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TRICRITICAL POINT IN LATTICE QED*

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The four-dimensional U(1) lattice gauge theory with the action $-\sum_p (\beta \cos \theta_p + \gamma \cos 2\theta_p)$ is studied by Monte Carlo simulation along the phase transition line separating the confinement and the Coulomb phases. The discontinuity of $\langle \cos \theta_p \rangle$, determined in the interval $0.2 \leq \gamma \leq 0.5$, is extrapolated according to a power law and shown to vanish at the tricritical point $\beta_{TCP} = 1.09 \pm 0.04$, $\gamma_{TCP} = -0.11 \pm 0.05$ (errors are systematic). A negative value of γ_{TCP} means that the phase transition in lattice QED with Wilson action ($\gamma = 0$) is of first order.

1. Introduction

Since the early days of lattice gauge theories lattice QED in 4 dimensions has been the subject of many investigations, both by analytic [1-6] and numerical [6-17] methods. One motivation has been the comparison of this lattice field theory with the familiar continuum electrodynamics. Another motivation has been of more technical nature. Lattice QED has played a major role in the development of calculational methods in lattice gauge theories, in particular in devising tools to handle the deconfining phase transition. This transition, being practically inaccessible in low-temperature lattice QCD, can be comfortably looked at from both sides in several versions of lattice QED. For example, one can investigate the mechanism underlying this transition [2-6] or observe in numerical calculations the vanishing of the string tension and the onset of the Coulomb potential between static charges [9-14]. In particular, the order of the phase transition can be studied.

Several Monte Carlo studies [7, 8, 10, 12, 13] suggested that the deconfining phase transition in lattice QED with Wilson action is of second order. For the extended Wilson action

$$S = -\sum_p (\beta \cos \theta_p + \gamma \cos 2\theta_p) ,$$

$$\theta_p = \text{plaquette angle} , \qquad \beta, \gamma = \text{coupling parameters} , \tag{1.1}$$

part of the phase boundary between the confinement and the Coulomb phases, in the interval $0.3 < \gamma < 1$, has been found, however, to be of first order [15]. The

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change of the order along a phase transition line is theoretically a rather complicated phenomenon. The experience from the models in statistical mechanics provides a warning that the vicinity of the tricritical point, where this change takes place, harbours various pitfalls for Monte Carlo calculations on finite lattices and within limited computer time. Thus, the presence of the tricritical point suggests that, with increased computer resources, a detailed reanalysis of the phase transition line in lattice QED with the action (1.1) might be worth while.

We have calculated the discontinuity ΔW_p of the mean plaquette value $W_p = \langle \cos \theta_p \rangle$ along the part $0.2 \leq \gamma \leq 0.5$ of the first-order phase transition line separating the confinement and Coulomb phases. For this part of the phase boundary ΔW_p turns out to be independent of the lattice size for 10^4 to 14^4 lattices. For γ decreasing below 0.2, phase flips occur with increasing frequency and make the measurement of ΔW_p rather unprecise. Furthermore, our estimates indicate increasing dependence of ΔW_p on lattice size with decreasing γ. In order to avoid both the phase flips and finite size effects we have therefore restricted long Monte Carlo runs required for a precise determination of ΔW_p to the values of γ above 0.2.

The values of ΔW_p have been fitted by the power law

$$\Delta W_p(\gamma) = A(\gamma - \gamma_{TCP})^{\beta_u}, \tag{1.2}$$

which is suggested by a scaling hypothesis in the tricritical region. We assume such a law, found in statistical physics models, to be valid also in U(1) lattice gauge theory. As the fit is very good, we can use it for an extrapolation of ΔW_p to lower values of γ. The value γ_{TCP} of γ where ΔW_p vanishes, determines the position of the tricritical point in the plane of couplings β, γ:

$$\beta_{TCP} = 1.09 \pm 0.04, \quad \gamma_{TCP} = -0.11 \pm 0.05 . \tag{1.3}$$

The subsidiary tricritical exponent β_u and the crossover exponent ϕ determining the shape of the phase boundary in the vicinity of the tricritical point (it will be defined in sect. 2), have the values

$$\beta_u = 1.7 \pm 0.2 , \quad \phi = 1.5 \pm 0.3 . \tag{1.4}$$

A negative value of γ_{TCP} means that lattice QED with Wilson action (coupling $\gamma = 0$ in (1.1)) has a deconfining transition of first order. The discontinuity $\Delta W_p = 0.016$ is very small, but nonzero. This improves on earlier results which have been consistent with a second-order phase transition.

It should be stressed that the difference between first- and second-order transitions in the vicinity of a tricritical point is very small and the misinterpretation of the order of the phase transition in this region does not have serious practical consequences. It is more important to realize that lattice QED with Wilson action has a deconfining phase transition dominated by a tricritical point, and not by an ordinary critical point. The exponents determined in the calculations with Wilson action are likely to be the tricritical, and not the critical ones.

To explain this difference, and to introduce the concept of tricritical points, we give a brief summary of the theory of tricritical points and of the numerical experience with them in statistical mechanics in sect. 2. In sects. 3–5 we describe the shape of the first-order phase boundary and the dependence of ΔW_p both on γ and on the lattice size. Sect. 6 contains the actual calculation of ΔW_p on lattices 10^4–14^4 and the localization of the tricritical point. The discussion of our results and some remarks about difficulties of numerical calculations in the vicinity of tricritical points follow in sect. 7.

2. Tricritical regions in statistical mechanics systems

Various physical systems and models in statistical mechanics, like He^3/He^4 mixture or metamagnets, have a line of first-order phase transitions changing in the tricritical point into a critical line of second-order phase transitions. In the vicinity of the tricritical point, the tricritical region, the behaviour of a system has to be described by a scaling theory which is more elaborate than the scaling theory of ordinary critical points. We include, for the reader's convenience, a brief summary of the properties of tricritical points which are known in statistical mechanics. A comprehensive recent review [18] and several papers [19–21] are recommended for further information.

One of the simplest models with tricritical behaviour is the Ising antiferromagnet with competing antiferromagnetic nn (nearest neighbour) and ferromagnetic nnn (next-to-nearest neighbour) couplings. The hamiltonian in d dimensions is

$$\mathcal{H} = \sum_{\text{nn}} \sigma_i \sigma_j - \sum_{\text{nnn}} \sigma_i \sigma_k + H \sum_i \sigma_i ,$$

$$\sigma_i = \pm 1 , \qquad i = (i_1, \ldots, i_d) . \tag{2.1}$$

H is here the nonordering magnetic field. The free energy is a function of H and of the temperature T, $F(T, H)$. For low values of T and H the spins tend to assume alternating orientations, constituting the antiferromagnetic phase with low magnetization. When H increases above a specific value, the wrongly oriented spins flip, and a paramagnetic phase with large magnetization arises. The corresponding phase transition is of first order at temperatures below the tricritical temperature T_{TCP}, and of second order for temperatures between T_{TCP} and the Néel temperature T_N (fig. 1a).

There is a formal analogy of this model with a ferromagnet, consisting in a twofold degeneracy of the ground state of the antiferromagnet: even spins oriented up and odd ones oriented down, or vice versa. This spontaneous symmetry breaking is characterized by the sign of the staggered magnetization

$$M_{\text{stg}} = \left\langle \sum_i (-1)^{i_1 + \cdots + i_d} \sigma_i \right\rangle . \tag{2.2}$$

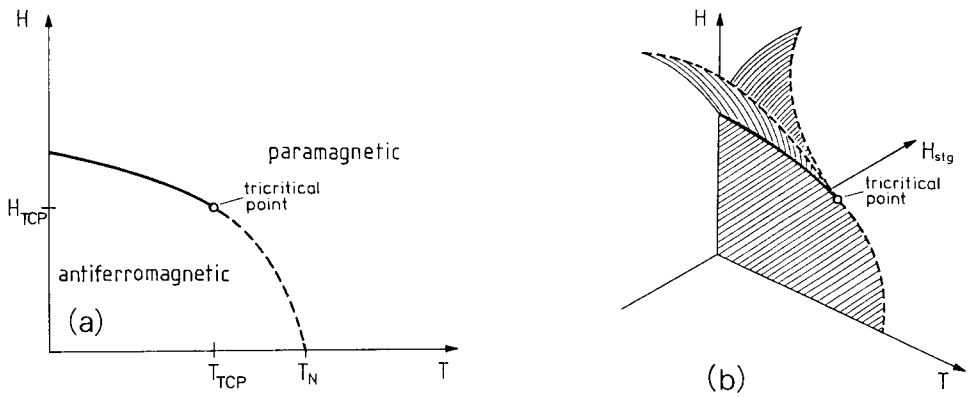

Fig. 1. Phase diagram of Ising antiferromagnet (2.1) (a) in the space of two parameters T, H, and (b) in the space of three parameters T, H, H_{stg}. The figures are adapted from ref. [18].

The degeneracy can be lifted by introducing an additional, staggered magnetic field of constant absolute value H_{stg} and with alternating sign on the lattice sites

$$\mathcal{H} \to \mathcal{H} + H_{stg} \sum_i (-1)^{i_1 + \cdots + i_d} \sigma_i . \tag{2.3}$$

H_{stg} is the thermodynamical conjugate of the order parameter M_{stg},

$$M_{stg} = -\frac{\partial}{\partial H_{stg}} F(T, H, H_{stg}) . \tag{2.4}$$

In the space of the three variables T, H and H_{stg} (fig. 1b) the first-order part of the phase boundary in the $H_{stg} = 0$ plane turns out to be a line of triple points. In this line three coexistence surfaces, separating the paramagnetic and two antiferromagnetic phases, intersect. The tricritical point is the intersection of the three critical lines forming boundaries of the coexistence surfaces.

A similar analysis can be performed for other systems with tricritical points and generalized to multi-dimensional phase diagrams. The most important conclusion is already apparent: any scaling hypothesis in the tricritical region has to take into account the presence of several critical lines emerging from the tricritical point, and the scaling laws and scaling functions will thus differ from the ordinary critical ones. Two new sets of critical exponents have to be introduced [19]. One set describes the behaviour of thermodynamical quantities when the tricritical point is approached along a path which is not asymptotically parallel to the phase boundary (tricritical exponents, labeled by index t). The other set is associated with the approach to the tricritical point along the triple line (subsidiary critical exponents, labeled by index u). For example, in the approach at constant $H = H_{TCP}$ the correlation length behaves as

$$\xi \sim |T - T_{TCP}|^{-\nu_t} . \tag{2.5}$$

The exponent ν_t differs, in general, from the ordinary critical exponent ν. A discontinuity along the triple line (e.g. the magnetization) behaves as

$$\Delta M \sim |T - T_{TCP}|^{\beta_u} . \tag{2.6}$$

For a description of the scaling hypothesis usually made in the tricritical region, we restrict ourselves to the 2-dimensional $T-H$ plane of the Ising antiferromagnet (fig. 1a). A suitable orthogonal coordinate system λ, g is introduced (fig. 2), with the origin in the tricritical point and with the $g=0$ line tangent to the critical line. The singular part F_{sing} of the free energy $F(\lambda, g) = F(T, H)$ is assumed to obey the scaling law [18, 19]

$$F_{sing}(\lambda, g) = \lambda^{\phi(2-\alpha_t)} F_{sc}^{\pm}\left(\frac{g}{|\lambda|^{\phi}}\right), \qquad \pm = \text{sign } \lambda . \tag{2.7}$$

Here ϕ is an additional, so-called crossover exponent, assumed to be greater than 1. $F_{sc}(x)$ is the scaling function and α_t is the thermal exponent.

The crossover exponent ϕ plays a multiple role. Firstly, the scaling function $F_{sc}(x)$ should reproduce the singularities of the free energy on both the critical and the coexistence lines by means of a singular behaviour at some points $x = x_{\pm}$. Thus, the scaling hypothesis (2.7) requires that, in the vicinity of the tricritical point, both the critical and the coexistence lines have shapes determined by ϕ:

$$g = x_{\pm}|\lambda|^{\phi}, \qquad \pm = \text{sign } \lambda . \tag{2.8}$$

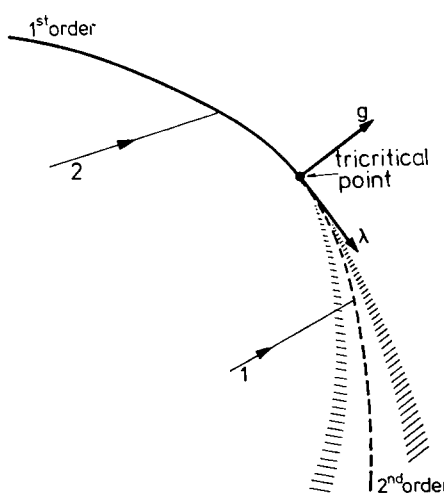

Fig. 2. Shape of the crossover regions (shaded areas) implied by the scaling hypothesis for free energy, eq. (2.7). Line 1 is an example of a path towards the second-order transition line, which is dominated by the tricritical point until a very small distance from the phase transition. This may cause misinterpretations of critical exponents in Monte Carlo calculations. On a path as line 2 towards the first-order phase boundary, the correlation length may grow because the distance to the tricritical point decreases. On a finite lattice the phase transition on this line might thus resemble a second-order transition.

As a consequence both lines are asymptotically parallel at the tricritical point, a first indication of the difficulties to localize this point precisely.

Secondly, the exponents characterizing the behaviour of observables change from their tricritical to their critical values in the so-called crossover regions. Again, the changes of the behaviour have to take place at some hopefully narrow intervals of the argument x of the scaling function $F_{sc}(x)$. Thus the crossover regions are approximately of the form

$$g \simeq A_k |\lambda|^{\phi}, \tag{2.9}$$

with various amplitudes A_k, but with a common power law determined by the crossover exponent ϕ.

Such shapes of the crossover regions, confirmed by Monte Carlo calculations [20], mean that the tricritical behaviour occurs even in some parts of the $T-H$ plane whose distance from the phase boundary is small in comparison with their distance from the tricritical point (fig. 2). This may cause confusion in Monte Carlo calculations on finite lattices. For example, a calculation in the vicinity of the critical line, even if finite size scaling is used, can produce the tricritical exponents rather than the critical ones if the crossover region (shaded in fig. 2) has not yet been passed (line 1 in fig. 2). Moreover, a first-order phase transition can be misinterpreted as a second-order one, since the correlation length, the specific heat peak etc. grow, when the first-order transition line is approached along a path on which the distance from the tricritical point decreases (line 2 in fig. 2).

Furthermore, in the vicinity of the tricritical point the dynamical difficulties of the second-order transitions (large correlation length, critical slowing down) combine with the metastability properties of the first-order transitions. Even in very long Monte Carlo runs on finite lattices a doubly peaked distribution of some measured quantity may suggest the existence of metastable states, though the transition is of second order [21]. The only warning may be a slow decrease of the distance of the peaks with growing lattice size.

3. Estimates of the mean plaquette discontinuity for $\gamma \geq 0$

Earlier Monte Carlo investigations [15] of the phase boundary between the confinement and the Coulomb phases in lattice QED with extended Wilson action (1.1) indicated in the plane of couplings β, γ the presence of a first-order transition on a line between the branching point at $\beta \simeq 0.5$, $\gamma \simeq 1$ and the point $\beta \simeq 0.8$, $\gamma \simeq 0.3$ (fig. 3). Along this line a discontinuity in the latent heat was observed on a 4^4 lattice. The phase boundary was followed to large negative values of γ [15, 16], where the transition is apparently of higher order. For $\gamma = 0$ (Wilson action) several calculations indicated a transition of second order [7, 8, 10, 12, 13]. However, recently clear signs of metastability phenomena on a 6^4 lattice and a hysteresis on a 16^4 lattice have

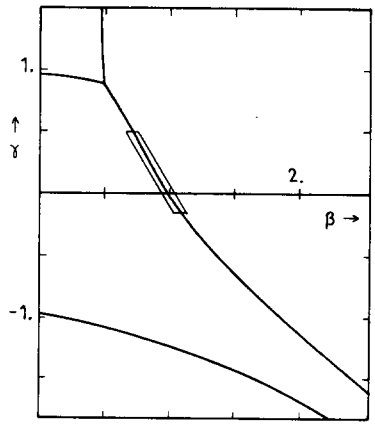

Fig. 3. Phase diagram of lattice QED with extended Wilson action, obtained in refs. [15, 16]. The couplings β, γ are defined in eq. (1.1). The parallelogram indicates the part of the phase boundary we are investigating in this paper (see fig. 11).

been found even for $\gamma = 0$ [14]. This raises the question of the precise position of the tricritical point in the $\beta - \gamma$ plane.

We have decided to calculate, on the phase boundary, the difference ΔW_p of the mean plaquette values in both phases,

$$\Delta W_p = W_p^{\text{Coulomb phase}} - W_p^{\text{confinement phase}},$$

$$W_p = \langle \cos \theta_p \rangle, \tag{3.1}$$

for several values of γ with a precision high enough to allow conclusions about the value of γ for which ΔW_p vanishes. This requires small errors and a careful proof that ΔW_p is independent of the lattice size.

It is clear that both requirements are difficult to satisfy in the vicinity of the tricritical point, and we had to choose the values of γ and the lattice sizes L for which the calculations were to be performed, judiciously. Therefore, we first made an estimate of the measurability of ΔW_p and of its dependence both on γ and L. The estimates have been performed for even L in the interval $L = 4$–12 at 8 values of γ in the interval $0.05 \leq \gamma \leq 0.5$. For $\gamma = 0$ the even lattices up to $L = 16$ have been used.

To speed up the program, we have approximated the U(1) group by Z(32) in all our calculations. This approximation is sufficient for the range of β and γ values we are working in [10]. We have also verified by using Z(100) [14] that this approximation does not change the properties of the phase transition we are studying. Our vectorized Metropolis Monte Carlo program performs one link update every 1.2 μsec on the CYBER 205 with two pipelines. We have spent about 40 h of computer time to obtain the data presented in this paper, not counting various test runs and trials.

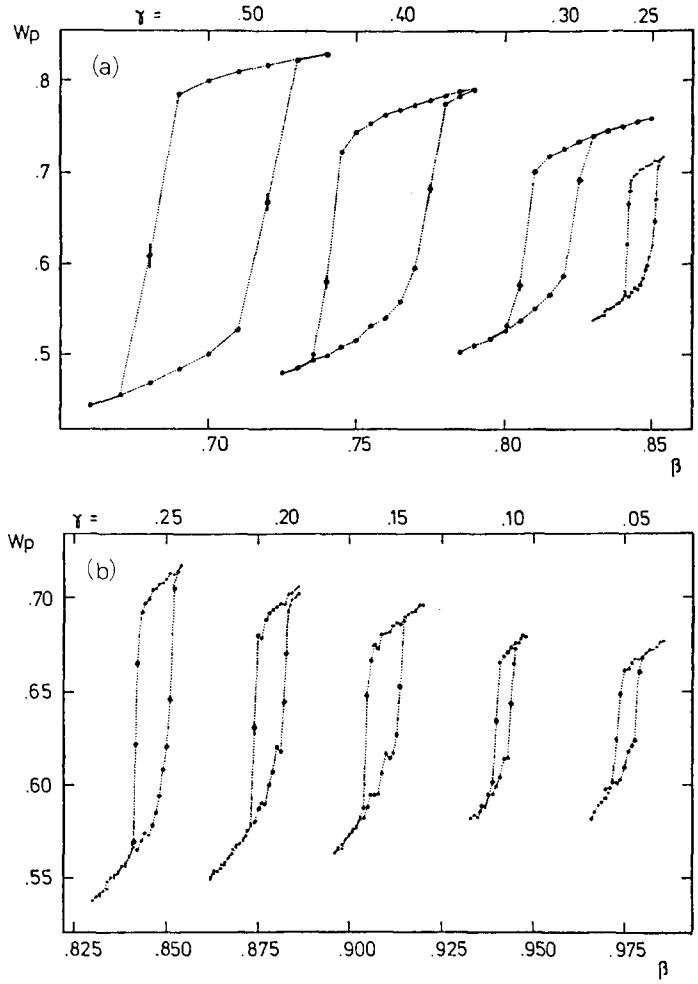

Fig. 4. Hystereses obtained in thermal runs on a 12^4 lattice. Each data point represents 960 Monte Carlo sweeps at fixed β. The hysteresis at $\gamma = 0.25$ is shown twice, allowing the comparison of different vertical scales in figs. 4a and 4b.

As from earlier work [15, 16] the position of the phase transition line $\beta_T(\gamma)$ is known only with insufficient precision, we first made, for each fixed value of γ, relatively fast thermal runs in β in both directions attempting to produce the hystereses on an 8^4 lattice. Their positions and widths provided an improved knowledge of $\beta_T(\gamma)$, permitting new, better positioned thermal runs with smaller steps in β and a larger number of sweeps per step. The resulting hystereses for $\gamma \geq 0.05$ on a 12^4 lattice with 960 sweeps per point are shown in fig. 4. As expected, both the heights and the widths of the hystereses decrease with decreasing γ. One can estimate ΔW_p by determining the heights of the hystereses at their middle points.

The precision of this procedure is insufficient, however. The fluctuations of W_p increase with decreasing γ, in particular on the hot (more disordered) branches (lower values of W_p), indicating long time correlations and increasing correlation length.

On smaller lattices the hystereses get even worse, and for smaller γ they cannot be produced with reasonably small fluctuations at all, since the system starts to flip between two metastable states. For $\gamma = 0$ this happens even on a 12^4 lattice. In these cases it is better to perform long iteration runs at fixed β and to plot the distribution of the values of W_p averaged over 10 successive configurations. An example of such a plot for $\gamma = 0.15$, $\beta = 0.887$ on a 4^4 lattice is shown in fig. 5. The doubly peaked structure of these plots has been used for an estimate of ΔW_p from the distance between the peaks, again with a rather poor precision.

From these estimates of ΔW_p and from the evolution of W_p during the Monte Carlo iteration, which has been recorded in all our calculations, several observations can be made:

(i) If for a given γ on a lattice of some size the flips between metastable states occur frequently, the calculation of ΔW_p will be rather imprecise and even a substantial increase in the iteration time does not improve the result. Only an increase of the lattice size, which suppresses the flips, helps.

(ii) As far as the estimated values of ΔW_p are reliable, they mostly show a decrease of ΔW_p with the lattice size. At $\gamma = 0$ the decrease from $L = 6$ to $L = 16$ is about 50% (figs. 3 and 4b in ref. [14]). For larger γ the decrease gets smaller and for $\gamma \geq 0.2$ ΔW_p does not seem to decrease any more for $L \geq 8$.

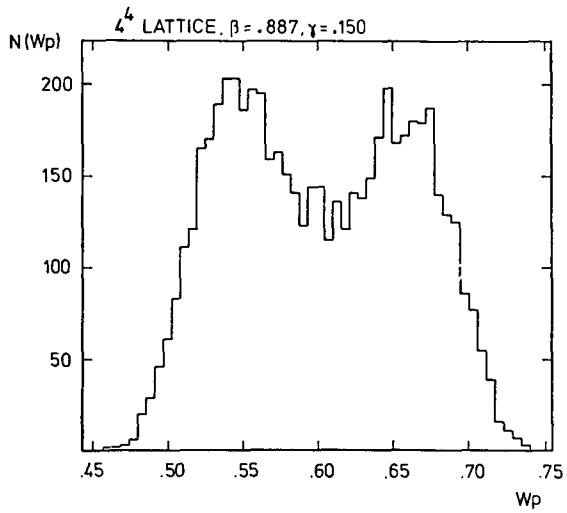

Fig. 5. Distribution of the values of W_p in a Monte Carlo run of 57 600 sweeps at $\beta = 0.887$ and $\gamma = 0.15$ on a 4^4 lattice. Each value of W_p is the average over 10 successive configurations.

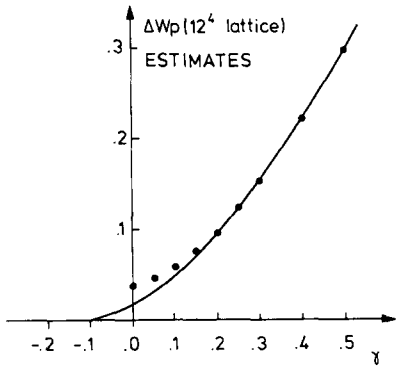

Fig. 6. Estimates of the discontinuity ΔW_p based mostly on the height of the hystereses on a 12^4 lattice, shown in fig. 4. For $\gamma < 0.2$ the values of ΔW_p lie above the power law curve, indicating a finite size effect.

(iii) We have made several attempts to extrapolate the values of ΔW_p to $L = \infty$ for γ fixed at some value in the interval $0 \leq \gamma < 0.2$, either by plotting ΔW_p as a function of L or as a function of the width of individual peaks in the doubly peaked distributions (this width should approach zero for $L \to \infty$). As any extrapolation requires reliable and precise data in a broader range of L, the difficulty of obtaining such data for smaller L prevents such extrapolations in practice.

(iv) From all our estimates of ΔW_p we show only the values obtained on 12^4 lattices in fig. 6. We stress that both the systematic and statistical errors in these estimates are not under sufficient control. Nevertheless, we have compared them with the power law (1.2). As seen in fig. 6, the five points with $\gamma \geq 0.2$ obey the power law quite well, whereas for $\gamma < 0.2$ the values of ΔW_p are clearly too high. This suggests that for $L = 12$ the asymptotic values of ΔW_p for $\gamma < 0.2$ have not yet been achieved and the deviation from the power law is caused by finite size effects.

From these observations we have drawn the conclusion that there is a fair chance of calculating ΔW_p precisely and to demonstrate its independence of L for $\gamma \geq 0.2$ and on lattices $L \geq 8$. We have chosen the values $\gamma = 0.2, 0.25, 0.3, 0.4, 0.5$. Higher values of γ, for which the calculation would be very easy, are of little use for an extrapolation by means of the power law (1.2) which is assumed to be valid only in the vicinity of the tricritical point.

4. Study of the hysteresis branches

For a precise calculation of ΔW_p at a phase transition point β_T it is necessary to determine the dependence of ΔW_p on β for fixed γ in the vicinity of β_T. As β_T is known with finite precision only, the knowledge of this dependence allows us to determine the error in ΔW_p caused by the uncertainty in β_T and may eventually indicate the necessity to improve the value of β_T.

We have performed high statistics thermal runs on an 8^4 lattice at all 5 values of γ chosen in sect. 3 above. For each γ the range of values of β and the steps in β

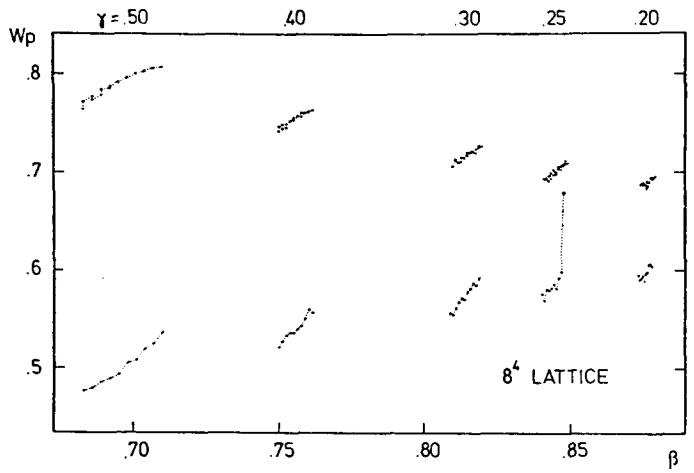

Fig. 7. The hysteresis branches obtained in high statistics thermal runs on an 8^4 lattice. The branches are not parallel and their slopes, listed in table 1, increase with decreasing γ.

have been chosen so that each branch of the hysteresis has been covered by 10 points. At each point 4800 Monte Carlo sweeps have been performed, starting on the stable side of each branch with the corresponding hot or cold initial configurations and throwing away the first 2000 sweeps. Knowing the positions and widths of the hystereses from previous runs, we have terminated the high statistics thermal runs on the metastable sides of the branches shortly before the transitions occur. Points containing jumps from the unstable to stable states would be of no use. Some branches have been calculated several times, starting also on the unstable side of the branch or with different random number generator seed.

The resulting hysteresis branches are shown in fig. 7. They have several important properties:

(i) The cold and hot branches at any γ are not parallel, the hot branch (the lower one) is steeper.

(ii) The slopes of both branches grow rapidly with decreasing γ. Fortunately, their difference grows much slower. The values of the slopes are listed in table 1.

(iii) Thermal runs on both branches show very long time correlations. This is illustrated in fig. 8 showing several thermal runs for $\gamma = 0.3$. Runs differing in the seed of the random number generator can produce distinctly shifted branches.

The difference in the slopes of the cold and hot branches does not allow us to determine ΔW_p simply from the distance of the branches with sufficient precision.

5. Mixed start runs

It has become apparent that the key to precise values of ΔW_p lies in a precise location of the phase transition points $\beta_T(\gamma)$. For $\gamma = 0.2$ and $\gamma = 0.25$ on a 12^4

TABLE 1

Summary of numerical results for the phase transition in the β-γ plane

γ	β_T ± syst. error	Slopes of cold/hot branches	W_p on cold/hot branches	ΔW_p ± syst. error
0.50	0.702	1.2	0.802	0.295
	±0.001	2.3	0.507	±0.001
0.40	0.759	1.7	0.760	0.218
	±0.001	3.0	0.542	±0.001
0.30	0.816	1.9	0.719	0.150
	±0.002	3.6	0.569	±0.003
0.25	0.846	2.5	0.702	0.119
	±0.002	4.2	0.583	±0.003
0.20	0.877	2.9	0.689	0.096
	±0.002	4.8	0.593	±0.004
0.15	0.909			
	±0.002			
0.10	0.942			
	±0.002			
0.05	0.977			
	±0.002			
0.00	1.0106			
	±0.0018			
−0.05	1.045			
	±0.003			
−0.10	1.084			
	±0.003			
−0.15	1.126			
	±0.002			

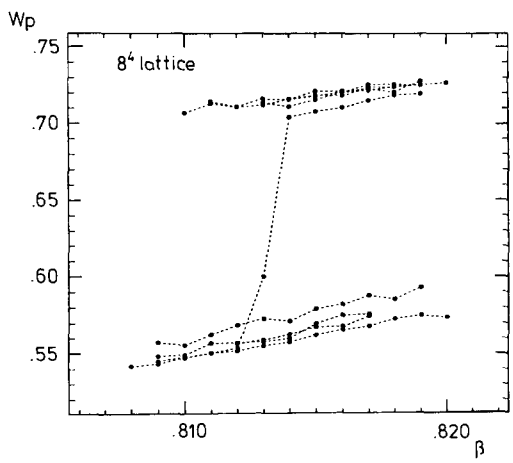

Fig. 8. Several high-statistics thermal runs at $\gamma = 0.3$ on an 8^4 lattice. They show very long time correlations

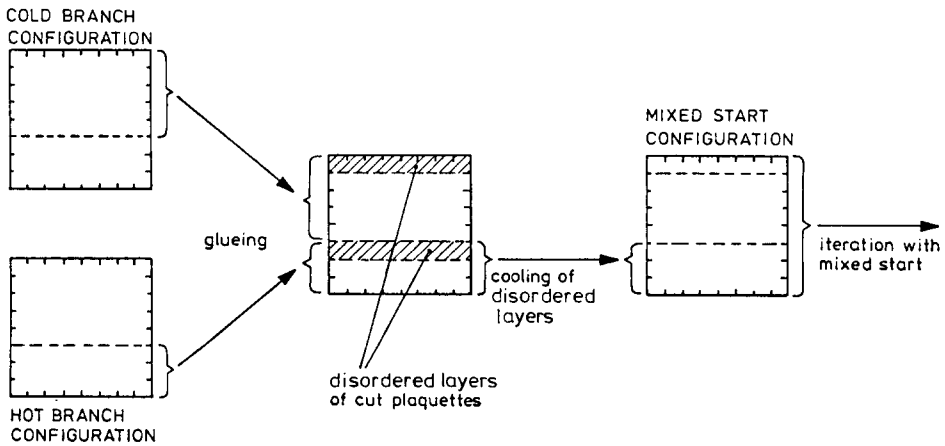

Fig. 9. The mixed start configurations are produced by the glueing of two metastable configurations. The layers of cut plaquettes are cooled by iteration of the hot (more disordered) branch part of the glued configuration.

lattice the hystereses are sufficiently narrow to yield β_T with an error of about 0.002. However, for $\gamma \geq 0.3$ they are too broad. We have succeeded to improve the values of $\beta_T(\gamma)$ substantially by the following mixed start run procedure:

For a given γ, a value of β is chosen approximately in the middle of the hysteresis. For this β a pair of configurations in metastable thermal equilibrium is produced, one on each branch of the hysteresis, by suitable thermal runs. Then the configurations are cut perpendicularly to one of the lattice axes next to the middle point of this axis. The larger part of the configuration from the cold branch is glued together with the smaller one from the hot branch. The resulting configuration thus consists of $(\frac{1}{2}L+1)$ cold and $(\frac{1}{2}L-1)$ hot branch configuration layers of link variables (fig. 9). The glueing produces two layers of cut plaquettes containing links from different configurations. These two layers are therefore disordered, making the mean value of W_p in the glued configuration, in spite of unsymmetric cutting, closer to its value in the hot branch configuration than in the cold one. This is repaired by performing a few dozen sweeps on the hot branch part of the glued configuration only, with the cold branch part acting as a fixed boundary condition. This moves the mean value of W_p of the configuration approximately to the middle position between the cold and the hot branch values.

This cooling of the disordered layers produces a configuration which contains germs of the metastable equilibrium configurations on both branches and gives an approximately equal chance to both of them to override the whole configuration during the following iteration. The equality of chances is important, since any bias might shift $\beta_T(\gamma)$, determined in this way on a lattice of a size not substantially larger than the correlation length. We have used eight configurations glued in this way (each axis has been cut twice, unsymmetrically) from a pair of original cold

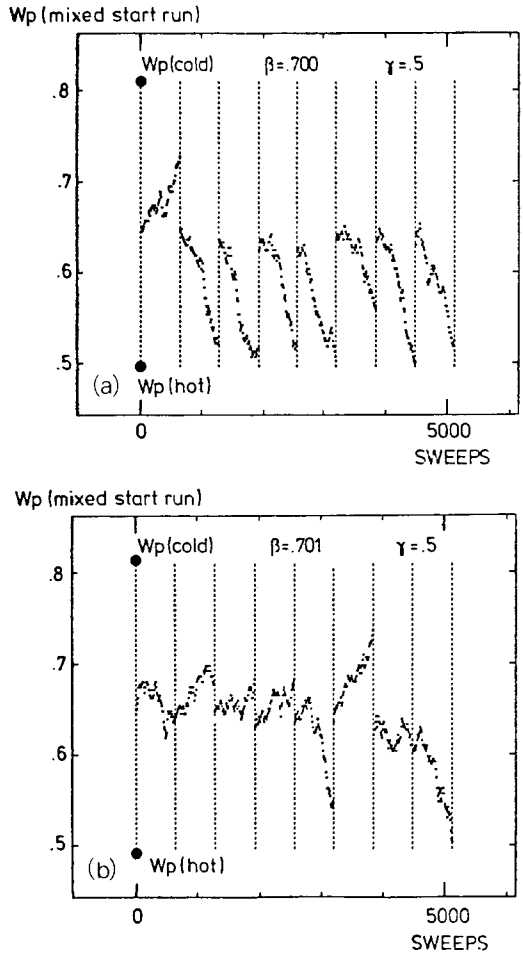

Fig. 10. Mixed start runs on an 8^4 lattice at two slightly different values of β in the hysteresis region for $\gamma = 0.5$. At $\beta = 0.700$ (a) the system evolves predominantly towards the hot branch. At $\beta = 0.701$ (b) neither of the branches wins, which indicates that the phase transition point has been reached. The eight runs at each value of β start from the mixed configurations produced by eight different cuts of a pair of metastable configurations.

and hot branch configurations and observed the development of the mean value of W_p for a few hundred sweeps. Then β has been slightly changed, a new pair of equilibrium configurations has been produced and the mixed start procedure has been repeated.

For most β values inside the hysteresis region nearly all glued configurations develop during the iteration into the same direction, towards one of the branches. But in a small interval of β they behave erratically, neither of the branches wins. We assume that $\beta_T(\gamma)$ lies within such an interval obtained on 8^4 or 10^4 lattices. For example for $\gamma = 0.5$ the location of the transition point, obtained in this way, is $\beta_T(0.5) = 0.702 \pm 0.001$. The (systematic) error is substantially smaller than the

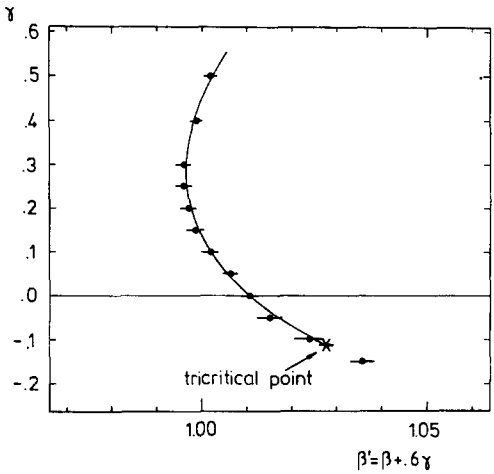

Fig. 11. Detailed picture of a part of the phase boundary line (see fig. 3), separating the confinement and the Coulomb phases. The curve starting at the tricritical point is the best fit to the data by means of the power law (2.8). The value of the crossover exponent ϕ given in eq. (1.4) was determined from this fit.

width of the hysteresis, which is about 0.04. In fig. 10 we show an example of two mixed start runs with β differing by 0.001 only.

For $\gamma < 0.3$ the mixed start runs on a 10^4 lattice do not improve on the already sufficiently small uncertainty of $\beta_T(\gamma)$ obtained from the hystereses on a 12^4 lattice. The values of $\beta_T(\gamma)$ obtained during our calculations are summarized in table 1 and in fig. 11.

6. Localization of the tricritical point

The very precise knowledge of the phase transition points $\beta_T(\gamma)$ makes it possible to perform, at a given γ, long iteration runs at $\beta_T(\gamma)$ on both branches and to calculate precisely the values of W_p and their difference. Actually we have often performed 2 or 3 such runs at β slightly different from $\beta_T(\gamma)$ (by about ± 0.001) and calculated W_p at $\beta_T(\gamma)$ by means of the known slopes of the branches. This increases the statistical independence of the data and diminishes a possible systematic error which might be caused by the small but still finite uncertainty of the values of $\beta_T(\gamma)$.

We performed these runs on 8^4–14^4 even lattices with the following total number of sweeps on each lattice:

for $L = 8$ 600 000 sweeps,

for $L = 10$ 370 000 sweeps,

for $L = 12$ 74 000 sweeps,

for $L = 14$ 98 000 sweeps.

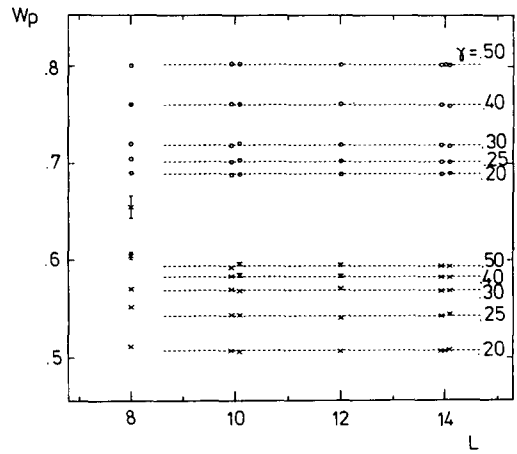

Fig. 12. Values of $W_p = \langle \cos \theta_p \rangle$ obtained at the phase transition points during Monte Carlo runs in both metastable states. The indicated errors are statistical. In most cases they are of the same size as the crosses and circles, which denote the hot and cold branch results, respectively. The $L = 8$ data on the hot branch are distorted by brief transitions into the other phase during long Monte Carlo runs. Therefore the $L = 8$ data have not been used for the calculation of the averages of W_p over the runs on various lattices indicated by the dashed lines.

The number of the sweeps performed at each γ increased gradually with decreasing γ. At $\gamma = 0.2$ the iterations have been about twice as long as at $\gamma = 0.5$.

The results are shown in fig. 12. Each point represents an independent run. The indicated errors are purely statistical, calculated as follows: each run is divided into several intervals longer than the observed time correlations, whose length did not exceed 1000 sweeps for all four lattice sizes. These intervals are about 5000 sweeps long on an 8^4 lattice, 2000 sweeps on a 10^4 lattice and 1000 sweeps long on 12^4 and 14^4 lattices. Except for the 14^4 lattice, their length decreases proportionally to the inverse volume, so that they represent statistically comparable measurements. The scattering of the values of W_p averaged over these intervals has been used to calculate the statistical error by the usual formulas.

As seen in fig. 12, for $L \geq 10$ the scattering of the points at each L is comparable to the statistical errors. This checks the consistency of the results from different runs. The data turn out to be independent of the lattice size L for $L \geq 10$ at all $\gamma \geq 0.2$.

Combining the runs at fixed L and calculating the difference between the cold and hot branches, one obtains the discontinuity ΔW_p as a function of L (fig. 13). Again, ΔW_p is independent of the lattice size for $L \geq 10$. The smaller values of ΔW_p on an 8^4 lattice are caused by attempts of the system to make a phase flip during the iterations, as can be seen from the record of the iteration history. Therefore we have not used the 8^4 lattice data. Fortunately, all the runs for $L \geq 10$ proved stable on both branches.

By averaging ΔW_p over the results obtained on 10^4, 12^4 and 14^4 lattices, we get the final values of ΔW_p. They are shown in fig. 14. The indicated errors are systematic,

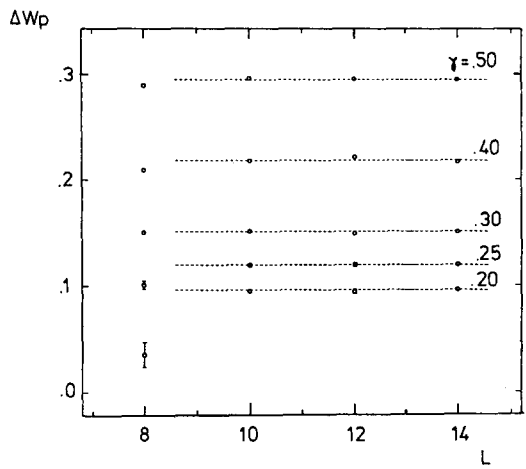

Fig. 13. The discontinuity ΔW_p as a function of the lattice size L. The $L=8$ data are distorted by brief phase transitions. The indicated errors are statistical. This figure demonstrates the independence of ΔW_p of the lattice size on 10^4, 12^4 and 14^4 lattices. The dashed straight lines show values of ΔW_p averaged over the lattices of these three sizes.

caused by the finite precision of the values of $\beta_T(\gamma)$, and have been calculated from the differences of slopes of the hysterese branches. The statistical errors, calculated by the above procedure, are smaller by a factor of about 6.

The form of the γ-dependence of ΔW_p with its apparent curvature already suggests that ΔW_p vanishes at some small negative $\gamma = \gamma_{TCP}$. A more precise localization of the tricritical point requires validity of the power law (1.2) in the entire region $\gamma_{TCP} \leqq \gamma \leqq 0.5$. In order to test this assumption, we have performed several fits to

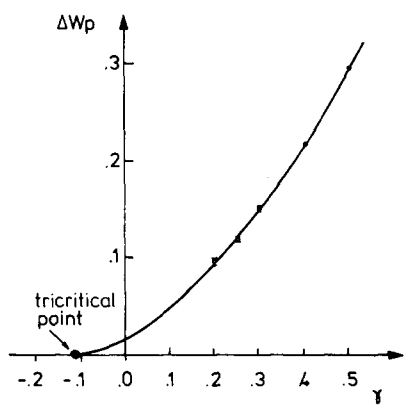

Fig. 14. The discontinuity ΔW_p of $\langle \cos \theta_p \rangle$ at the phase transition points $\beta_T(\gamma)$ for 5 values of γ, determined by Monte Carlo calculations on 10^4-14^4 lattices. The indicated errors are systematic, caused by a finite precision of the phase transition points. The curve is a fit to the data by means of the power law (1.2).

the 5 data points, omitting one or two of them at lower γ. The resulting values of γ_{TCP} varied in the interval $(-0.16, -0.06)$. This variation is of the same size as the errors associated with a MINUIT fit to all 5 points. This agreement verifies that the data is consistent with the power law (1.2). The values of the parameters determined by the MINUIT fit are

$$\gamma_{TCP} = -0.11 \pm 0.05\,, \qquad \beta_u = 1.7 \pm 0.2, \qquad A \simeq 0.68\,. \tag{6.1}$$

The quoted systematic errors are estimates based on variations of the fit procedure mentioned above. The resulting curve is shown in fig. 14. The data is fitted by the power law curve very well and supports the assumption of the validity of the power law (1.2) in the whole region $\gamma_{TCP} \leqslant \gamma \leqslant 0.5$. To check our result, we have performed the calculation and the extrapolation also for the discontinuity of $\langle \cos 2\theta_p \rangle$ and obtained the same value for γ_{TCP}.

From the known dependence of the phase transition points β_T on γ (table 1) the location of the tricritical point, eq. (1.3), has been derived. With this information at hand, we have also fitted the shape of the phase boundary using the expression (2.8). The fit is shown in fig. 11 and the value of the crossover exponent ϕ is given in eq. (1.4).

A negative value of γ_{TCP} means that the phase transition in lattice QED with Wilson action is of first order. More important, the growth of correlation length observed in this version of lattice QED, is caused by the vicinity of the tricritical point. The associated exponent $\nu_t \simeq 0.3$ describes the growth of the correlation length at least in the interval $0.9 < \beta < \beta_T$[8, 10–14]. This exponent is very probably the tricritical exponent and not the critical one. The critical exponent ν of lattice QED is not yet known, since also the phase transition in lattice QED with Villain action, for which several calculations have been performed [11, 17], is presumably influenced by the tricritical point [17].

7. Discussion

Assuming the validity of the power law (1.2), we have localized the tricritical point in lattice QED with extended Wilson action (1.1) by Monte Carlo simulation with a precision that allows us to conclude that the deconfining phase transition in the case of the Wilson action is of first order.

We have found that the discontinuity ΔW_p is well described by the power law (1.2) for $0.2 \leqslant \gamma \leqslant 0.5$. This supports the use of the power law for an extrapolation of ΔW_p in γ down to γ_{TCP}. Nevertheless, one can think of logarithmic corrections and/or of non-leading power law terms. Even in the case that such corrections are present, the shape of the data in fig. 14 indicates that γ_{TCP} is negative. The precise location of the tricritical point might change, however.

Our results are consistent with the properties of tricritical points as they are known from the study of systems with global symmetries. From the point of view of lattice

gauge theories, several interesting questions arise: how is the presence of a tricritical point reflected in the properties of the system in the Coulomb phase where the correlation length is infinite? How much do the critical and tricritical exponents differ? Where are the crossover regions? Furthermore, it would be interesting to generalize the action (1.1) by introducing a new relevant coupling parameter and to study a three-dimensional phase diagram analogous to fig. 1b.

The U(1) phase transition has always been regarded as a prototype of the deconfining phase transitions in lattice gauge theories. Its properties, studied in our work, thus might be of some relevance to numerical studies of the deconfining phase transitions both at zero and high temperature.

It is obvious that a growth of the correlation length when the phase transition point is approached, does not necessarily mean that the transition is of higher order. The correlation length may grow on a path towards a first-order phase boundary if the distance to the tricritical point on the path decreases. This is what presumably happens in lattice QED with Wilson action, where the string tension jumps to zero from some finite value [14]. Such a jump can, however, only be seen on lattices larger than the maximal correlation length. Earlier warnings are provided by the occurrence of metastability phenomena.

On the other hand, these metastability phenomena do not prove that the transition is of first order. They might be just a finite size effect. We have observed double peaks, not unlike those shown in fig. 5, on an 8^4 lattice even for $\gamma = -0.5$. Another, very drastic example can be found in ref. [21]. Furthermore, the metastable behaviour might be difficult to observe on small lattices, where transitions between metastable states are frequent. An increase of the lattice size makes the metastability more apparent, but simultaneously the distance between the metastable states, e.g. the latent heat or ΔW_p in our case, may decrease with lattice size. What actually happens on an infinite lattice is one of the tricky questions typical for tricritical points.

In spin systems the most convenient method to overcome the difficulties related to the vicinity of a tricritical point seems to be the Monte Carlo renormalization group method [21]. Hopefully, this method will be helpful also in the tricritical regions in lattice gauge theories.

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