Generalised 1D Brownian motion

This problem is about generalised one-dimensional Brownian motion, which is mathematically similar to the treatment of collisional processes in plasmas. We consider a heavy Brownian particle (dust) in a background of many light particles (molecules) where the background has temperature T. The background particles randomly hit the Brownian particle and influence its dynamics by collisions.

1. First consider the collisionless case for Newtonian Mechanics of a single Brownian particle of mass M at position $X(t)$ and velocity $V(t)$. (Hint: don't think in a complicated way)

   (a) What are the equations of motion for $X(t)$ and $V(t)$ if a force $F(X(t))$ is acting on the particle? Write them in the form

   $$\frac{d}{dt}X(t) = \ldots, \tag{1}$$
   $$\frac{d}{dt}V(t) = \ldots. \tag{2}$$

   (b) If we approximate the differential timestep dt by a small but finite step Δt starting at time t, what are the approximate changes in position $\Delta X_{\mathrm{orb}} = X(t+\Delta t) - X(t)$ and velocity $\Delta V_{\mathrm{orb}} = V(t+\Delta t) - V(t)$ after the time Δt has passed?

   $$\Delta X_{\mathrm{orb}} = \ldots, \tag{3}$$
   $$\Delta V_{\mathrm{orb}} = \ldots. \tag{4}$$

2. Now we consider an ensemble of non-interacting identical Brownian particles, still neglecting collisions with the background. In the force-free case $F = 0$, the evolution of their probability density $f(x, v, t)$ in phase-space with coordinates (x, v) is given by

   $$\frac{\partial f(x,v,t)}{\partial t} + v\frac{\partial f(x,v,t)}{\partial x} = 0. \tag{5}$$

   To distinguish the (x, v) used as an independent variables here, they are denoted by small letters. This is in contrast to the orbit quantities $X(t)$, $V(t)$ in the particle picture, being functions of time.

   (a) What is the mathematical and physical connection to the particle picture of 1.? (Hint: use a total time derivative of f with $x = X(t)$, $v = V(t)$ from 1a with $F = 0$)

   (b) What changes in Eq. (5) if we introduce a force $F(x)$?

3. If we allow collisions of the ensemble of Brownian particles with the background (not with each other), Eq. (5) is changed to

$$\frac{\partial f(x,v,t)}{\partial t} + v\frac{\partial f(x,v,t)}{\partial x} = \nu_c \frac{\partial}{\partial v}\left(v_T^2 \frac{\partial f(x,v,t)}{\partial v} + v\, f(x,v,t)\right) . \tag{6}$$

Here, the zero on the right-hand side in Eq. (5) has been replaced by a Fokker-Planck collision operator, where the collision frequency ν_c measures how frequently collisions occur in time, and the thermal velocity $v_T = \sqrt{2T/M}$ is the average velocity magnitude of the Brownian particle ensemble if it had the same temperature T as the background.

   (a) For a homogenous system with $\partial f/\partial x = 0$, what is the stationary distribution f_∞ for which f does not change in time anymore?

   (b) What is its physical meaning of f_∞? Does ν_c appear in f_∞? Why/why not? Explain mathematically and physically.

   (c) What is the stationary distribution in the general case with $\partial f/\partial x \neq 0$ with a conservative force $F(x)$ acting as in 2b)?

4. Switching back from the ensemble to the particle picture, we can represent the orbit of the Brownian particle including randomisation by collisions with the background using

$$V(t+\Delta t) - V(t) = \Delta V_{\mathrm{orb}} - \nu_c V(t)\Delta t + \sqrt{2\nu_c v_T^2}\Theta\sqrt{\Delta t}\,. \tag{7}$$

Here, Δt is a sufficiently small time-step, Θ is a random number from the standard normal distribution and ΔV_{orb} is taken from 1b).

   (a) Based on the similar form of ΔV_{orb} of 1b), how can the second term on the right-hand side of Eq. (7) be interpreted in terms of a force acting on the Brownian particle?

   (b) Where does this so-called drag force appear in the Fokker-Planck equation? (Hint: move a term from the operator on the right-hand side to the left-hand side)

   (c) How does an increase or decrease of the collision frequency ν_c affect the system's approach towards the stationary state over time?