Dynamics and thermalization in isolated quantum systems

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QCD Hadronization and the Statistical Model
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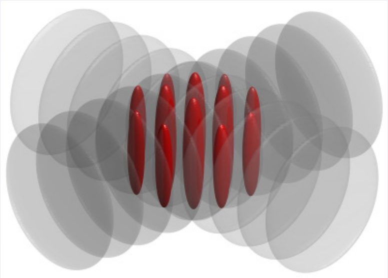
With comments
by H.G. Evertz

Outline

Experiments with ultracold gases in 1D



Effective one-dimensional δ potential

M. Olshanii, PRL **81**, 938 (1998).

$$U_{1D}(x) = g_{1D}\delta(x)$$

where

$$g_{1D} = \frac{2\hbar a_s \omega_\perp}{1 - C a_s \sqrt{\frac{m\omega_\perp}{2\hbar}}}$$

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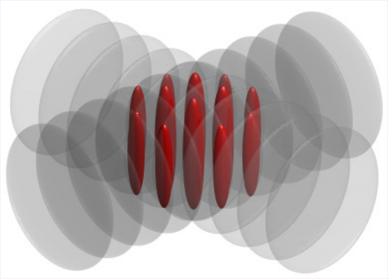

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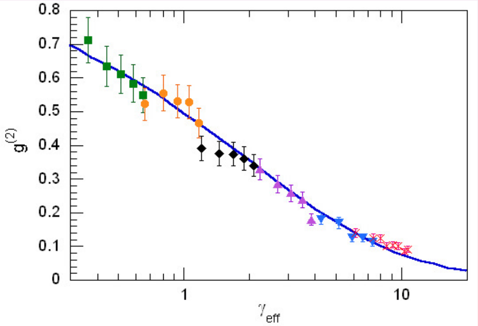
Girardeau '60, Lieb and Liniger '63

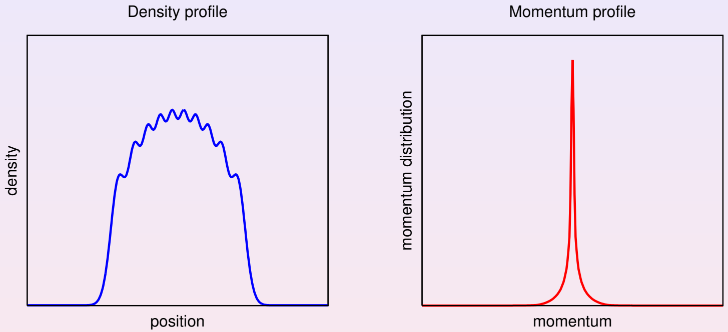
T. Kinoshita, T. Wenger, and D. S. Weiss, Science **305**, 1125 (2004).

T. Kinoshita, T. Wenger, and D. S. Weiss, Phys. Rev. Lett. **95**, 190406 (2005).

$\gamma_{\text{eff}} = \frac{m g_{1D}}{\hbar^2 \rho}$



Absence of thermalization in 1D?



T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440**, 900 (2006).

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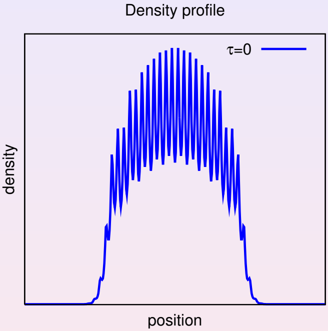

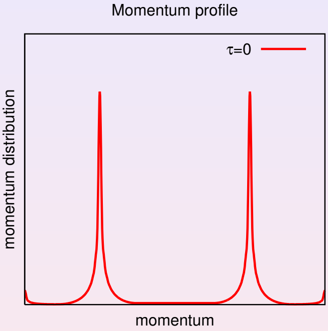

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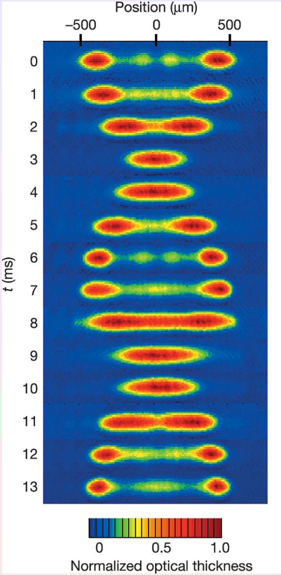
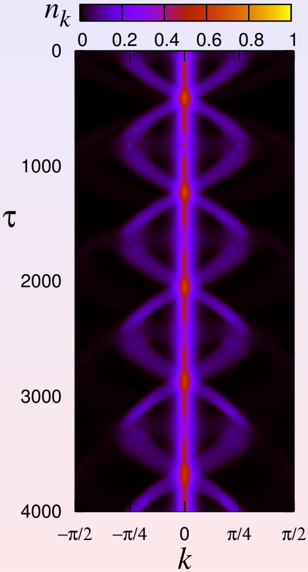
Experiment

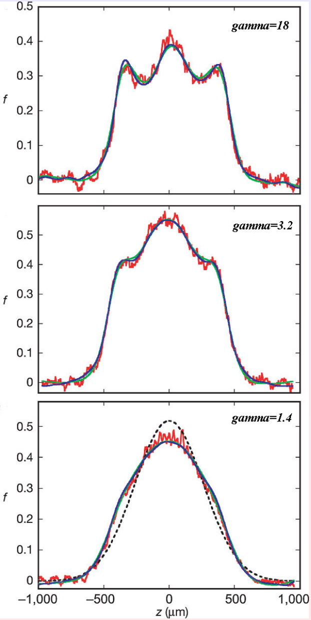




Theory

Periodic movement: no thermalization

Absence of thermalization in 1D?



T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440**, 900 (2006).

$$\gamma = \frac{m g_{1D}}{\hbar^2 \rho}$$

g_{1D}: Interaction strength
ρ: One-dimensional density

If $\gamma \gg 1$ the system is in the strongly correlated Tonks-Girardeau regime

If $\gamma \ll 1$ the system is in the weakly interacting regime

Gring *et al.*, Science **337**, 1318 (2012).

Outline

Classical physics:

Chaotic evolution --> ergodicity (uniform)
--> thermal description

(Exception: integrable systems:
many conserved quantities
--> orbits in phase space,
not ergodic)

Quantum physics:

NOT ergodic ! (only a tiny part of Hilbert space is relevant)

Is there thermalization of an isolated system?

In what sense ?

Need to consider states, observables, matrix elements

(Integrable systems again do not thermalize)

Exact results from quantum mechanics

If the initial state is not an eigenstate of $\widehat{H}$

$$|\psi_0\rangle \neq |\alpha\rangle \quad \text{where} \quad \widehat{H}|\alpha\rangle = E_\alpha|\alpha\rangle \quad \text{and} \quad E_0 = \langle\psi_0|\widehat{H}|\psi_0\rangle,$$

then a generic observable O will evolve in time following

$$O(\tau) \equiv \langle\psi(\tau)|\widehat{O}|\psi(\tau)\rangle \quad \text{where} \quad |\psi(\tau)\rangle = e^{-i\widehat{H}\tau}|\psi_0\rangle.$$

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What is it that we call thermalization?

Thermal expectation value depends only on **a few parameters** !

$$\overline{O(\tau)} = O(E_0) = O(T) = O(T, \mu).$$

"diagonal ensemble" =? microcan. =? canonical =? grand canonical ensemble (in therm.dyn. limit)

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One can rewrite

$$O(\tau) = \sum_{\alpha',\alpha} C^\star_{\alpha'} C_\alpha e^{i(E_{\alpha'}-E_\alpha)\tau} O_{\alpha'\alpha} \quad \text{where} \quad |\psi_0\rangle = \sum_\alpha C_\alpha|\alpha\rangle,$$

and taking the infinite time average (diagonal ensemble) (=definition(!) of "diag. ensemble")

(could also integrate from some later time)

$$\overline{O(\tau)} = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau d\tau'\langle\Psi(\tau')|\hat{O}|\Psi(\tau')\rangle \boxed{=} \sum_\alpha |C_\alpha|^2 O_{\alpha\alpha} \equiv \langle\hat{O}\rangle_{\text{diag}}$$

(since all oscillating terms vanish after integral)

which depends on the initial conditions through $C_\alpha = \langle\alpha|\psi_0\rangle$.

Many parameters !

Width of the energy density after a sudden quench

Initial state $|\psi_0\rangle = \sum_\alpha C_\alpha |\alpha\rangle$ is an eigenstate of $\widehat{H}_0$. At $\tau = 0$ (Example)

"Global quench" $\boxed{\widehat{H}_0 \to \widehat{H} = \widehat{H}_0 + \widehat{W}}$ with $\widehat{W} = \sum_j \hat{w}(j)$ and $\widehat{H}|\alpha\rangle = E_\alpha|\alpha\rangle$.

MR, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).

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The width of the weighted energy density ΔE is then Delta E = sqrt(<H^2> - <H>^2) =

$$\Delta E = \sqrt{\sum_\alpha E_\alpha^2 |C_\alpha|^2 - (\sum_\alpha E_\alpha |C_\alpha|^2)^2} = \sqrt{\langle\psi_0|\widehat{W}^2|\psi_0\rangle - \langle\psi_0|\widehat{W}|\psi_0\rangle^2},$$

or

$$\Delta E = \sqrt{\sum_{j_1, j_2 \in \sigma} [\langle\psi_0|\hat{w}(j_1)\hat{w}(j_2)|\psi_0\rangle - \langle\psi_0|\hat{w}(j_1)|\psi_0\rangle\langle\psi_0|\hat{w}(j_2)|\psi_0\rangle]} \overset{N\to\infty}{\propto} \sqrt{N},$$

(unless terms in the sum are highly correlated!)

where N is the total number of lattice sites.

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where N is the total number of lattice sites.
Since the width W of the full energy spectrum is $\propto N$

$$\Delta\epsilon = \frac{\Delta E}{W} \overset{N\to\infty}{\propto} \frac{1}{\sqrt{N}},$$

so, as in any thermal ensemble, $\Delta\epsilon$ vanishes in the thermodynamic limit.

MR, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).

Outline

Relaxation dynamics of hard-core bosons in 2D

Hard-core boson Hamiltonian

(Equivalent to spin 1/2 Quantum Heisenberg model)

$$\widehat{H} = -J \sum_{\langle i,j \rangle} \left(\hat{b}_i^\dagger \hat{b}_j + \text{H.c.} \right) + U \sum_{\langle i,j \rangle} \hat{n}_i \hat{n}_j, \qquad \hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

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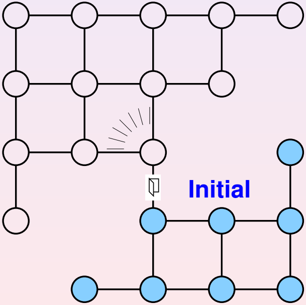
Nonequilibrium dynamics in 2D



Weak n.n. $U = 0.1J$

$N_b = 5$ bosons

$N = 21$ lattice sites

Hilbert space: $D = 20349$

All states are used!

Initial state: single occupation number state.
During the time evolution, ALL occupation number states occur.

Hard-core boson Hamiltonian

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"One can rewrite

(quote from page 8)

$$O(\tau) = \sum_{\alpha',\alpha} C^\star_{\alpha'} C_\alpha e^{i(E_{\alpha'} - E_\alpha)\tau} O_{\alpha'\alpha} \quad \text{where} \quad |\psi_0\rangle = \sum_\alpha C_\alpha |\alpha\rangle,$$

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Relaxation dynamics of hard-core bosons in 2D

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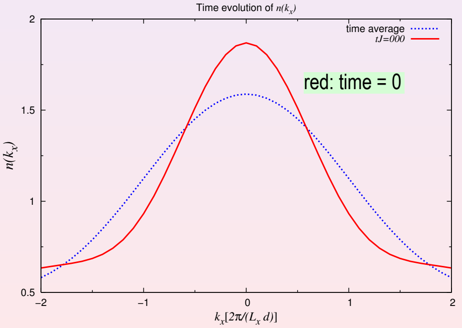
Nonequilibrium dynamics in 2D

n_k = < c^dagger_k c_k >



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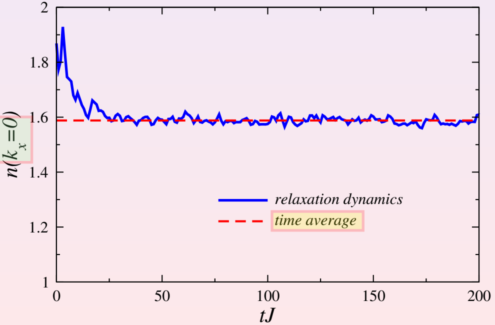

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Outline

Statistical description after relaxation

Canonical calculation

$$O = \mathrm{Tr}\left\{\hat{O}\hat{\rho}\right\}$$

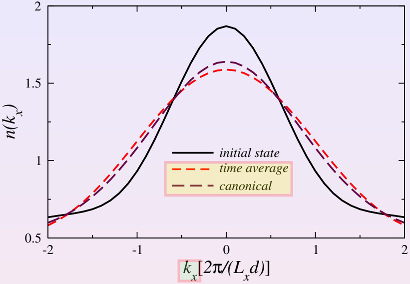
$$\hat{\rho} = Z^{-1}\exp\left(-\hat{H}/k_B T\right)$$

$$Z = \mathrm{Tr}\left\{\exp\left(-\hat{H}/k_B T\right)\right\}$$

$$E_{\cancel{0}} = \mathrm{Tr}\left\{\hat{H}\hat{\rho}\right\} \qquad T = 1.9J$$

(T: best match)

Momentum distribution



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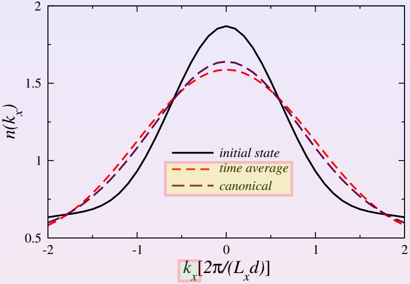


Microcanonical calculation

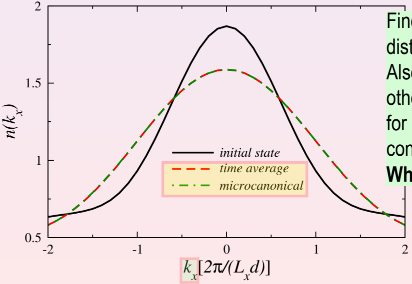
$$O = \frac{1}{N_{states}} \sum_{\alpha} \langle \Psi_\alpha | \hat{O} | \Psi_\alpha \rangle$$

= eigenstates

with $E_0 - \Delta E < E_\alpha < E_0 + \Delta E$

N_{states} : # of states in the window

E_0: energy of initial state !
Delta E: energy width of initial state (small!)



Finding same distribution ! Also in many other systems, for many initial conditions.
Why ?

Outline

Eigenstate thermalization hypothesis

Paradox?

$$\sum_{\boxed{\alpha}} \boxed{|C_\alpha|^2} O_{\alpha\alpha} \stackrel{!?}{=} \langle O \rangle_{\text{microcan.}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{\boxed{|E_0-E_\alpha|<\Delta E}} \boxed{}\ O_{\alpha\alpha} \qquad (1)$$

Left hand side: Depends on the initial conditions through $C_\alpha = \langle \Psi_\alpha | \psi_I \rangle$
Right hand side: Depends only on the initial energy

Eigenstate thermalization hypothesis

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$$\sum_{\alpha} |C_\alpha|^2 O_{\alpha\alpha} = \langle O\rangle_{\text{microcan.}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{|E_0-E_\alpha|<\Delta E} O_{\alpha\alpha}$$

Left hand side: Depends on the initial conditions through $C_\alpha = \langle\Psi_\alpha|\psi_I\rangle$
Right hand side: Depends only on the initial energy

0) Usually required: C_alpha significant only in a small energy window around E_0
(or suitable cancellations, see e.g. page 16)

i) ~~For physically relevant initial conditions, $|C_\alpha|^2$ practically do not fluctuate. (in alpha)~~ (not typically true, see next page)

ii) Large (and uncorrelated) (with Oaa) fluctuations occur in ~~both $O_{\alpha\alpha}$ and~~ $|C_\alpha|^2$. ~~A physically relevant~~ Then the initial state performs an unbiased sampling of $O_{\alpha\alpha}$. i.e., then the equality (1) is true.

MR and M. Srednicki, PRL **108**, 110601 (2012).

or iii) The matrix elements O_aa of the observable are almost constant in the relevant energy range.

Then equality (1) is true AND also equal to exp. value <O> in a SINGLE eigenstate within the energy window !!

Paradox?

$$\sum_{\alpha} |C_\alpha|^2 O_{\alpha\alpha} = \langle O \rangle_{\text{microcan.}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{|E_0 - E_\alpha| < \Delta E} O_{\alpha\alpha}$$

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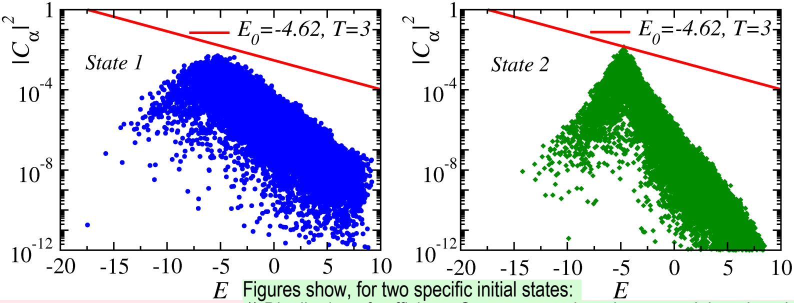

Figures show, for two specific initial states:
1) Distribution of cofficients C_a are smooth and exponential, and peaked at E_0
2) A thermal state with the same <E> (red line) has a very different distribution

MR, PRA **82**, 037601 (2010).

Eigenstate thermalization hypothesis

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Left hand side: Depends on the initial conditions through $C_\alpha = \langle \Psi_\alpha | \psi_I \rangle$
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Potential explanations:

i) For physically relevant initial conditions, $|C_\alpha|^2$ practically do not fluctuate.

ii) Large (and uncorrelated) fluctuations occur in both $O_{\alpha\alpha}$ and $|C_\alpha|^2$. A physically relevant initial state performs an unbiased sampling of $O_{\alpha\alpha}$.
MR and M. Srednicki, PRL **108**, 110601 (2012).

(see p.15 for commented version)

Eigenstate thermalization hypothesis

Paradox?

$$\sum_{\alpha} |C_\alpha|^2 O_{\alpha\alpha} = \langle O \rangle_{\text{microcan.}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{|E_0 - E_\alpha| < \Delta E} O_{\alpha\alpha}$$

Left hand side: Depends on the initial conditions through $C_\alpha = \langle \Psi_\alpha | \psi_I \rangle$
Right hand side: Depends only on the initial energy

Eigenstate thermalization hypothesis (ETH)
[J. M. Deutsch, PRA **43** 2046 (1991); M. Srednicki, PRE **50**, 888 (1994); MR, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).]

This is a consequence of (iii) on page 15

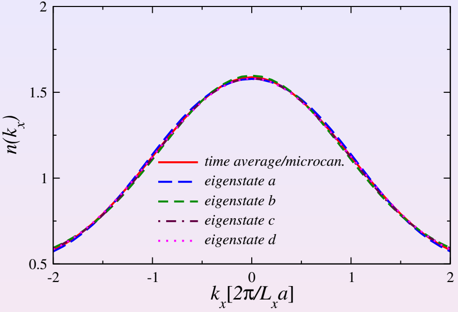
iii) The expectation value $\langle \Psi_\alpha | \hat{O} | \Psi_\alpha \rangle$ of a few-body observable $\widehat{O}$ in an eigenstate of the Hamiltonian $|\Psi_\alpha\rangle$, with energy E_α, of a large interacting many-body system equals the thermal average of $\widehat{O}$ at the mean energy E_α:

$$\langle \Psi_\alpha | \hat{O} | \Psi_\alpha \rangle = \langle O \rangle_{\text{microcan.}}(E_\alpha)$$

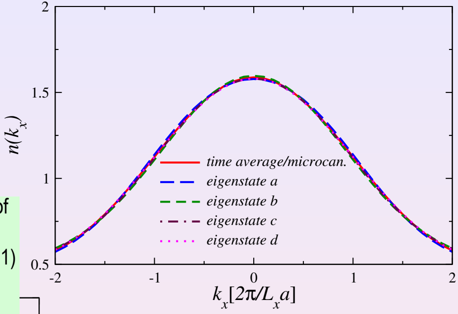
Eigenstate thermalization hypothesis

Momentum distribution

Eigenstates $a - d$ are the ones with energies closest to E_0



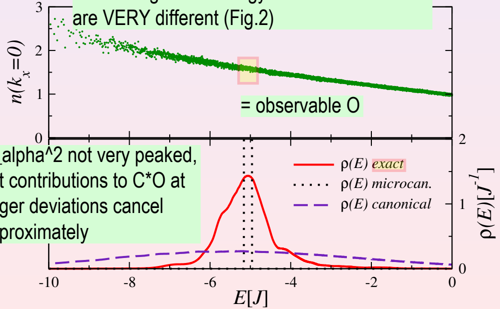
Eigenstate thermalization hypothesis



Momentum distribution

Eigenstates $a-d$ are the ones with energies closest to E_0

Figures show: the momentum distributions of (1) time average, (2) microcan., and (3) several eigenstates are very similar (Fig.1) even though their energy distributions are VERY different (Fig.2)



= observable O

C_alpha^2 not very peaked, but contributions to C*O at larger deviations cancel approximately

$n(k_x=0)$ vs energy

$\rho(E) = \mathrm{P(E)} \times \mathrm{dens.\ stat.}$

$P(E)_{\text{exact}} \to |C_\alpha|^2$ ("exact" refers to time average)

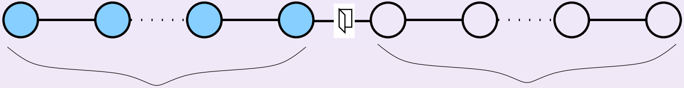
$P(E)_{\text{mic}} \to \text{constant}$

$P(E)_{\text{can}} \to \exp\left(-E/k_B T\right)$

One-dimensional integrable case

Similar experiment in one dimension

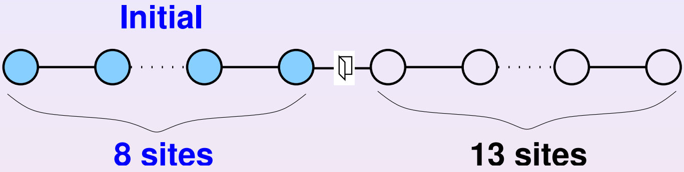
Initial



8 sites **13 sites**

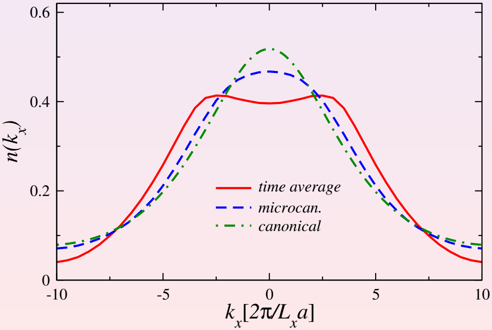
One-dimensional integrable case

Similar experiment in one dimension



Time average vs Stat. Mech.

No thermalization!

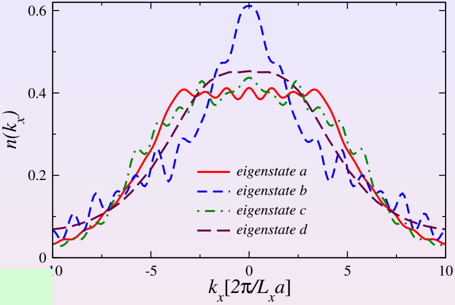


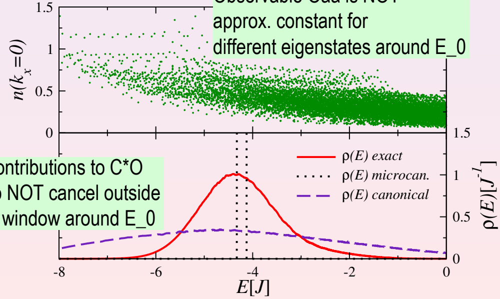
Breakdown of eigenstate thermalization

Momentum distribution

Eigenstates $a - d$ are the ones with energies closest to E_0

Very different from page 16





Observable Oaa is NOT approx. constant for different eigenstates around E_0

Contributions to C*O do NOT cancel outside of window around E_0

$n(k_x = 0)$ vs energy

$\rho(E) = \mathrm{P(E)} \times \text{dens. stat.}$

$P(E)_{\text{exact}} \rightarrow |C_\alpha|^2$ ("exact" refers to time average)

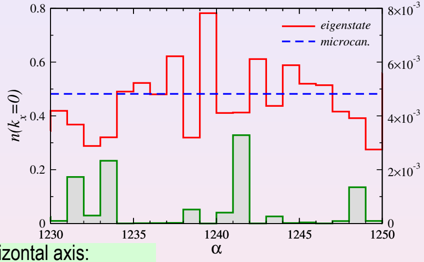
$P(E)_{\text{mic}} \rightarrow \text{constant}$

$P(E)_{\text{can}} \rightarrow \exp(-E/k_B T)$

Integrable vs Nonintegrable cases

Correlations between $n(k)$ and C_α

1D (integrable) case

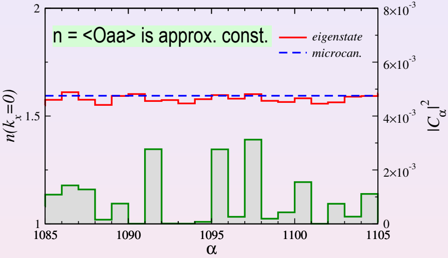


horizontal axis:
successive eigenstates
(Green bars: C_alpha)

$\Downarrow$

Conservation laws play a role in integrable models.

2D (nonintegrable) case



$\Downarrow$

Correlations are not relevant, ~~and they are not present~~!

Transition between integrability and nonintegrability:

MR, PRL **103**, 100403 (2009); PRA **80**, 053607 (2009).

Outline

(not an integrable model)

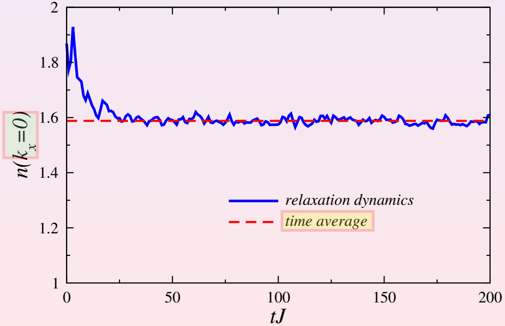
Hard-core boson Hamiltonian

$$\widehat{H} = -J \sum_{\langle i,j \rangle} \left(\hat{b}_i^\dagger \hat{b}_j + \text{H.c.} \right) + U \sum_{\langle i,j \rangle} \hat{n}_i \hat{n}_j, \qquad \hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

(Same slide as last part of "page 11")

(Equivalent to spin 1/2 Quantum Heisenberg model)

Nonequilibrium dynamics in 2D



Weak n.n. $U = 0.1J$

$N_b = 5$ bosons

$N = 21$ lattice sites

Hilbert space: $D = 20349$

All states are used!

Time fluctuations

Are they small because of dephasing?

$$\langle\hat{O}(t)\rangle - \overline{\langle\hat{O}(t)\rangle} = \sum_{\substack{\alpha',\alpha \\ \alpha'\neq\alpha}} C^\star_{\alpha'} C_\alpha e^{i(E_{\alpha'}-E_\alpha)t} O_{\alpha'\alpha} \sim \sum_{\substack{\alpha',\alpha \\ \alpha'\neq\alpha}} \frac{e^{i(E_{\alpha'}-E_\alpha)t}}{N_{\text{states}}} O_{\alpha'\alpha}$$

$$\sim \frac{\sqrt{N^2_{\text{states}}}}{N_{\text{states}}} O^{\text{typical}}_{\alpha'\alpha} \sim O^{\text{typical}}_{\alpha'\alpha}$$

sqrt(N^2) results when the exponential
is randomly distributed (why should it ?)
Then all phases cancel,
i.e. only alpha=alpha' contributes

Time fluctuations

Are they small because of dephasing?

$$\langle \hat{O}(t)\rangle - \overline{\langle \hat{O}(t)\rangle} = \sum_{\substack{\alpha',\alpha \\ \alpha'\neq\alpha}} C^{\star}_{\alpha'} C_{\alpha} e^{i(E_{\alpha'}-E_{\alpha})t} O_{\alpha'\alpha} \sim \sum_{\substack{\alpha',\alpha \\ \alpha'\neq\alpha}} \frac{e^{i(E_{\alpha'}-E_{\alpha})t}}{N_{\text{states}}} O_{\alpha'\alpha}$$

$$\sim \frac{\sqrt{N^2_{\text{states}}}}{N_{\text{states}}} O^{\text{typical}}_{\alpha'\alpha} \sim O^{\text{typical}}_{\alpha'\alpha}$$

Time average of $\langle \hat{O}\rangle$

$$\overline{\langle \hat{O}\rangle} = \sum_{\alpha} |C_{\alpha}|^2 O_{\alpha\alpha}$$

$$\sim \sum_{\alpha} \frac{1}{N_{\text{states}}} O_{\alpha\alpha} \sim O^{\text{typical}}_{\alpha\alpha}$$

Time fluctuations

Are they small because of dephasing?

$$\langle\hat O(t)\rangle - \overline{\langle\hat O(t)\rangle} = \sum_{\substack{\alpha',\alpha \\ \alpha'\neq\alpha}} C^\star_{\alpha'} C_\alpha e^{i(E_{\alpha'}-E_\alpha)t} O_{\alpha'\alpha} \sim \sum_{\substack{\alpha',\alpha \\ \alpha'\neq\alpha}} \frac{e^{i(E_{\alpha'}-E_\alpha)t}}{N_{\text{states}}} O_{\alpha'\alpha}$$

NOT the time average!
(see page 8)

from subtraction

(when C_alpha is sharply peaked)

$$\sim \frac{\sqrt{N^2_{\text{states}}}}{N_{\text{states}}} O^{\text{typical}}_{\alpha'\alpha} \sim O^{\text{typical}}_{\alpha'\alpha}$$

Time average of $\langle\hat O\rangle$

$$\overline{\langle\hat O\rangle} = \sum_{\alpha} |C_\alpha|^2 O_{\alpha\alpha}$$

(when ETH is valid):

$$\sim \sum_{\alpha} \frac{1}{N_{\text{states}}} O_{\alpha\alpha} \sim O^{\text{typical}}_{\alpha\alpha}$$

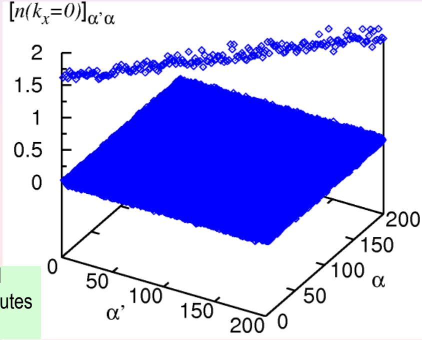
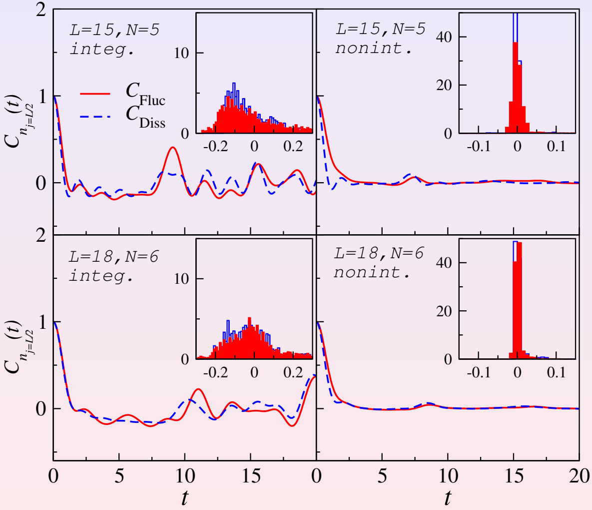


Figure shows that indeed only alpha=alpha' contributes to the expectation value

Fluctuation-dissipation theorem (dipolar bosons)

Occupation in the center of the trap ($n_{j=L/2}$)



Hamiltonian

$$\hat{H} = -J \sum_{j=1}^{L-1} \left(\hat{b}_j^\dagger \hat{b}_{j+1} + \text{H.c.} \right)$$

$$+V \sum_{j<l} \underbrace{\frac{\hat{n}_j \hat{n}_l}{|j-l|^3}}_{\text{"dipolar IA"}} + g \sum_j \underbrace{x_j^2}_{\text{trap}} \hat{n}_j$$

magnetic atoms, polar molecules

Relaxation dynamics

$$O(t) = C(t) O(t=0)$$

where

$$C(t) = \frac{\overline{O(t+t')O(t')}}{\overline{(O(t'))^2}}$$

Srednicki, JPA **32**, 1163 (1999).

E. Khatami, G. Pupillo, M. Srednicki, and MR, PRL **111**, 050403 (2013).

Outline

Bose-Fermi mapping

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + \sum_i v_i \, \hat{n}_i$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

Bose-Fermi mapping

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + \sum_i v_i \ \hat{n}_i$$

Here: no density-density interaction !

Constraints on the bosonic operators

Since n_i = b^dagger_i b_i, this Hamiltonian is bilinear in creation and annihilation operators and the eigenstates are products of single particle states. (Special case v_i=const: momentum space states)

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

$$\Downarrow$$

Map to spins and then to fermions (Jordan-Wigner transformation)

$$\sigma_i^+ = \hat{f}_i^\dagger \prod_{\beta=1}^{i-1} e^{-i\pi \hat{f}_\beta^\dagger \hat{f}_\beta}, \quad \sigma_i^- = \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i$$

= (-1)^(number of fermions before site i)
==> in 1d provides for anticommutation

$$\Downarrow$$

Signs cancel as long as fermions cannot hop past each other !

Mapping results in

Non-interacting fermion Hamiltonian

$$\hat{H}_F = -J \sum_i \left(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.} \right) + \sum_i v_i \ \hat{n}_i^f$$

(This is an integrable model)

One-particle density matrix

One-particle Green's function

$$G_{ij} = \langle \Psi_{HCB} | \sigma_i^- \sigma_j^+ | \Psi_{HCB} \rangle = \langle \Psi_F | \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i \hat{f}_j^\dagger \prod_{\gamma=1}^{j-1} e^{-i\pi \hat{f}_\gamma^\dagger \hat{f}_\gamma} | \Psi_F \rangle$$

For non-neighboring sites i,j each interchange of fermions gives a sign -> need to cancel by e-factors

$$\Downarrow$$

Time evolution

$$|\Psi_F(\tau)\rangle = e^{-i\hat{H}_F \tau/\hbar} |\Psi_F^I\rangle = \prod_{\delta=1}^{N} \sum_{\sigma=1}^{L} P_{\sigma\delta}(\tau) \hat{f}_\sigma^\dagger \, |0\rangle$$

One-particle density matrix

One-particle Green's function

$$G_{ij} = \langle \Psi_{HCB} | \sigma_i^- \sigma_j^+ | \Psi_{HCB} \rangle = \langle \Psi_F | \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i \hat{f}_j^\dagger \prod_{\gamma=1}^{j-1} e^{-i\pi \hat{f}_\gamma^\dagger \hat{f}_\gamma} | \Psi_F \rangle$$

For non-neighboring sites i,j
each interchange of fermions
gives a sign -> cancelled
by e-factors

$$\Downarrow$$

Time evolution

from an initial Fock state

remains a Fock state,
which can be written as:

$$|\Psi_F(\tau)\rangle = e^{-i\hat{H}_F \tau/\hbar} |\Psi_F^I\rangle = \prod_{\delta=1}^{N} \sum_{\sigma=1}^{L} P_{\sigma\delta}(\tau) \hat{f}_\sigma^\dagger \, |0\rangle$$

$$\Downarrow$$

Fock state: prod_k (c^dagger_k)^(n_k) /0>
= prod_k sum_x e^(i k x n_k) c^dagger_x /0>
= prod_n (sum_x e^(i k_n x) c^dagger_x /0>
= prod_n sum_x P_(k_n x) c^dagger_x /0>

Exact Green's function

$$G_{ij}(\tau) = \det \left[\left(\mathbf{P}^l(\tau) \right)^\dagger \mathbf{P}^r(\tau) \right]$$

Here "k" numbers the
single particle eigenstates
(= momentum states when
v_i=const in Hamiltonian)

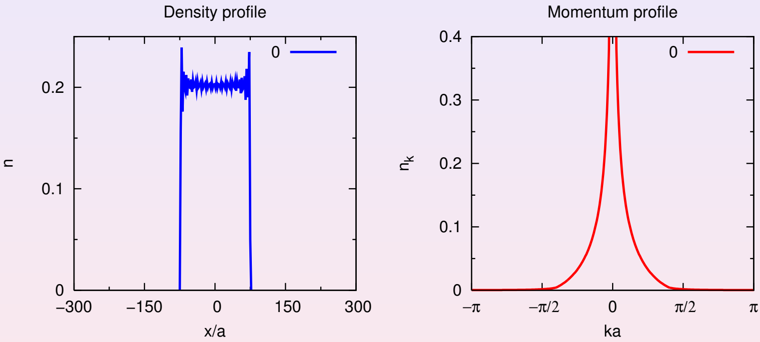
Computation time $\sim L^2 N^3$

$$3000 \text{ lattice sites}, \qquad 300 \text{ particles}$$

MR and A. Muramatsu, PRL **93**, 230404 (2004); PRL **94**, 240403 (2005).

Relaxation dynamics in an integrable system

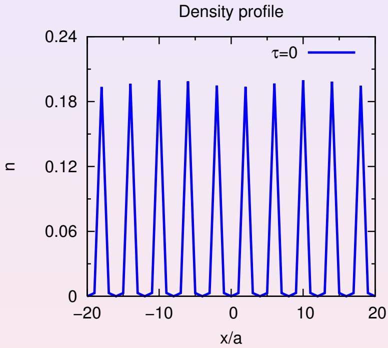
Initial state:
center part of system filled at approx. constant density

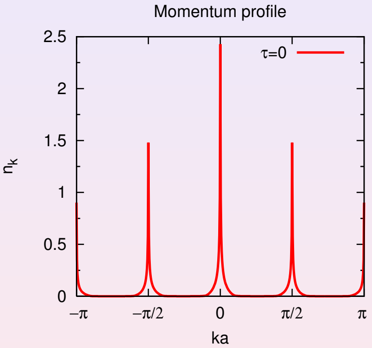


MR, V. Dunjko, V. Yurovsky, and M. Olshanii, PRL **98**, 050405 (2007).

Relaxation dynamics in an integrable system

State after long time relaxation:
combination of (a few) eigenstates





MR, V. Dunjko, V. Yurovsky, and M. Olshanii, PRL **98**, 050405 (2007).

Outline

Statistical description after relaxation (integrable system)

Thermal equilibrium

$$\hat{\rho} = Z^{-1} \exp\left[-\left(\hat{H} - \mu \hat{N}_b\right)/k_B T\right]$$

$$Z = \mathrm{Tr}\left\{\exp\left[-\left(\hat{H} - \mu \hat{N}_b\right)/k_B T\right]\right\}$$

$$E = \mathrm{Tr}\left\{\hat{H}\hat{\rho}\right\}, \quad N_b = \mathrm{Tr}\left\{\hat{N}_b \hat{\rho}\right\}$$

MR, PRA **72**, 063607 (2005).

Thermal equilibrium

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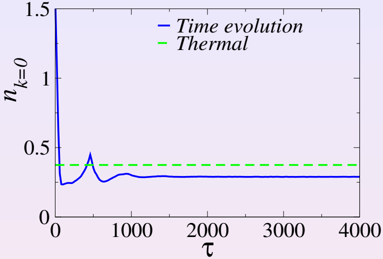
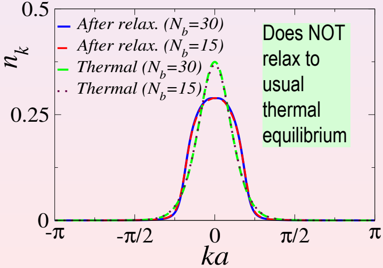
MR, PRA **72**, 063607 (2005).

Evolution of $n_{k=0}$



n_k after relaxation



Does NOT relax to usual thermal equilibrium

Thermal equilibrium

$$\hat{\rho} = Z^{-1} \exp\left[-\left(\hat{H} - \mu \hat{N}_b\right)/k_B T\right]$$

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MR, PRA **72**, 063607 (2005).

Integrals of motion

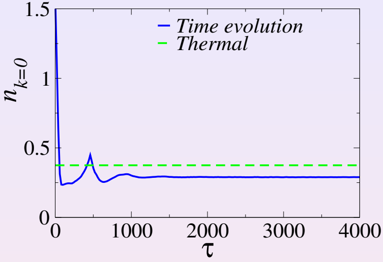
(underlying noninteracting fermions)

$$\hat{H}_F \hat{\gamma}_m^{f\dagger}|0\rangle = E_m \hat{\gamma}_m^{f\dagger}|0\rangle$$

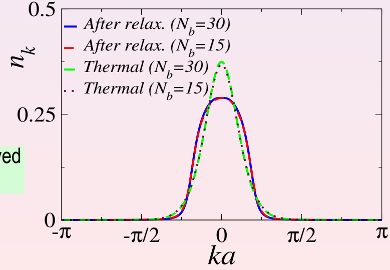
$$\left\{\hat{I}_m^f\right\} = \left\{\hat{\gamma}_m^{f\dagger}\hat{\gamma}_m^f\right\}$$

Many local conserved quantities I^f_m

Evolution of $n_{k=0}$



n_k after relaxation



Statistical description after relaxation

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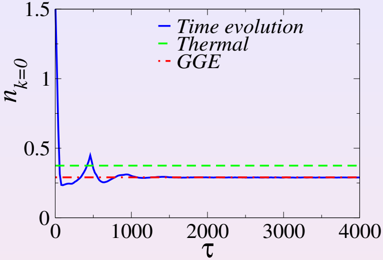
MR, PRA **72**, 063607 (2005).

Evolution of $n_{k=0}$



Generalized Gibbs ensemble

$$\hat{\rho}_c = Z_c^{-1} \exp\left[-\sum_m \lambda_m \hat{I}_m\right]$$

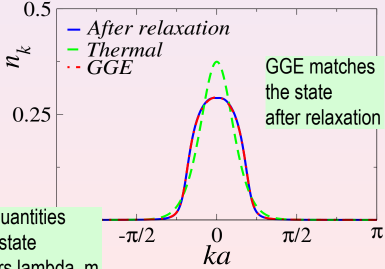
instead of e^(- H/T)

$$Z_c = \mathrm{Tr}\left\{\exp\left[-\sum_m \lambda_m \hat{I}_m\right]\right\}$$

$$\langle \hat{I}_m \rangle_{\tau=0} = \mathrm{Tr}\left\{\hat{I}_m \hat{\rho}_c\right\}$$

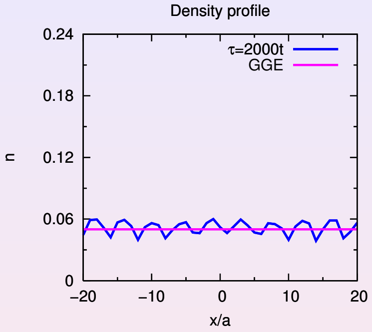
Values of local quantities I_m in the initial state fix the parameters lambda_m

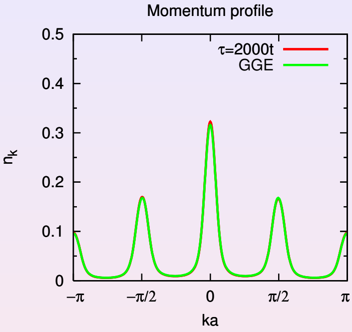
n_k after relaxation



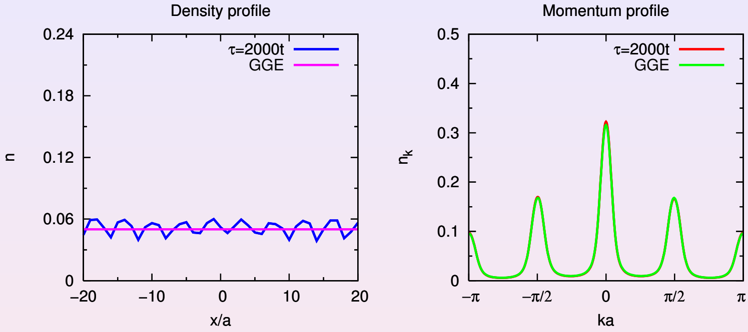
GGE matches the state after relaxation

Statistical description after relaxation





Statistical description after relaxation



Why does the GGE work?

Generalized eigenstate thermalization:

A. C. Cassidy, C. W. Clark, and MR, Phys. Rev. Lett. **106**, 140405 (2011).

K. He, L. F. Santos, T. M. Wright, and MR, Phys. Rev. A **87**, 063637 (2013).

J.-S. Caux and F. H. L. Essler, Phys. Rev. Lett. 110, 257203 (2013).

Summary

- Thermalization occurs in generic isolated systems
  ★ Finite size effects

Summary

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  ★ Finite size effects
- Eigenstate thermalization hypothesis
  ★ $\langle \Psi_\alpha | \hat{O} | \Psi_\alpha \rangle = \langle O \rangle_{\text{microcan.}}(E_\alpha)$

Summary

- Thermalization occurs in generic isolated systems
  ★ Finite size effects
- Eigenstate thermalization hypothesis
  ★ $\langle\Psi_\alpha|\hat{O}|\Psi_\alpha\rangle = \langle O\rangle_{\text{microcan.}}(E_\alpha)$
- Thermalization and ETH break down close integrability (finite system)
  ★ Quantum equivalent of KAM?

Summary

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- Small time fluctuations $\leftarrow$ smallness of off-diagonal elements

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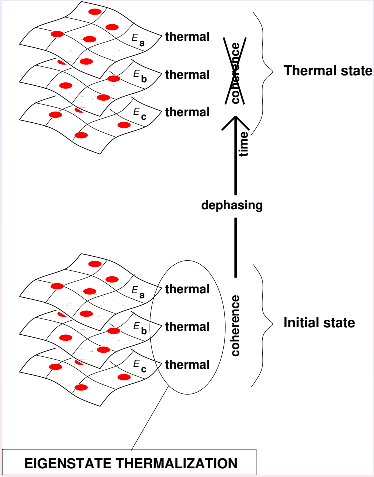
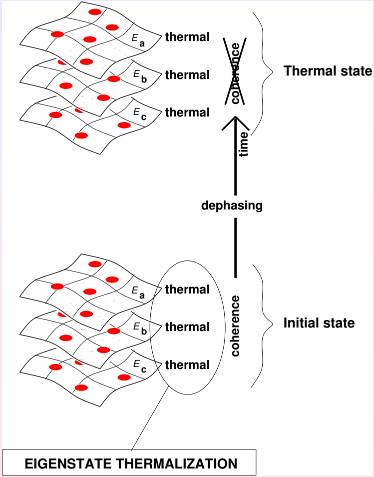


Summary

- Thermalization occurs in generic isolated systems
  ★ Finite size effects
- Eigenstate thermalization hypothesis
  ★ $\langle \Psi_\alpha | \hat{O} | \Psi_\alpha \rangle = \langle O \rangle_{\text{microcan.}}(E_\alpha)$
- Thermalization and ETH break down close integrability (finite system)
  ★ Quantum equivalent of KAM?
- Small time fluctuations $\leftarrow$ smallness of off-diagonal elements
- Time plays only an auxiliary role
- Integrable systems are different (Generalized Gibbs ensemble)

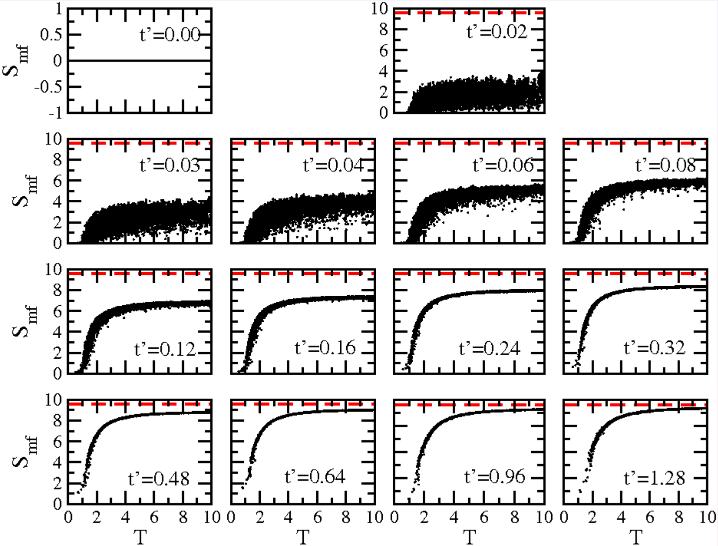


Collaborators

- Vanja Dunjko (U Mass Boston)
- Alejandro Muramatsu (Stuttgart U)
- Maxim Olshanii (U Mass Boston)
- Lea F. Santos (Yeshiva U)
- Mark Srednicki (UC Santa Barbara)
- Former group members: Kai He, Ehsan Khatami

Supported by:

Information entropy ($S_j = -\sum_{k=1}^{D} |c_j^k|^2 \ln |c_j^k|^2$)



L.F. Santos and MR, PRE **81**, 036206 (2010); PRE **82**, 031130 (2010).