

Chapter 10

Entangled states

The possibility to entangle several degrees of freedom is central to quantum mechanics and distinguishes it drastically from classical physics. In this chapter we shall discuss some aspects and applications of entanglement. A **pure** quantum state with two sets of degrees of freedom $\mathcal{A}$ and $\mathcal{B}$ is called a product state (see also chap. 9) when it is of the form

$$|\psi_{total}\rangle = |\varphi\rangle_{\mathcal{A}} \otimes |\chi\rangle_{\mathcal{B}}. \quad (10.1)$$

A state is entangled when it is of the form

$$|\psi\rangle = \sum_n c_n |\varphi_n\rangle_{\mathcal{A}} \otimes |\chi_n\rangle_{\mathcal{B}} \quad (10.2)$$

with more than one term in the sum that cannot be factorized into the form of a product state.

In the previous chapter we noticed that when one investigates only one of two entangled subsystems, interference can vanish. In chaps. 10.1 and 10.2 we will treat consequences for the quantum mechanical measurement process itself and for macroscopically occurring decoherence. In chap. 10.3 we shall see that one cannot explain certain experimentally verified quantum mechanical correlations by just using classical physics, even when one allows for infinitely many so-called hidden variables. In chap. 10.4 we will look at applications of entanglement in the field of quantum information, and at quantum teleportation as an example.

10.1 The measurement process

We examine in more detail what happens during a measurement. Let the quantum mechanical system S be in a pure state $|\psi\rangle$. We measure an observable A with corresponding operator (analyzer) $\hat{A}$. Let the spectrum of $\hat{A}$ be discrete, with eigenvectors $|n\rangle$:

$$\hat{A} |n\rangle = a_n |n\rangle. \quad (10.3)$$

One can write the state in the eigenbasis of $\hat{A}$:

$$|\psi\rangle = \sum_n c_n |n\rangle. \quad (10.4)$$

As a simple example, we consider the case that only two eigenvectors of $\hat{A}$ contribute to $|\psi\rangle$:

$$|\psi\rangle = c_1 |1\rangle + c_2 |2\rangle. \quad (10.5)$$

The density operator of this pure state is

$$\begin{aligned} \hat{\rho}_0 = |\psi\rangle \langle\psi| &= (c_1 |1\rangle + c_2 |2\rangle)(c_1^* \langle 1| + c_2^* \langle 2|) \\ &\hat{=} \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{pmatrix}. \end{aligned} \quad (10.6)$$

During a measurement, the system interacts with a measurement apparatus M. Afterwards, the result of the measurement is read off by an observer. The quantum mechanical postulates stipulate that the result of a measurement of A will always be one of the eigenvalues of $\hat{A}$, each with probability $|\langle n|\psi\rangle|^2 = |c_n|^2$.

Numerous difficult questions arise in trying to interpret this statement. One might read the postulate such that directly after the interaction with the measurement apparatus, the system is in the mixed state

$$\hat{\rho}_I = \sum_n |c_n|^2 |n\rangle \langle n|. \quad (10.7)$$

see below

In our example this has a matrix representation

$$\rho_I = \begin{pmatrix} |c_1|^2 & 0 \\ 0 & |c_2|^2 \end{pmatrix}. \quad (10.8)$$

When reading off the result, the observer then apparently obtains one of the possible outcomes with the corresponding probability $|c_i|^2$, similar to a probability in classical statistics. The observer then knows that the system is in the single eigenstate $|n\rangle$, i.e. $\hat{\rho}$ is further reduced to

$$\hat{\rho}_{II} = |n\rangle\langle n|. \quad (10.9)$$

In the quantum mechanical state $|\psi\rangle$, the contributing states $|1\rangle$ and $|2\rangle$ can interfere in a suitable experimental setup (see chap. 9.1) via off-diagonal terms $\langle 1|\hat{B}|2\rangle = \text{tr}(\hat{B}|2\rangle\langle 1|)$ of some observable $\hat{B}$. For interference to occur, the density operator (10.6) needs to contain off-diagonal terms like $|2\rangle\langle 1| \hat{=} c_1^*c_2$. Note that in (10.8), only classical probabilities remain, the off-diagonal terms are missing.

The transitions from $\hat{\rho}_0$ to $\hat{\rho}_I$ and then to $\hat{\rho}_{II}$ (or directly $\hat{\rho}_0$ to $\hat{\rho}_{II}$) are very difficult to interpret. The so-called Kopenhagen interpretation of quantum mechanics talks about the „collapse of the wave function“ ($\hat{\rho}_0$ to $\hat{\rho}_{II}$, or equivalently $|\psi\rangle$ to $|n\rangle$), as opposed to the otherwise unitary time evolution according to the Schrödinger equation (footnote p.263).

One drastic example is Schrödinger's famous cat (1935): it is hidden in a box. During the experiment, the cat is supposed to be killed with probability $\frac{1}{2}$ in the box by some poison (for example triggered by a radioactive decay). One only knows whether the cat is dead or alive after opening the box. Immediately before, one seems to be forced to specify the state of the cat as

$$\frac{1}{\sqrt{2}} (|\text{alive}\rangle + |\text{dead}\rangle).$$

This is very strange. Is this a macroscopic superposition of a live and a dead cat? Does that make sense? Is the fate of the cat decided at the moment of observation? Should one also include the observer into the quantum mechanical system? Does one have to take account of interactions with the environment? How does one get from microscopic quantum mechanics to macroscopic determinism? How does one handle the probability interpretation? These and related questions are known as the „measurement problem“. Despite decades of considerable effort, there is still no generally accepted interpretation.

10.1. The measurement process

von Neumann

Interestingly, some of the questions can be solved when one also treats the time evolution of the measurement apparatus quantum mechanically. Let $|M_0\rangle$ be its state before the measurement. The total state of system and measurement apparatus before the measurement is then

$$|\psi\rangle \otimes |M_0\rangle \equiv |\psi, M_0\rangle. \quad (10.10)$$

It factorizes into a part $|\psi\rangle$ of the system and the part $|M_0\rangle$ of the measurement apparatus. Let M be such a measurement apparatus in the following sense: When S is in the state $|k\rangle$ before a measurement, then afterwards it shall still be in state $|k\rangle$ and the measurement apparatus gets into a state $|M_k\rangle$ (e.g. with a macroscopic pointer position "k"). We now consider the quantum mechanical time evolution of the total system, unitarily via the time evolution operator:

$$|\psi, M_0\rangle = \sum_n c_n |n, M_0\rangle \longrightarrow |\psi_{total}\rangle = \sum_n c_n |n, M_n\rangle. \quad (10.11)$$

Note that the final state is now entangled between system and measurement apparatus; it does not factorize anymore. It still is a pure state, however.

Afterwards, the observer registers (measures) the pointer position of the measuring apparatus. The observer does *not* have direct access to the system S , but just interacts with the measuring apparatus. His/her situation is thus described by a reduced density matrix which contains only the states of the *measuring apparatus*:

$$\begin{aligned} \tilde{\rho}_{\text{meas}} &= \text{tr}_{\text{system}} |\psi_{total}\rangle \langle \psi_{total}| \\ &= \text{tr}_{\text{system}} \sum_{n,m} c_n c_m^* |n, M_n\rangle \langle m, M_m| \\ &= \sum_n |c_n|^2 |M_n\rangle \langle M_n|. \end{aligned} \quad (10.12)$$

diagonal because $\langle n|m \rangle = \delta_{nm}$

This corresponds to a matrix which contains the expected probabilities $|c_n|^2$ for the individual results on the diagonal, but does not contain interference terms. The transition from the quantum mechanical situation with interference to classical probabilities ($\hat{\rho}_0$ to $\hat{\rho}_I$) therefore happens by focusing

¹More precisely, in a state that is in one-to-one correspondence with $|k\rangle$. For simplicity, we still use the name $|k\rangle$.

and interacting with only the measurement apparatus² when reading off the result, and not with the total system.³

These considerations shed light on how one gets from ρ_D to ρ_E , but unfortunately not on how to interpret the presence of probabilities instead of classical determinism. In chap. 10.3 we shall see that a seemingly natural interpretation as a statistical mixture of individual realizations with predetermined results ("hidden variables") has been ruled out by an ingenious setup of experiments.

10.2 Decoherence

There is another very important mechanism which drastically reduces quantum mechanical interference effects in the macroscopic world, the so-called decoherence, due to interactions with the many degrees of freedom of the outside world.

For simplicity we again consider a particle (system) in the pure state

$$|\psi\rangle_S = c_1 |1\rangle_S + c_2 |2\rangle_S,$$

with the density matrix (10.6).

It interacts with the environment for example by scattering of (usually numerous) photons from the particle. Let the photon be in state $|i\rangle_P$ before such a scattering event. The state of the photon after scattering will depend on the state of the particle (e.g. on its location). Schematically, we

²For Schrödinger's experiment, one could also consider the cat to be a macroscopic measurement apparatus which detects whether the decay of the atom has taken place. Note that for the observer, Schrödinger's cat is then not in the pure state $\frac{1}{\sqrt{2}}(|\text{alive}\rangle + |\text{dead}\rangle)$ but in a mixed state like eq.(10.12)

³There actually is a physical difference between (a) the Copenhagen view with a collapse of the wave function at every measurement and (b) the present description with unitary time evolution and a reduced density matrix, namely that the unitary time evolution could in principle be inverted after the measurement to re-obtain the initial state, whereas this is impossible after a collapse. An experimental verification or disproof of the wave function collapse might therefore be possible! Note however, that surprisingly a collapse vs. unitary evolution of the wave function has no influence on measurement outcomes at later times when one considers the joint probabilities with the first measurement outcome (see book by Basdevant and Dalibard, p.110)

10.2. Decoherence

can write

$$|n\rangle_S \otimes |i\rangle_P \rightarrow |\tilde{n}\rangle_S \otimes |n\rangle_P.$$

We have denoted the state of the particle after the scattering by $|\tilde{n}\rangle_S$. Let us assume that before the scattering, the particle is in a product state with the photon. After the scattering we then obtain (completely analogously to the measurement discussed before)

$$|\psi\rangle_S \otimes |i\rangle_P = \left(\sum_n c_n |n\rangle_S \right) \otimes |i\rangle_P \rightarrow |\psi_{total}\rangle = \sum_n c_n (|\tilde{n}\rangle_S \otimes |n\rangle_P). \quad (10.13)$$

This is another pure state, but it is now entangled between particle and photon.

If the contributing final states of the photon are orthogonal to each other, $\langle 1|2\rangle_P = 0$, then the reduced density matrix of the system is

$$\tilde{\rho}_S = \text{tr}_P |\psi_{total}\rangle \langle \psi_{total}| = \dots \hat{=} \begin{pmatrix} |c_1|^2 & 0 \\ 0 & |c_2|^2 \end{pmatrix}. \quad (10.14)$$

The interference terms are gone!

More realistically, let us assume that the final states $|n\rangle_P$ of the photon are not orthogonal. Their overlap $\langle 1|2\rangle_P$, which will then appear in the off-diagonal elements, is usually small, especially when the states $|1\rangle_S$ and $|2\rangle_S$ of the system are macroscopically different (e.g. different locations). When another photon scatters on $|\psi_{total}\rangle$, with final states $|n_P^{(2)}\rangle$, then the off-diagonal elements of the density matrix contain the product $\langle 1^{(1)}|2^{(1)}\rangle_P \cdot \langle 1^{(2)}|2^{(2)}\rangle_P \ll 1$. Each additional scattering event reduces the size of the off-diagonal elements. The total system of particles and all photons becomes a pure but highly entangled states with many terms in the sum. With more and more scattering events, the reduced density matrix, on the other hand, becomes more and more similar to (10.8): the system loses its coherence.

A real system usually contains macroscopically many photons and other particles, which interact on very small time scales. Thus one automatically gets the transition from microscopic to macroscopic physics: the microscopic quantum mechanical interference effects get lost very quickly for

macroscopic objects for which one cannot measure all the entangled degrees of freedom together. It is therefore usually difficult to keep quantum mechanical coherence in an experiment. However, there are some spectacular exceptions, where macroscopically many particles are in the same quantum state, which is therefore stable, most notably suprafluidity and superconductivity.

10.3 EPR-paradoxon and Bell's inequalities

Einstein was always unhappy with the fact that quantum mechanics only predicts probabilities, as can be seen from his famous quote

I cannot for a moment believe that [god] plays dice and makes use of 'telepathic' means (as the current quantum theory alleges he does).

Maybe the outcome of measurements is actually predetermined by some hidden parameters? Then quantum mechanics would actually be deterministic, like classical mechanics. In 1935, Einstein, Podolsky und Rosen (EPR) proposed a Gedanken-experiment on this issue. Several decades later, in 1964, the theorist John Bell proposed a setup which allows to distinguish between quantum mechanics and *all* such theories with hidden variables, even with infinitely many (!). Many subsequent experiments clearly verified quantum mechanics and ruled out the hidden variables.

We consider a modified version of the EPR experiment which is due to D. Bohm. Let there be a particle at rest with spin zero, which decays into two particles A and B, each with spin 1/2, which move in opposite directions.⁴ Quantum mechanically, A and B are entangled (see below).



We wait until the particles are far from each other and then we measure their spin. The measurements shall be so close in time that they cannot influence each other by any interaction not faster than the speed of light.⁵

When one only examines particle A, one will find spins $+\frac{\hbar}{2}$ und $-\frac{\hbar}{2}$ with equal probability with respect to any direction $\vec{n}$. Similarly when one only

⁴Similar experiments can be done with photons.

⁵This has indeed become possible experimentally since the 1980s.

10.3. EPR-paradoxon and Bell's inequalities

measures particle B. Because of angular momentum conservation, the total system of A and B still has to have spin zero. For example, when a measurement of the spin of particle A in z-direction results in $+\frac{\hbar}{2}$, then a measurement of the spin of particle B in the same direction must result in the value $-\frac{\hbar}{2}$. One can therefore predict the result of the second measurement.

Note, however, that such a correlation can also occur classically, for example when one distributes a red and a black ball among two containers: when the first container turns out to contain the red ball, then one can predict the contents of the second container to be black with certainty. Different from the classical case, the quantum mechanical case contains additional correlations, e.g. for measurements in the x-direction of spin.

Bell's distinction between quantum mechanics and classical physics with hidden variables requires measurements of the two particles with respect to different spin-directions, as we shall soon show. We will analyze such experiments in two different ways, first assuming hidden variables, and then with the rules of quantum mechanics.

10.3.1 Locally realistic theories (hypothetical)

EPR define a theory to be "realistic" when the following is true:

When one can predict the value of a physical observable with certainty, without disturbing the system, then there is an element of physical reality [i.e. a hidden variable] that corresponds to this quantity.

We now assume that the correct theory of the world is "realistic" in this sense.

In addition we assume "locality", i.e. that there is no interaction faster than the speed of light.

We analyze the case mentioned above, namely that the measurement of S_z for particle A gave the value $+\frac{\hbar}{2}$. Then we can predict the result for particle B with certainty, without disturbing B. With the criterion quoted above, this means that particle B indeed possesses a property " $S_z = -\frac{\hbar}{2}$ ".

We could also measure S_x at A and then predict S_x at B correctly. Because of the assumed locality, the choice of measurement direction⁶ at A cannot influence the results at B. Therefore B must also have a property " S_x ".

The contradiction with quantum mechanics, where $[\hat{S}_z, \hat{S}_x] \neq 0$ implies that one cannot have sharp values for both directions, is the "paradox" of EPR.

The above considerations are valid for any direction $\vec{n}$ and for switched roles of A and B. In a "local realistic theory" the results of measurements must therefore be predetermined after the decay, for every of the infinitely many directions of measurement (!).

so many, parameters.

Note that these considerations and experimental results already rule out the simplest version of a hidden variable, namely that each of the particles possesses a fixed direction of spin after the decay. If this direction was $\pm \vec{e}_z$ for example, then the measurement results with respect to $\vec{e}_x$ would not be correlated. Measurement results with respect to the same direction at A and B then would not have the experimentally verified perfect correlation.

!

Bell's inequality

In 1964, John Bell managed to derive an experimentally testable consequence of a generic theory with hidden variables.

!

He considered measurements of the spin of A and B in different directions. Each individual measurement will produce a result $\pm \frac{\hbar}{2}$. We define the correlation function

$$C(\vec{a}, \vec{b}) := P_{++} + P_{--} - P_{+-} - P_{-+}, \quad (10.15)$$

where for example P_{+-} is the probability to find $+\frac{\hbar}{2}$ upon measuring A in the direction $\vec{a}$ and to find $-\frac{\hbar}{2}$ upon measuring B in the direction $\vec{b}$ for the same decay.

We will show:

⁶It was indeed varied very quickly locally and randomly in experiments.

10.3. EPR-paradoxon and Bell's inequalities

BELL'S INEQUALITY

$$|C(\vec{a}, \vec{b}) - C(\vec{a}, \vec{c})| \leq 1 + C(\vec{b}, \vec{c}) \quad (10.16)$$

holds for a local realistic theory.

Proof: In each of the three directions $\vec{a}, \vec{b}, \vec{c}$ the result at A can be $\pm \frac{\hbar}{2}$. We denote the relative frequency of these 8 cases by $n(\alpha_A, \beta_A, \gamma_A)$, with $\alpha_A, \beta_A, \gamma_A = \pm 1$. The assumption of a realistic theory means that each of these 8 numbers has a fixed value, with

$$\sum_{\alpha_A, \beta_A, \gamma_A} n(\alpha_A, \beta_A, \gamma_A) = 1.$$

One can then read the correlation function as $C(\vec{a}, \vec{b}) = \langle \alpha_A \beta_B \rangle$. Because the total spin is zero, the value β_B for measuring at B in direction $\vec{b}$ is the opposite of the value β_A for measuring at A with respect to the same direction, i.e. $\beta_B = -\beta_A$. The correlation $\langle \alpha_A \beta_B \rangle$ then becomes

$$C(\vec{a}, \vec{b}) = \sum_{\alpha_A, \beta_A, \gamma_A} n(\alpha_A, \beta_A, \gamma_A) \alpha_A \cdot (-\beta_A),$$

and thus (we omit the index A now):

$$\begin{aligned} C(\vec{a}, \vec{b}) - C(\vec{a}, \vec{c}) &= - \sum_{\alpha, \beta, \gamma} n(\alpha, \beta, \gamma) \alpha(\beta - \gamma) \\ &= - \sum_{\alpha, \beta, \gamma} n(\alpha, \beta, \gamma) \alpha\beta(1 - \beta\gamma) \end{aligned}$$

since $\beta^2 = 1$. Because of $\alpha\beta = \pm 1$ we get

$$|C(\vec{a}, \vec{b}) - C(\vec{a}, \vec{c})| \leq \sum_{\alpha, \beta, \gamma} n(\alpha, \beta, \gamma) (1 - \beta\gamma) = 1 + C(\vec{b}, \vec{c}),$$

as claimed.

10.3.2 Quantum mechanical analysis



Singlet

The two particles A and B come from the decay of a particle without spin. Their total quantum mechanical states must therefore also have total spin zero, i.e. it must remain constant upon applying the rotation operator

$$e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \vec{J}}$$

with $\vec{J}$ the total operator of angular momentum

$$\vec{J} = \vec{S}_A + \vec{S}_B, \quad (10.17)$$

where $\vec{S}_A$ only acts on particle A and $\vec{S}_B$ only on particle B.

A basis for the combined spin state of A and B consists of the products of the individual basis vectors:

$$|\uparrow, \uparrow\rangle, |\downarrow, \downarrow\rangle, |\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle,$$

where the first position is for particle A and the second for particle B, and $|\uparrow\rangle$ and $|\downarrow\rangle$ are, e.g., $|+z\rangle$ and $|-z\rangle$. The desired linear combination which is invariant under rotations is the singlet state

$$|\psi_S\rangle = \frac{1}{\sqrt{2}} (|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle). \quad (10.18)$$

Note that because of this rotation invariance, the singlet state has an identical looking representation with respect to any basis direction $\vec{n}$.

The rotation invariance can be verified by applying $(\vec{J})^2$ and using that A and B each have spin $j = \frac{1}{2}$. For A this implies

$$(\vec{S}_A)^2 |\pm z\rangle_A = \hbar^2 \frac{1}{2} \left(\frac{1}{2} + 1 \right) |\pm z\rangle_A = \hbar^2 \frac{3}{4} |\pm z\rangle_A.$$

and for the total state

$$\begin{aligned} (\vec{J})^2 |\psi_S\rangle &= (\vec{S}_A + \vec{S}_B)^2 |\psi_S\rangle \\ &= \left((\vec{S}_A)^2 + (\vec{S}_B)^2 + 2\vec{S}_A \cdot \vec{S}_B \right) |\psi_S\rangle. \end{aligned} \quad (10.19)$$

10.3. EPR-paradoxon and Bell's inequalities

By using the Pauli matrices one sees that

$$\hat{S}_A^x \hat{S}_B^x |\psi_S\rangle = \hat{S}_A^y \hat{S}_B^y |\psi_S\rangle = \hat{S}_A^z \hat{S}_B^z |\psi_S\rangle = -\frac{\hbar^2}{4} |\psi_S\rangle,$$

and the contributions in eq.(10.19) add up to zero. Therefore the singlet is indeed an eigenstate of $(\vec{J})^2$, with eigenvalue $\hbar^2 j(j+1) = 0$, i.e. with $j = 0$, and thus rotation invariant.

One can choose the singlet as one basis state for a system of two spin 1/2 particles. The three other basis states $|\uparrow, \uparrow\rangle$, $|\downarrow, \downarrow\rangle$ and $\frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle)$ together form the so-called spin-triplet (j=1) with total spin $1 \cdot \hbar$ and with $m = 1, m = 0, m = -1$.

It is important that the singlet is an entangled state. An EPR analysis can also be done with the $m = 0$ state of the triplet or with entangled states of photons.

Quantum mechanical correlations

The calculation of the correlations $C(\vec{a}, \vec{b})$ is easiest when we define $\vec{a}$ as our z-Achse, i.e. $|\pm a\rangle \equiv |\pm z\rangle$. The basis vectors $|\pm b\rangle$ can be written in the z-basis (2.28):

$$\begin{aligned} |+\vec{b}\rangle &= \cos\left(\frac{\theta}{2}\right) |+\vec{z}\rangle + e^{i\varphi} \sin\left(\frac{\theta}{2}\right) |-\vec{z}\rangle \\ |-\vec{b}\rangle &= \sin\left(\frac{\theta}{2}\right) |+\vec{z}\rangle - e^{i\varphi} \cos\left(\frac{\theta}{2}\right) |-\vec{z}\rangle, \end{aligned}$$

with θ and φ the polar coordinates of $\vec{b}$. For the state $|\psi_S\rangle$, the probability P_{++} of eq.(10.15) is

$$\begin{aligned} P_{++} &\equiv \left| \langle \psi_S | +a, +b \rangle \right|^2 \\ &= \left| \frac{1}{\sqrt{2}} \langle +z, \underbrace{-z}_{\text{particle A}} + \underbrace{+z}_{\text{particle B}}, +b \rangle - \frac{1}{\sqrt{2}} \langle -z, +z | +z, +b \rangle \right|^2 \\ &= \left| \frac{1}{\sqrt{2}} \underbrace{A\langle +z | +z \rangle_A}_{=1} B\langle -z | +b \rangle_B - \underbrace{A\langle -z | +z \rangle_A}_{=0} B\langle -z | +b \rangle_B \right|^2 \\ &= \frac{1}{2} \sin^2 \frac{\theta}{2}, \end{aligned} \tag{10.20}$$

with only the first term of the singlet contributing. P_{--} gives the same result. For P_{+-} we obtain similarly

$$\left| \langle \psi_S | +a, -b \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \langle +z, -z | +z, -b \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} {}_B \langle -z | -b \rangle_B \right|^2 = \frac{1}{2} \cos^2 \frac{\theta}{2},$$

and the same for P_{-+} . The complete correlation function is then

$$\begin{aligned} \underline{C(\vec{a}, \vec{b})} &= P_{++} + P_{--} - P_{+-} - P_{-+} \\ &= \sin^2 \frac{\theta}{2} - \cos^2 \frac{\theta}{2} \\ &= -\cos \theta = \underline{-\vec{a} \cdot \vec{b}}. \end{aligned} \quad (10.21) \quad !$$

10.3.3 Comparison

We now choose the directions

$$\vec{a} = (0, 0, 1), \quad \vec{b} = (0, 1, 0), \quad \vec{c} = \frac{1}{\sqrt{2}}(0, 1, 1).$$

Then the quantum mechanical prediction for the left side of Bell's inequality (10.16) is

$$|C(\vec{a}, \vec{b}) - C(\vec{a}, \vec{c})| = \left| 0 + \frac{1}{\sqrt{2}} \right| \simeq 0.707. \quad \text{QM result}$$

This is much larger than the quantum mechanical result for the right side

$$1 + C(\vec{b}, \vec{c}) = 1 - \frac{1}{\sqrt{2}} \simeq 0.293. \quad \text{QM result}$$

and therefore contradicts Bell's inequality, which was derived assuming a local realistic theory.

Only experiments can decide what is true in nature. The results of numerous experiments, with ever increasing accuracy, is very clear: they are compatible with the quantum mechanical result, but Bell's inequality is violated. Therefore a local realistic theory in the sense of EPR is incompatible with experimental results!

10.4 Quantum information

In recent years, the so-called quantum information theory has been developing rapidly, partially because of ever improving experimental possibilities.

One aspect of its applications concern the test of quantum mechanical predictions on a microscopic scale, almost unimaginable when quantum mechanics was developed. Other applications are about using quantum mechanical effects for new applications. Such effects are strongest with two-state-systems, so-called qubits, which have a variety of practical realisations.

While a classical bit can only take values 0 and 1, a qubit

$$\frac{1}{\sqrt{1 + |\alpha|^2}} \left(|0\rangle + \alpha |1\rangle \right)$$

has a continuous parameter α and therefore contains more information.

Amazing new applications can be reached by entangling and manipulating several (ideally many) qubits. It is important and also very difficult to do the manipulations coherently, such that e.g. the phase information in the parameters α of the qubits is not lost.

One of the spectacular applications is quantum cryptography, which is already used commercially. Another one is quantum teleportation (on the level of a few qubits), which we will discuss below. A still quite distant goal is a quantum computer, which requires the manipulation of many qubits over a long time by so-called quantum gates. The idea is to use entanglement in order to reach a theoretically exponential speedup with respect to the same number of classical bits. However, it is still not clear whether decoherence effects can (even in principle) be kept sufficiently small.

10.5 Entanglement measure

Entanglement between different degrees of freedom is characteristic for quantum mechanics and essential for applications of quantum information. Without entanglement there are only product states, which cannot

produce interference effects between qubits. It is therefore relevant to ask how one can quantify entanglement.

10.5.1 Schmidt decomposition of a pure state

Consider any quantum-mechanical system and two arbitrarily chosen subsystems A and B (like "system" and "environment" before).

Let $|j\rangle_A$ be orthonormal basis states of subsystem A, and $|k\rangle_B$ those of subsystem B. Then a general pure state of the total system is a linear combination

$$|\Psi\rangle = \sum_{j,k} c_{jk} |j\rangle_A |k\rangle_B, \quad (10.22)$$

where c_{jk} are coefficients. We now regard c_{jk} as a matrix and look at its singular value decomposition:

$$(c_{jk}) = \tilde{U} \tilde{D} \tilde{V}^\dagger, \quad \text{with } \tilde{U} \text{ and } \tilde{V} \text{ unitary.} \quad (10.23)$$

Written in matrix components, this becomes $c_{jk} = \sum_{\alpha=1}^{\chi} \lambda_{\alpha} \tilde{U}_{j\alpha} (\tilde{V}_{\alpha k})^\dagger$, where χ is the rank of the matrix c_{jk} , the so called Schmidt-rank. Since $\tilde{U}$ and $\tilde{V}$ are unitary, it is possible to perform two basis transformations: $|A\rangle_{\alpha} := \sum_j \tilde{U}_{j\alpha} |j\rangle_A$ and $|B\rangle_{\alpha} := \sum_k (\tilde{V}^\dagger)_{\alpha k} |k\rangle_B$ and to express the state $|\Psi\rangle$ in the new basis:

$$|\Psi\rangle = \sum_{\alpha=1}^{\chi} \lambda_{\alpha} |A\rangle_{\alpha} |B\rangle_{\alpha}, \quad \text{with } \chi \leq \min(\dim(A), \dim(B)). \quad (10.24)$$

This Schmidt-decomposition of a general state $|\Psi\rangle$ always exists! The rank χ , i.e. the number of terms in the sum, is at most as large as the Hilbert space dimension of the smaller subsystem.

The normalization $\langle\Psi|\Psi\rangle = 1$ implies

$$\sum_{\alpha} \lambda_{\alpha}^2 = 1. \quad (10.25)$$

10.5. Entanglement measure

10.5.2 Reduced Density Matrix of a pure state

The total density matrix of a *pure* state is just $\hat{\rho} = |\Psi\rangle\langle\Psi|$.

With the Schmidt decomposition, this becomes

$$\hat{\rho} = \sum_{\alpha=1}^{\chi} \sum_{\beta=1}^{\chi} \lambda_{\alpha} \lambda_{\beta} \underbrace{|A\rangle_{\alpha}} \underbrace{|B\rangle_{\alpha}} \underbrace{\langle A|_{\beta}} \underbrace{\langle B|_{\beta}}.$$

The *reduced density matrix* for subsystem A is obtained by taking the trace over subsystem B :

$$\hat{\rho}_A = \text{tr}_B \hat{\rho} = \sum_{\gamma} \underbrace{\langle B|}_{\gamma} \hat{\rho} \underbrace{|B\rangle}_{\gamma}. \quad (10.26)$$

Since the bases $|A\rangle$ and $|B\rangle$ are orthonormal, only $\alpha = \beta = \gamma$ contributes and one gets:

$$\hat{\rho}_A = \sum_{\gamma=1}^{\chi} \lambda_{\gamma}^2 \underbrace{|A\rangle_{\gamma} \langle A|}_{\gamma} \quad (10.27)$$

This is the spectral representation of the reduced density matrix, with eigenvalues (weights, probabilities) $p_{\gamma} = \lambda_{\gamma}^2$ and eigenvectors $|A\rangle_{\gamma}$. Thus the Schmidt decomposition immediately implies the reduced density matrix !

The reduced density matrix for sub-system B is defined similarly, with the same Schmidt rank χ , and

$$\hat{\rho}_B = \sum_{\gamma=1}^{\chi} \lambda_{\gamma}^2 \underbrace{|B\rangle_{\gamma} \langle B|}_{\gamma} \quad (10.28)$$

10.5.3 Von Neumann entanglement entropy

The most important measure for the entanglement of two states was already invented by von Neumann. It is similar in definition to the entropy of a statistical system. There the density matrix is diagonal in the classical states. Its entries are the Boltzmann weights $p_i \sim \exp(-E_i/(k_B T))$. The classical entropy of a statistical system can be written as

$$S = -\text{tr}(\hat{\rho} \ln \hat{\rho}). \quad (10.29)$$

For a quantum mechanical system in a pure state, the von Neumann entanglement entropy between two arbitrary subsystems A and B is defined in a similar way via the reduced density matrix:

$$S_A := -\text{tr}_A(\hat{\rho}_A \ln \hat{\rho}_A) \quad (10.30)$$

The reduced density matrix $\hat{\rho}_A$ is diagonal in the basis $|A\rangle$:

$$\hat{\rho}_A = \sum_{\gamma=1}^{\chi} \lambda_{\gamma}^2 |A\rangle_{\gamma} \langle A|.$$

One can therefore take the logarithm separately for each component and the von Neumann entropy reduces to

$$S_A = - \sum_{\gamma=1}^{\chi} \lambda_{\gamma}^2 \ln \lambda_{\gamma}^2 = S_B. \quad (10.31)$$

Thus the entanglement entropy of subsystem A relative to B is the same as that of subsystem B relative to A. Note that when the reduced density matrix corresponds to a pure state (i.e. $\chi = 1$ and $\lambda = 1$), then the entanglement entropy is zero, since $\ln 1 = 0$.

(see below)

One can show that S_A takes its maximum possible value when all λ_{γ} are of equal value, i.e. $\lambda_{\gamma}^2 = \frac{1}{\chi}$ (since $\sum_{\gamma} \lambda_{\gamma}^2 = 1$). Then the reduced density matrix is just a normalized identity matrix:

Then:
$$\hat{\rho}_A = \frac{1}{\chi} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad (10.32)$$

10.6. No-Cloning-Theorem

and S_A is given by:

Then:
$$S_A = -\chi \left(\frac{1}{\chi} \ln \frac{1}{\chi} \right) = \ln \chi \quad (\text{max. possible value}) \quad (10.33)$$

Examples:

$$\hat{S} = |A\rangle\langle B| + |B\rangle\langle A|$$

- For a product state $|\psi\rangle = |A\rangle|B\rangle$ we have $\hat{\rho}_A = |A\rangle\langle A|$, thus $\chi = 1$, $\lambda = 1$, $\ln \lambda = 0$ and $S_A = 0$, as expected.
- For a singlet, the expression

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B)$$

is already the Schmidt-decomposition, with $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{2}}$. The reduced density matrix is

$$\begin{aligned} \hat{\rho}_A &= \sum_{\alpha=1}^2 \lambda_{\alpha}^2 |A\rangle_{\alpha} \langle A| = \frac{1}{2} (|\uparrow\rangle_A \langle \uparrow| + |\downarrow\rangle_A \langle \downarrow|) \\ &\hat{=} \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_A = \frac{1}{2} \mathbb{I}_A. \end{aligned}$$

The entanglement entropy is $S_A = -2(\frac{1}{2} \ln \frac{1}{2}) = \ln 2 = \ln \chi$. Since $\chi = 2$ is also the dimension of the Hilbert spaces $\{\uparrow, \downarrow\}$ of the individual particles, they are maximally entangled in a singlet and can therefore not also be entangled with any other particle. !

10.6 No-Cloning-Theorem

Classical bits can be copied and reused with high reliability. Very efficient error correction algorithms are then possible, which make classical computers largely immune against occasionally occurring bit errors.

However, for qubits a reliable copy of an arbitrary state is not possible, as we shall show now. Manipulations of any quantum mechanical system are done by applying unitary operators (time evolution operators). An operator $\hat{U}$ which would copy the state $|\psi\rangle$ of a qubit A to a qubit B, itself initially in the state $|i\rangle$, would have to achieve

$$\hat{U} (|\psi\rangle_A |i\rangle_B) = |\psi\rangle_A |\psi\rangle_B,$$

and similarly for a different initial state $|\varphi\rangle_A$:

$$\hat{U} \left(|\varphi\rangle_A |i\rangle_B \right) = |\varphi\rangle_A |\varphi\rangle_B.$$

The scalar product of the two left sides has to be equal to the scalar product of the two right sides:

$$\left({}_A\langle\psi| {}_B\langle i| \right) \hat{U}^\dagger \hat{U} \left(|\varphi\rangle_A |i\rangle_B \right) = {}_A\langle\psi| {}_B\langle\psi| |\varphi\rangle_A |\varphi\rangle_B.$$

Because of $\hat{U}^\dagger \hat{U} = \hat{\mathbb{I}}$, the left side also has to be equal to ${}_A\langle\psi| {}_B\langle i| |\varphi\rangle_A |i\rangle_B$. This implies

$${}_A\langle\psi|\varphi\rangle_A {}_B\langle\psi|\varphi\rangle_B = {}_A\langle\psi|\varphi\rangle_A \underbrace{{}_B\langle i|i\rangle_B}_{=1}.$$

$|\varphi\rangle_B, |\psi\rangle_B$ are to be copies of $|\varphi\rangle_A, |\psi\rangle_A$. Thus ${}_A\langle\psi|\varphi\rangle_A = {}_B\langle\psi|\varphi\rangle_B$ and

$$\langle\psi|\varphi\rangle^2 = \langle\psi|\varphi\rangle$$

with the only solutions $\langle\psi|\varphi\rangle = 1$ (i.e. $|\psi\rangle = |\varphi\rangle$) and $\langle\psi|\varphi\rangle = 0$. Therefore, in addition to making a copy of some specific state $|\psi\rangle$ one can at most copy additional orthogonal states $|\varphi\rangle$, but not arbitrary qubits. One can also show that mixed states cannot be copied either.

10.7 Quantum teleportation

A method for “quantum teleportation” was first proposed by Peres et al. in 1993. The first experimental realizations were achieved in 1997 in Innsbruck by A. Zeilinger and his group, and at about the same time also by a group in Great Britain. Because of the complexity of the actual experiments, we will concentrate on the basic idea.

The basic setup is shown in Fig. 10.7. The goal is to transfer the quantum mechanical state

$$|\varphi\rangle = a|0\rangle + b|1\rangle, \quad (10.34)$$

of a particle “C” onto a different particle “B”. There are two experimenters, traditionally called “Alice” and “Bob”. Alice has the original particle C whose state shall eventually be transferred to particle B at the location of Bob. Because of the no-cloning theorem, after the experiment, particle C

10.7. Quantum teleportation

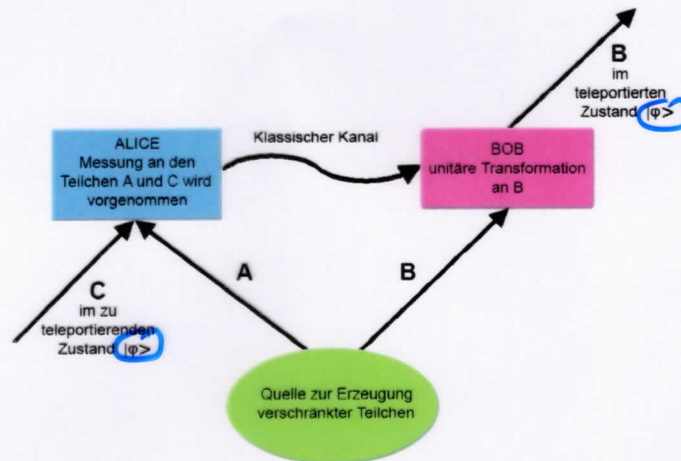


Figure 10.1: Basic scheme of quantum teleportation (Christina Weiss, Bachelor Thesis 2007)

cannot be in the original state anymore. Note that in this experiment particles are not teleported, different from science fiction stories. Instead “only” a quantum mechanical state is transferred between already existing particles.

The course of events in quantum teleportation is as follows: Alice possesses a particle C, in an unknown (!) state $|\varphi\rangle$ to be teleported. Now some source generates a pair of particles A and B (usually photons) in a specific entangled state. Particle A is transferred to Alice and particle B to Bob. Then Alice performs a specific measurement on particles A and C, which produces an entangled state of these two particles. She transmits the result of this measurement to Bob via a classical channel. In response to the measurement result, Bob now has to do a transformation on particle B (like e.g. $|0\rangle \leftrightarrow |1\rangle$). Afterwards, particle B will be in the desired state $|\varphi\rangle$.

We will now discuss the states and operators. We need the so-called Bell

Bell

basis for the state of two qubits A and B:

$$\begin{aligned} |\Psi^-\rangle_{AB} &:= \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B) \\ |\Psi^+\rangle_{AB} &:= \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B) \\ |\Phi^-\rangle_{AB} &:= \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B) \\ |\Phi^+\rangle_{AB} &:= \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B) \end{aligned}$$

This basis is similar to the basis of singlet (similar to $|\Psi^-\rangle$) and triplet discussed before, with the difference that it contains only entangled states.

Let the source produce the entangled state $|\Psi^+\rangle_{AB}$. This can be done for example by generating two entangled photons from a single high energy photon in a birefringent crystal. The initial state of the the total system of particles A, B, and C is then

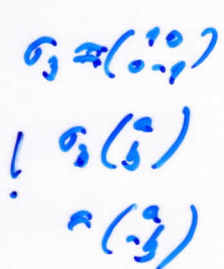
$$|\psi\rangle_{total} = |\varphi\rangle_C \cdot |\Psi^+\rangle_{AB} \quad (10.35)$$

One can write this same state in a different way, namely in the Bell-basis of A and C, multiplied by states of B. A short calculation shows that

$$\begin{aligned} 2 |\psi\rangle_{total} &= |\Psi^+\rangle_{AC} \cdot |\varphi\rangle_B \\ &+ |\Psi^-\rangle_{AC} \cdot \sigma_3 |\varphi\rangle_B \\ &+ |\Phi^+\rangle_{AC} \cdot \sigma_1 |\varphi\rangle_B \\ &+ |\Phi^-\rangle_{AC} \cdot (-i\sigma_2) |\varphi\rangle_B \end{aligned} \quad (10.36)$$

Now Alice does a measurement which distinguishes the four Bell states of A and C. This is very challenging experimentally. According to the postulates of quantum mechanics, after the measurement A and C are in the corresponding state. They have thus be entangled by the measurement. As we can see from (10.36), this also fixes the state of particle B. Alice transmits the outcome of her measurement to Bob by classical communication. Depending on that outcome, Bob either need not do anything (in case of $|\Psi^+\rangle_{AC}$) or he still has to suitably manipulate the state of particle B.

In the first experiment in 1997, only the state $|\Psi^-\rangle_{AC}$ could be detected, by making use of the antisymmetry of the wave function. Thus a "teleportation" was only achieved in a quarter of all cases. Also, the transformation on particle B was actually left out.



10.7. Quantum teleportation

Nowadays, twenty years later, teleportation in the sense described has been demonstrated for far more difficult cases, e.g. for entangled pairs of particles, or for distances of more than 100km, etc. This area of research is on constant development, similar to quantum information theory as a whole.