C and we finally have

$$\xi(K) \sim 2^{m^*} = 2^{\frac{2K}{\ln(2)}} = e^{\ln\left(2^{\frac{2K}{\ln(2)}}\right)} = e^{2K} \ .$$

$$\xi(K) \sim e^{2K} \ . \tag{3.31}$$

So the bottom line is that the correlation length increases exponentially with decreasing temperature and becomes infinite at $T = 0$.

Finally, we compute the free energy for $h = 0$. We start out from the partition function for the original system and integrate out the spins on the odd sites and use equation (3.13) [p. 49]

$$Z_N(K, C) = \mathrm{tr}\Big\{e^{-\beta H}\Big\} = \mathrm{tr}\Big\{e^{K\sum_{\langle ij\rangle} S_i S_j + NC}\Big\} = Z_{N/2}(K', C') \ ,$$

with K' given in equation (3.26) [p. 52] and C' given in equation (3.25) [p. 52]

$$K^{(1)} = \frac{1}{2}\ln\Big(\cosh\big(2K^{(0)}\big)\Big) \tag{3.32}$$

$$C' = 2C + K^{(1)} + \ln(2) \ . \tag{3.33}$$

Next we express the partition function slightly different, by pulling C in front

$$Z_N(K,C) = e^{NC} Z_N^{(0)}(K) = e^{\frac{N}{2}C'} Z_{N/2}^{(0)}(K') \; . \tag{3.34}$$

and taking the logarithm per lattice site (apart from $k_B T$ the free energy per lattice site)

$$f(K^{(0)},C^{(0)}) = \frac{\ln\big(Z_N(K^{(0)},C^{(0)})\big)}{N} = C^{(0)} + \underbrace{\frac{\ln\big(Z_N^{(0)}(K^{(0)})\big)}{N}}_{:=\tilde{f}(K^{(0)})}$$

$$f(K^{(0)},C^{(0)}) = C^{(0)} + \tilde{f}(K^{(0)}) \; . \tag{3.35}$$

On the other hand, based on equation (3.34) we obtain

$$\begin{aligned} f(K^{(0)},C^{(0)}) &= \frac{1}{N}\left(\frac{N}{2}C' + \ln\big(Z_{N/2}^{(0)}(K^{(1)})\big)\right) \\ &= \frac{1}{2}\left(C' + \frac{\ln\big(Z_{N/2}^{(0)}(K^{(1)})\big)}{N/2}\right) \\ f(K^{(0)},C^{(0)}) &= \frac{1}{2}\Big(C' + \tilde{f}(K^{(1)})\Big) \; . \end{aligned}$$

Comparison with equation (3.35) yields

$$\tilde{f}(K^{(0)}) = \frac{1}{2}\Big(C^{(1)} - 2C^{(0)} + \tilde{f}(K^{(1)})\Big)$$

Inserting equation (3.32) [previous page] finally yields

$$\tilde{f}(K^{(0)}) = \frac{1}{2}\Big(K^{(1)} + \ln(2) + \tilde{f}(K^{(1)})\Big) \tag{3.36}$$

A second iteration yields obviously

$$\tilde{f}(K^{(1)}) = \frac{1}{2}\Big(K^{(2)} + \ln(2) + \tilde{f}(K^{(2)})\Big) \; .$$

Inserting into equation (3.36) yields

$$\begin{aligned} \tilde{f}(K^{(0)}) &= \frac{1}{2}\left(K^{(1)} + \ln(2) + \frac{1}{2}\Big[K^{(2)} + \ln(2) + \tilde{f}(K^{2})\Big]\right) \\ &= \frac{K^{(1)} + ln(2)}{2^1} + \frac{K^{(2)} + ln(2)}{2^2} + \frac{\tilde{f}(K^{(2)})}{2^2} \; . \end{aligned}$$

Clearly, this leads after m iterations to

$$\begin{aligned}
f(K,C) &= \sum_{\nu=1}^{m} \frac{\ln(2)}{2^\nu} + \sum_{\nu=1}^{m} \frac{K^{(\nu)}}{2^\nu} + \frac{\tilde{f}(K^{(m)})}{2^m} \\
&= \ln(2) \sum_{\nu=1}^{m} \frac{1}{2^\nu} + \sum_{\nu=1}^{m} \frac{K^{(\nu)}}{2^\nu} + \frac{\tilde{f}(K^{(m)})}{2^m} \\
&= \ln(2) \sum_{\nu=1}^{m} \frac{1}{2^\nu} + \sum_{\nu=1}^{m} \frac{K^{(\nu)}}{2^\nu} + \frac{\tilde{f}(K^{(m)})}{2^m}
\end{aligned}$$

We have seen, that $K^{(n)} \to 0$ for $n \to \infty$. The partition function for $K \to 0$ can be obtained analytically as

$$Z_N(K \to 0) = 2^N \ ,$$

hence, according to the definition $\tilde{f} = \ln(Z)/N$, we obtain

$$\tilde{f}(K \to 0) = \frac{\ln(Z_N)(K \to 0)}{N} = \ln(2)$$

So, if we perform an infinite number of renormalization steps, then

$$\begin{aligned}
\tilde{f}(K^{(0)}) &= \ln(2) \underbrace{\left(\sum_{\nu=0}^{\infty} \left(\frac{1}{2}\right)^\nu - 1 \right)}_{=1} + \sum_{\nu=1}^{\infty} \frac{K^{(\nu)}}{2^\nu} + \underbrace{\lim_{L\to\infty} \frac{\ln(2)}{2^L}}_{=0} \\
&= \ln(2) + \sum_{\nu=1}^{\infty} \frac{K^{(\nu)}}{2^\nu} \ .
\end{aligned}$$

We have seen before that the exact result is given by

$$\begin{aligned}
Z_N &= d_1^N \\
\tilde{f}(K) &= \ln(d_1) \\
d_1 &= 2\cosh\big(K^0\big) \ ,
\end{aligned}$$

Hence, since $C^{(0)} = 0$

$$f(K^{(0)}, C^{(0)}) = \tilde{f}(K^{(0)}) = \ln(2) + \ln\big(\cosh\big(K^{(0)}\big)\big) \ .$$

Numerical comparison shows that both results agree, i.e.

$$\sum_{\nu=1}^{\infty} \frac{K^{(\nu)}}{2^\nu} = \ln\big(\cosh\big(K^{(0)}\big)\big)$$

with

$$K^{(n)} = \frac{1}{2} \ln \left(\cosh\left(2K^{(n-1)}\right) \right) .$$

In the 1D case, no approximations where necessary for the RNG procedure.

3.2 2D Ising

We decompose the sc-lattice into an A-B-lattice, i.e. sites belong either to sub-lattice A (blue circles with even indices) or B (red crossed with odd indices). All nearest neighbors of a point of sub-lattice A belong to sub-lattice B and vice versa.

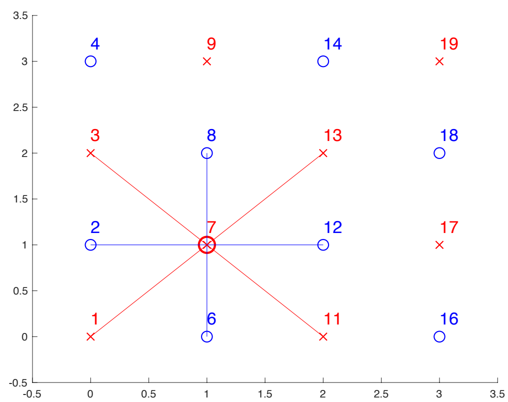

Figure 3.2: **Illustration of the RG schem for 2D Ising.**
t

The goal is again to decimate the lattice by integrating out sublattice B, i.e. to compute the reduced density matrix for sub-lattice A. The reduce density matrix on sub-lattice A is

$$\rho_A = \sum_{\{S_i\} \in B} e^{-\beta H}$$

We consider all terms of the hamiltonian that contain the spin on site 7, which is part of sub-lattice B.

$$\begin{aligned} -\beta H_7 &= a + hS_7 + K\mathcal{S}_7^{\text{nn}} \\ \mathcal{S}_7^{\text{nn}} &= S_2 + S_6 + S_8 + S_{12} \ . \end{aligned}$$

Next we separate the contribution of this spin from the rest

$$\rho_A = \sum_{\{S_i\}\in B/S_7} e^{-\beta H_{\not 7}} Z_7$$
$$Z_7 = \sum_{S_7} e^{-\beta H_7}$$

The trace over S_7 yields

$$Z_7 := \sum_{S_0} e^{a+S_7\left(K\,\mathcal{S}_7^{\mathrm{nn}}+L\,\mathcal{S}_7^{\mathrm{nnn}}+h\right)} = e^a\cdot 2\cosh\left(K\mathcal{S}_7^{\mathrm{nn}}+h\right)\,. \tag{3.37}$$

We want to express this function again as

$$e^{-\beta\tilde{H}}\,.$$

The most general exponential form for this expression, since $S_i^2 = 1$, is given by

$$Z_7 = \exp\left(a' + h'\mathcal{S} + \overline{K}\sum_{ij}{}' S_iS_j + b\sum_{ijk}{}' S_iS_jS_k + c\,S_1S_2S_3S_4\right)\,. \tag{3.38}$$

The indices in the sums are from the set $I_7^{nn} = \{2, 6, 8, 12\}$, and the prims indicate that no indices must occur twice and all the indices shall be ordered, e.g. $S_i < S_j < S_k$. We have used the symmetry that equation (**??**) [p. **??**] is invariant against interchange of any two indices $i \leftrightarrow j$. Otherwise , each product of spins would have its own prefactor.
In the double sum there are also the products S_6S_8 and S_2S_{12}, which belong to next-nearest neighbour sites on the decimated lattice, which only consists of sub-lattice A (blue circles). I.e. starting from nn coupling the decimation also introduces nnn coupling and also term with three and four spins. The latter will turn out negligible but the nnn coupling is relevant. To obtain the parameter mapping in an RG step, we include the nnn terms also in the original hamiltonian, i.e.

$$-\beta H_7 = a + hS_7 + K\mathcal{S}_7^{\mathrm{nn}} + L\mathcal{S}_7^{\mathrm{nnn}}$$
$$\mathcal{S}_7^{\mathrm{nn}} = S_2 + S_6 + S_8 + S_{12}$$
$$\mathcal{S}_7^{\mathrm{nnn}} = S_1 + S_3 + S_{11} + S_{13}\,.$$

Again, we separate the contribution of this spin from the rest

$$\rho_A = \sum_{\{S_i\}\in B/S_7} e^{-\beta H_{\not 7}} Z_7$$
$$Z_7 = \sum_{S_7} e^{-\beta H_7}$$

The trace over S_7 now yields

$$Z_7 := \sum_{S_0} e^{a+S_7\left(K\,\mathcal{S}_7^{\mathrm{nn}}+L\,\mathcal{S}_7^{\mathrm{nnn}}\right)} = e^a \cdot 2\cosh\left(K\mathcal{S}_7^{\mathrm{nn}} + L\mathcal{S}_7^{\mathrm{nnn}} + h\right)\,. \qquad (3.39)$$

Here we have a problem, since $\mathcal{S}_7^{\mathrm{nnn}}$ contains spins of sublattice B (e.g. S_{13}) that need to be integrated out. It is contained in the residual hamiltonian in the Ising from H_7, but in addition, we get the factor of equation (3.39) . In this combination the sum over S_{13} is not as trivial as that over S_7, which we had before. Moreover, the complexity increases, as Z_7 also contains other spins of sub-lattice B (S_1 say) and after the trace over S_{13} the expression containing S_1 is getting even more complicated, etc. Hence, to keep thing manageable, we replace $\mathcal{S}_7^{\mathrm{nnn}}$ by its mean value, which is zero. Then we are left with

$$Z_7 := \sum_{S_0} e^{a+S_7\left(K\,\mathcal{S}_7^{\mathrm{nn}}+L\,\mathcal{S}_7^{\mathrm{nnn}}\right)} = e^a \cdot 2\cosh\left(K\mathcal{S}_7^{\mathrm{nn}} + h\right)\,. \qquad (3.40)$$

Aa we will see soon, that does not mean that the original nnn coupling has no influence at all. We can use equation (3.38) [previous page] to express Z_7 in an exponential form

$$Z_7 = \exp\left(a' + h'\mathcal{S} + \overline{K}\sum_{ij}{}' S_iS_j + \sum_{ijk}{}' S_iS_jS_k + c\,S_1S_2S_3S_4\right)\,.$$

As before, the indices in the sums are from the set $I_7^{nn} = \{2, 6, 8, 12\}$, The term with 3 spins can equivalently be written as

$$\sum_{ij}{}' S_iS_jS_k = \mathcal{S}\cdot S_1S_2S_3S_4$$

proof

$$\begin{aligned}\sum_{ij}{}' S_iS_jS_k &= S_1S_2S_3 + S_1S_2S_4 + S_1S_3S_4 + S_2S_3S_4\\ &= S_1S_2S_3S_4^2 + S_1S_2S_2^2S_4 + S_1S_2^2S_3S_4 + S_1^2S_2S_3S_4\\ &= S_1S_2S_3S_4\left(S_4 + S_3 + S_2 + S_1\right)\end{aligned}$$

q.e.d. ✓

So we have the constraint

$$a + \ln\Big(2\cosh\big[K\mathcal{S} + h\big]\Big) = a' + h'\mathcal{S} + \overline{K}\sum_{ij}{}' S_i S_j + \Big(b\mathcal{S} + c\Big) S_1 S_2 S_3 S_4 \tag{3.41}$$

Now we consider the possible spin configurations. We begin with $+ + ++$ which yields the condition

$$1)\quad (+ + ++):\quad a + \ln\Big(2\cosh\big[4K + h\big]\Big) = a' + 4h' + 6\overline{K} + \big(4b + c\big)\cdot 1$$

$$2)\quad (+ + +-):\quad a + \ln\Big(2\cosh\big[2K + h\big]\Big) = a' + 2h' + 0\overline{K} + \big(2b + c\big)\cdot(-1)$$

$$3)\quad (+ + --):\quad a + \ln\Big(2\cosh\big[0K + h\big]\Big) = a' + 0h' - 2\overline{K} + \big(0b + c\big)\cdot 1$$

Since the geometric position of the spins does not enter in equation (3.41) we get the same equations if we permute the spins.

The remaining spin configurations are obtained by the transformation $S_i \to -S_i$, which changes the sign in the terms with an odd number of spins. I.e. 3) is invariant, and for the other two we obtain

$$1')\quad (- - --):\quad a + \ln\Big(2\cosh\big[-4K + h\big]\Big) = a' - 4h' + 6\overline{K} + \big(-4b + c\big)\cdot 1$$

$$2')\quad (- - -+):\quad a + \ln\Big(2\cosh\big[-2K + h\big]\Big) = a' - 2h' + 0\overline{K} + \big(-2b + c\big)\cdot(-1)$$

We simplify these equations a bit further

$$
\begin{aligned}
&1) \quad (++++): && \ln\Big(2\cosh\big[4K+h\big]\Big) = \Delta a + 4h' + 6\overline{K} + 4b + c\\
&1') \quad (----): && \ln\Big(2\cosh\big[-4K+h\big]\Big) = \Delta a - 4h' + 6\overline{K} - 4b + c\\
&2) \quad (+++-): && \ln\Big(2\cosh\big[2K+h\big]\Big) = \Delta a + 2h' - 2b - c\\
&2') \quad (---+): && \ln\Big(2\cosh\big[-2K+h\big]\Big) = \Delta a - 2h' + 2b - c\\
&3) \quad (++--): && \ln\Big(2\cosh\big[h\big]\Big) = \Delta a - 2\overline{K} + c\ ,
\end{aligned}
$$

with $\Delta a = a' - a$.
When we add or subtract equation 1 and 1' and likewise for 2 and 2', we obtain

$$
\begin{aligned}
&a): && \ln\Big(4\cosh\big[4K+h\big]\cosh\big[4K-h\big]\Big) = 2\Big(\Delta a + 6\overline{K} + c\Big)\\
&a'): && \ln\Big(\cosh\big[4K+h\big]/\cosh\big[4K-h\big]\Big) = 8\big(h' + b\big)\\
&b): && 2\ln\Big(4\cosh\big[2K+h\big]\cosh\big[2K-h\big]\Big) = 4\Big(\Delta a - c\Big)\\
&b'): && 2\ln\Big(\cosh\big[2K+h\big]/\cosh\big[2K-h\big]\Big) = 8\Big(h' - b\Big)\ .
\end{aligned}
$$

Equation 3) yields

$$
c)\Delta a + c = \ln\Big(2\cosh\big[h\big]\Big) + 2\overline{K}
$$

Inserting c) into a) gives

$$
\begin{aligned}
\tilde{a}): \quad \ln\Big(4\cosh\big[4K+h\big]\cosh\big[4K-h\big]\Big) &= 2\ln\Big(2\cosh\big[h\big]\Big) + 16\overline{K}\\
16\overline{K} &= \ln\left(\frac{\cosh\big[4K+h\big]\cosh\big[4K-h\big]}{\cos^2(\big[h\big]}\right)
\end{aligned}
$$

We see that $\overline{K}$ only depends on K and h If we add a') and b') we obtain

$$
16h' = \ln\left(\frac{\cosh\big[4K+h\big]\cosh^2\big[2K+h\big]}{\cosh\big[4K-h\big]\cosh^2\big[2K-h\big]}\right)\ .
$$

Also h' only depends on h and K. Especially for $h = 0$, we find $h' = 0$. So the renormalization does not introduce a B-field, when we start with $B = 0$. In the following we will consider only the case $h = 0$. Then we obtain

$$\overline{K} = \frac{1}{16} \ln\big(\cosh^2[4K]\big) = \frac{1}{8} \ln(\cosh[4K]) \ .$$

First approximation, ignoring nnn interaction

$\overline{K}$ describes the nn coupling in the decimated lattice (e.g. for the spins S_8 and S_{12}. Integrating out S_{13} also mediates a coupling between theses spins. AS the contributions add up, we already have for the new nn coupling between site 8 and 12

$$K' = 2\overline{K} \ .$$

Moreover, the coupling for the new nn and nnn pairs is ferromagnet and support the same orientation. As a first approximation, we can therefore ignore the nnn coupling and increase the nn coupling by that constant. Hence

$$K' = 3\overline{K} = \frac{3}{8} \ \ln\Bigg(\cosh(4K) \Bigg) \ .$$

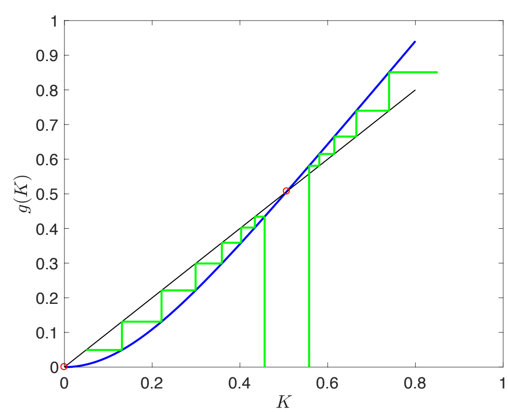

Figure 3.3: **Flux of the renormalization iterations.**
t

In figure 3.3 the mapping of the coupling parameter K during the renoirmalization steps is depicted. We find two fixed points, $K_1^* = 0$ and $K_2^* = 0.507$. The first one is stable, while the second one is unstable. If we start with a physical parameter $K > K_2^*$ the interaction leads to $K = \infty$, while if we