# ON THE (MODIFIED) KADOMTSEV-PETVIASHVILI HIERARCHY

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Abstract. We present a novel approach to the Kadomtsev-Petviashvili (KP) hierarchy and its modified counterpart, the mKP hierarchy based on factorizations of formal pseudo-differential operators and a matrix-valued Lax operator for the mKP hierarchy. As a result of this framework we obtain new Bäcklund transformations for the KP hierarchy and the possibility of transferring classes of KP solutions into those of mKP solutions, and vice versa. As an application of our techniques we provide a new derivation of soliton solutions of the KP and mKP equation.

## 1. Introduction

In this note we extend previous results on the Gelfand-Dickey (GD) and Drinfeld-Sokolov (DS) hierarchies and GD Bäcklund transformations in [8]-[13] to the nonlinear evolution equations of the Kadomtsev-Petviashvili (KP) and modified Kadomtsev-Petviashvili (mKP) hierarchy (see Section 2 for precise definitions of the (m)KP hierarchy). Our main new technique, when compared to the traditional approach to the KP hierarchy (see (2.16)), consists of replacing the usual first-order formal pseudo-differential Lax operator

$$L_1 = \partial_x + \sum_{j=-\infty}^{-1} u_j \partial_x^j \tag{1.1}$$

by an $n^{th}$-order formal pseudo-differential operator

$$L_n = \partial_x^n + \sum_{j=-\infty}^{n-2} q_j \partial_x^j, \quad n \geq 2. \tag{1.2}$$

This enables us to derive new KP Bäcklund transformations by studying factorizations of $L_n$ into $n-1$ first-order formal differential operators $A_k$, $1 \leq k \leq n-1$ and one first-order formal pseudo-differential operator $\tilde{A}_n$ of the type

$$L_n = \tilde{A}_n A_{n-1} \cdots A_2 A_1, \tag{1.3}$$

$$A_k = \partial_x + \eta_{k,x}, \quad 1 \leq k \leq n, \quad \sum_{k=1}^{n} \eta_{k,x} = 0, \tag{1.4}$$

$$\tilde{A}_n = A_n + \sum_{j=-\infty}^{-1} b_{n,j} \partial_x^j. \tag{1.5}$$

Associated with the factorization (1.3) we introduce the following matrix-valued Lax operator $\mathcal{M}_n$

$$\mathcal{M}_n = \begin{pmatrix} 0 & 0 & \dots & 0 & \tilde{A}_n \\ A_1 & 0 & \ddots & & 0 \\ 0 & A_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_{n-1} & 0 \end{pmatrix}, \quad n \geq 2 \tag{1.6}$$

for the mKP hierarchy (see (2.43)). The Miura-type identity

$$\mathcal{M}_n^n = \begin{pmatrix} L_{n,1} & 0 & \dots & 0 \\ 0 & L_{n,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & L_{n,n} \end{pmatrix}, \tag{1.7}$$

where

$$L_{n,k} = A_{k-1} \cdots A_2 A_1 \tilde{A}_n \cdots A_{k+1} A_k, \quad 1 \leq k \leq n \tag{1.8}$$

are of the form (1.2) (here indices are taken mod $n$), then implies in a manner well-known from GD and DS systems (see, e.g., [13] and the references therein) the following link between solutions of the KP and mKP hierarchy: any solution $(\underline{\eta}, \underline{b}_n) = (\eta_1, \dots, \eta_n, b_{n,j})_{j \leq -1}$ of the mKP hierarchy (2.43) yields $n$ solutions $\underline{q}_k = (q_{j,k})_{j \leq n-2}$, $1 \leq k \leq n$ of the KP hierarchy (2.16). Our main result in Theorem 2.5 and Corollary 2.6 reverses this procedure, i.e., given a solution $\underline{q}_1 = (q_{j,1})_{j \leq n-2}$ of the KP hierarchy we construct an associated solution $(\underline{\eta}, \underline{b}_n) = (\eta_1, \dots, \eta_n, b_{n,j})_{j \leq -1}$ of the mKP hierarchy and $n-1$ further solutions $\underline{q}_k = (q_{j,k})_{j \leq n-2}$, $2 \leq k \leq n$ of the KP hierarchy. (We note that $\frac{2}{n} q_{n-2}$ and $\eta_k$, $1 \leq k \leq n-1$ solve the KP equation (2.21) and mKP equation (2.52) in standard form.) In this way whole classes of solutions such as soliton solutions, rational solutions etc. can be transferred from the KP hierarchy to the mKP hierarchy and vice versa.

It must be pointed out at this occasion that the use of $L_n$ respectively $\mathcal{M}_n$ in the context of the KP respectively mKP hierarchy is not new but goes back to an observation in [21]. However, our particular factorization of $L_n$ into $n-1$ formal $1^{st}$-order differential operators and one formal $1^{st}$-order pseudo-differential operator was not studied in [21] and no connections to KP Bäcklund transformations were established. It is our use of $L_n$ instead of the traditional Lax operator $L_1$ (see, e.g., [3], [4], [5], [14], [17], [26], [29]) or the partial differential operator $L = \partial_x^2 + \partial_{t_2} + u$ (see, e.g., [11], [12], [19], [23], [24], Chapter 3) in conjunction with the matrix-valued Lax operator $\mathcal{M}_n$ for the mKP hierarchy which allows one to obtain $n-1$ further solutions of the KP hierarchy as opposed to just one further such solution in the context of $L_1$ or $L$.

The efficiency of our approach is illustrated by Example 2.9 where we provide a new derivation of soliton solutions of the KP and mKP equation. In particular, by choosing $n$ appropriately ($n = 2N+2$), our formalism allows one to construct the $N$-soliton KP and associated $(2N-1)$-soliton mKP solutions without recourse to formal pseudo-differential operators but solely within the class of formal differential operators.

In order to be widely applicable, we present our main results in Section 2 in a general algebraic framework.

Finally, we emphasize that the KP hierarchy plays an important role in a variety of different fields including modern string theory and in connection with the solution of the Schottky problem of compact Riemann surfaces [26], [32]. Moreover, a large variety of completely integrable nonlinear evolution equations can be derived by special reductions from the KP or mKP hierarchy [15], [30]. (For more complex systems requiring an extension of $L_n$ of the form $\sum_{j=-\infty}^{n} u_j \partial_x^j$ with $u_n \neq 1, u_{n-1} \neq 0$ in general, see, e.g., [18], [20], [36]. A suitable modification of our approach extends to this situation.)

## 2. KP and mKP hierarchies

We start by briefly reviewing the following algebraic framework (see, e.g., [1], [2], [5], Chapter 1, [6], [7], [13], [22], [25], [29], [31], [33]-[35] for details).

Let $A$ be a commutative differential algebra defined over $\mathbb{C}$ with unity 1 and a derivation $\partial : A \to A$ satisfying the following conditions:

- **(i):** $\partial$ is surjective on $A$ (i.e., for every $f \in A$ there exists a $g \in A$ such that $\partial g = f$).
- **(ii):** $A$ is closed under exponentiation (i.e., for any $f \in A$ the expression $\sum_{n=0}^{\infty} f^n/n! = e^f$ yields an element of $A$).

The polynomial algebra (algebra of formal differential operators) generated by $A \cup \{\xi\}$ is then given by

$$A[\xi] = \bigg\{ \sum_{j=0}^{N} a_j \xi^j \mid a_j \in A,\ 0 \leq j \leq N, N \in \mathbb{N}_0 \bigg\}, \tag{2.1}$$

where

$$\begin{aligned} &\xi^0 a = a, \quad \xi^j a = \sum_{\ell=0}^{j} \binom{j}{\ell} a^{(\ell)} \xi^{j-\ell}, \quad j \in \mathbb{N}, \\ &a^{(0)} = a, \quad a^{(\ell)} = \partial^\ell a, \qquad \ell \in \mathbb{N}, \quad a \in A. \end{aligned} \tag{2.2}$$

We also introduce the algebra of formal pseudo-differential operators with coefficients in $A$

$$A((\xi^{-1})) = \bigg\{ \sum_{j=-\infty}^{M} a_j \xi^j \mid a_j \in A,\ j \leq M, M \in \mathbb{Z} \bigg\} \tag{2.3}$$

with the extended Leibniz rule

$$\xi^{-j}a = \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{j+\ell-1}{\ell}a^{(\ell)}\xi^{-j-\ell}, \quad j \in \mathbb{N}, \text{Ↄ} \in \mathbb{A}. \tag{2.4}$$

For elements $S = \sum_{j=-\infty}^{M} s_j\xi^j \in A((\xi^{-1}))$ one writes

$$S_+ = \sum_{j=0}^{M} s_j\xi^j, \quad S_- = \sum_{j=-\infty}^{-1} s_j\xi^j, \qquad S = S_+ + S_- \tag{2.5}$$

and calls $S_+$ the (formal) differential operator part of $S$. The order of $S$ is defined by

$$\text{ord}(S) = \max\{j \in \mathbb{Z} \mid \sim\!\jmath \neq \nvdash\}. \tag{2.6}$$

Consider for a fixed $n \in \mathbb{N}$ an element of $A((\xi^{-1}))$ of the form

$$L_n = \xi^n + \sum_{j=-\infty}^{n-2} q_j\xi^j \in A((\xi^{-1})). \tag{2.7}$$

Then there exists an element $K_n = 1 + \sum\limits_{j=-\infty}^{-1} \chi_j\xi^j \in A((\xi^{-1}))$ (the formal dressing operator of Zakharov-Shabat [33], [37]) such that

$$L_n = K_n\xi^n K_n^{-1}. \tag{2.8}$$

Moreover, $K_n$ is unique up to right multiplication by a constant coefficient operator

$$M = 1 + \sum_{j=-\infty}^{-1} c_j\xi^j, \quad c_j = \text{ const}, \quad j \in \mathbb{N}. \tag{2.9}$$

On the subalgebra $B$ of $A$ generated by $q_j,\ j \leq n-2$, we associate the degree (weight)

$$\deg(q_j^{(\ell)}) = n + \ell - j, \qquad \ell \in \mathbb{N}_{\nvdash} \tag{2.10}$$

with $q_j^{(\ell)}$. $B$ becomes a $\mathbb{Z}$-graded algebra and $\partial$ is then homogeneous of degree 1. (In making use of the grading (2.10) it is implicitly assumed that there is no polynomial relation between the $q_j^{(l)}$.) Defining $\deg(\xi) = 1$, this grading naturally extends to $B[\xi]$ and $B((\xi^{-1}))$. $L_n$ is then homogeneous of degree $n$. (We recall that $K_n$, unlike $L_n$, is not an element of $B((\xi^{-1}))$.)

Next, for $L \in A((\xi^{-1}))$, we denote by $C_A(\{L\})$ the centralizer

$$C_A(\{L\}) = \Big\{P \in A((\xi^{-1})) \mid [P, L] = 0\Big\} \tag{2.11}$$

and by $Z(C_A(\{L\}))$ the center of the centralizer of $\{L\}$.

Let $P_{0,r} = \xi^r, r \in \mathbb{N}$, $P_{n,r} = K_nP_{0,r}K_n^{-1}$ then $P_{n,r} = (L_n)^{\frac{r}{n}}$ and $P_{n,r} \in C_A(\{L\})$, i.e., $[P_{n,r}, L_n] = 0$. Writing

$$(P_{n,r})_+ = (L_n^{\frac{r}{n}})_+ = \xi^r + p_{r-2}\xi^{r-2} + \ldots + p_0, \qquad r \in \mathbb{N}, \tag{2.12}$$

one obtains, e.g., for $r = 1, 2, 3$ :

$$(P_{n,1})_+ \;=\; \xi, \tag{2.13}$$

$$(P_{n,2})_+ \;=\; \xi^2 + \frac{2}{n} q_{n-2}, \tag{2.14}$$

$$(P_{n,3})_+ \;=\; \xi^3 + \frac{3}{n} q_{n-2}\xi + \frac{3}{n}\left(q_{n-3} + \frac{3-n}{2}\partial q_{n-2}\right). \tag{2.15}$$

Let the elements of the algebra $A$ depend on the parameters $t_r$, $r \in \mathbb{N}$. Then for any fixed n, the KP$_n$ hierarchy is defined by the system

$$\partial_{t_r} L_n = [(P_{n,r})_+, L_n], \qquad r \in \mathbb{N}. \tag{2.16}$$

In terms of the coefficients $q_j$ of $L_n$, (2.16) yields the KP$_n$ system

$$\begin{aligned} \mathrm{KP}_{n,r,j}(\underline{q}) \;&=\; q_{j,t_r} - \mathcal{E}_{\backslash,\nabla,|}(\underline{\amalg}) = \prime, \qquad \underline{\amalg} = \{\amalg_\ell\}_{-\infty<\ell\leq\backslash-\in}, \\ &\qquad -\infty < j \leq n-2, \quad r \in \mathbb{N}, \end{aligned} \tag{2.17}$$

where the $\mathcal{E}_{\backslash,\nabla,|}$ are differential polynomials in $q_\ell$ of degree $r + \ell - j$.

**Example 2.1.** *For $q_{n-2}$ and $q_{n-3}$ the equations for $r = 2$ read*

$$q_{n-2,t_2} \;=\; (2-n)\partial^2 q_{n-2} + 2\partial q_{n-3}, \tag{2.18}$$

$$\begin{aligned} q_{n-3,t_2} \;&=\; \partial^2 q_{n-3} + 2\partial q_{n-4} - \frac{1}{3}(n-1)(n-2)\partial^3 q_{n-2} \\ &\quad - \frac{2}{n}(n-2) q_{n-2}\partial q_{n-2}. \end{aligned} \tag{2.19}$$

*For $q_{n-2,t_3}$ one gets*

$$\begin{aligned} q_{n-2,t_3} \;&=\; \frac{1}{4}(n^2 - 6n + 9)\partial^3 q_{n-2} - \frac{3}{2}(n-3)\partial^2 q_{n-3} \\ &\quad + 3\partial q_{n-4} - \frac{3}{n}(n-3) q_{n-2}\partial q_{n-2}. \end{aligned} \tag{2.20}$$

*We can eliminate $q_{n-3}, q_{n-4}$ from (2.20) by (2.18) and (2.19) and writing $\tilde{q}_{n-2} = \frac{2}{n} q_{n-2}$ yields the KP equation in standard form*

$$\partial \tilde{q}_{n-2,t_3} = \frac{1}{4}\partial^4 \tilde{q}_{n-2} + \frac{3}{2}\partial(\tilde{q}_{n-2}\partial\tilde{q}_{n-2}) + \frac{3}{4}\tilde{q}_{n-2,t_2t_2}. \tag{2.21}$$

**Remark 2.2.** *(i) The traditional approach uses $n = 1, L_1 = \xi + \sum_{j=-\infty}^{-1} u_j \xi^j$. Then $L_n = (L_1)^n$, (i.e., $q_{n-2} = n\,u_{-1}$, etc.) and $[\partial_{t_r} - (P_{1,r})_+, L_1] = 0$ implies $[\partial_{t_r} - (P_{1,r})_+, (L_1)^n] = 0$. The opposite direction can be proven using the dressing operator $K_n$: assume*

$$[\partial_{t_r} - (P_{n,r})_+, L_n] = 0, \quad r \in \mathbb{N} \tag{2.22}$$

*and define $L_1 = L_n^{1/n}$. Then $P_{n,r} = P_{1,r}$ and (2.22) is equivalent to*

$$[(\partial_{t_r} K_n)K_n^{-1} - (P_{1,r})_+, L_n] = 0 \tag{2.23}$$

*respectively to*

$$[K_n^{-1}\left((\partial_{t_r} K_n)K_n^{-1} - (P_{1,r})_+\right) K_n, \xi^n] = 0. \tag{2.24}$$

*This immediately implies*

$$[K_n^{-1}\left((\partial_{t_r}K_n)K_n^{-1}-(P_{1,r})_+\right)K_n,\xi]=0 \tag{2.25}$$

*which is equivalent to*

$$[(\partial_{t_r}K_n)K_n^{-1}-(P_{1,r})_+,L_1]=0. \tag{2.26}$$

*Hence we obtain*

$$\partial_{t_r}L_1=\partial_{t_r}(K_n\xi K_n^{-1})=[(\partial_{t_r}K_n)K_n^{-1},L_1]=[(P_{1,r})_+,L_1], \quad r\in\mathbb{N}. \tag{2.27}$$

*However, the choice of $L_n$ with $n\geq 2$ is better suited for deriving Bäcklund transformations as will become clear in Corollary 2.6.*

*(ii) The reduction to the corresponding equations of the GD hierarchy now simply becomes $q_j=0$ for $j\leq -1$.*

*(iii) Since the equations for $r=2$ have the form*

$$q_{j,t_2}=\partial q_{j-1}-\tilde{\mathcal{E}}_{n,2,j}(q_{n-2},\dots,q_j), \quad j\leq n-2, \tag{2.28}$$

*where the $\tilde{\mathcal{E}}_{n,2,j}$ are differential polynomials in $(q_{n-2},\dots,q_j)$, one can express every $q_j$ in terms of $\partial^r\partial_{t_2}^s q_{n-2}$ with $r+2s+j-n+2=0,\ j\leq n-3,\ 0\leq s\leq n-2-j,$ $r\leq n-2-j$. This possibility of expressing all $q_j,\ j\leq n-3$ in terms of one function is well-known to be related to the $\tau$-function formalism underlying the (m)KP hierarchy (see, e.g., [3], [4], [5], Chapter 7, [16], [17], [28]).*

In order to generate the modified KP$_n$ hierarchy we consider the algebra $(A)^n$ of $n\times n$-matrices, $n\geq 2$ with entries in $A$ and similar to (2.1) and (2.3) we then define $(A[\xi])^n$ and $(A((\xi^{-1}))^n$. Let

$$A_k = e^{-\eta_k}\xi e^{\eta_k}=\xi+\partial\eta_k\in A[\xi], \quad 1\leq k\leq n, \qquad \sum_{k=1}^n\partial\eta_k=0, \tag{2.29}$$

$$B_n = \sum_{j=-\infty}^{-1} b_{n,j}\xi^j\in A((\xi^{-1})), \tag{2.30}$$

$$\tilde{A}_n = A_n+B_n, \tag{2.31}$$

and define

$$\mathcal{M}_n = \begin{pmatrix} 0 & 0 & \dots & 0 & \tilde{A}_n \\ A_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & A_{n-1} & 0 \end{pmatrix} \in \Big(A((\xi^{-1}))\Big)^n. \tag{2.32}$$

On the subalgebra $\tilde{B}$ of $A$ generated by $\eta_j,\ 1\leq j\leq n-1,\ b_{n,j},\ j\leq -1$ we associate the degree (weight)

$$\deg(\eta_j^{(\ell)})=\ell, \quad \deg(b_{n,j}^{(\ell)})=\ell-j+1, \quad \ell\in\mathbb{N}_0 \tag{2.33}$$

with $\eta_j^{(\ell)}$ and $b_{n,j}^{(\ell)}$. $\tilde{B}$ becomes a $\mathbb{Z}$-graded algebra and $\partial$ is then homogeneous of degree 1. This grading naturally extends to $\tilde{B}[\xi]$ and $\tilde{B}((\xi^{-1}))$ defining $\deg(\xi) = 1$. Hence $\mathcal{M}_n$ is homogeneous of degree 1. Then

$$\begin{aligned} (\mathcal{M}_n)^n &= \operatorname{diag}\left(\tilde{A}_n \cdots A_2 A_1, A_1 \tilde{A}_n \cdots A_2, \ldots, A_{n-1} \cdots A_2 A_1 \tilde{A}_n\right) \\ &= \operatorname{diag}\left(L_{n,1}, \ldots, L_{n,n}\right), \end{aligned} \tag{2.34}$$

where the $L_{n,k}, 1 \leq k \leq n$ are of the form

$$L_{n,k} = \xi^n + \sum_{j=-\infty}^{n-2} q_{j,k} \xi^j, \quad 1 \leq k \leq n, \tag{2.35}$$

$$\begin{aligned} q_{n-2,k} &= \partial^2 \left((n-1)\eta_k + (n-2)\eta_{k+1} + \ldots + \eta_{k+n-2}\right) \\ &\quad + (\partial\eta_1 \partial\eta_2 + \partial\eta_1 \partial\eta_3 + \ldots + \partial\eta_{n-1}\partial\eta_n) + b_{n,-1} \end{aligned} \tag{2.36}$$

and it is understood that indices are taken mod $n$. The expressions for $q_{n-2-j,k}$ have the form

$$\begin{aligned} q_{n-2-j,k} = b_{n,-1-j} + \mathcal{F}_{n,j,k}(\underline{\eta}, \underline{b}_{n,j}), \quad j \in \mathbb{N}_0, \ 1 \leq k \leq n, \\ \underline{\eta} = (\eta_k)_{1 \leq k \leq n}, \quad \underline{b}_{n,j} = (b_{n,-m})_{1 \leq m \leq j}, \end{aligned} \tag{2.37}$$

where the $\mathcal{F}_{n,j,k}$ are differential polynomials in $\eta_k$ of degree $j+2$ and in $b_{n,-m}$ of degree $j+1-m$.

Note that

$$q_{n-2,k+1} - q_{n-2,k} = -n\partial^2\eta_k, \qquad 1 \leq k \leq n, \tag{2.38}$$

(where indices are again taken mod $n$).

Define $\mathcal{Q}_{n,r}$ by

$$\mathcal{Q}_{n,r} = \operatorname{diag}\left(P_{n,r,1}, \ldots P_{n,r,n}\right), \quad P_{n,r,k} = (L_{n,k})^{\frac{r}{n}}, \quad 1 \leq k \leq n, \quad r \in \mathbb{N}, \tag{2.39}$$

i.e., (see (2.13), (2.14), (2.15)),

$$(P_{n,1,k})_+ = \xi, \tag{2.40}$$

$$(P_{n,2,k})_+ = \xi^2 + \frac{2}{n} q_{n-2,k}, \tag{2.41}$$

$$(P_{n,3,k})_+ = \xi^3 + \frac{3}{n} q_{n-2,k}\xi + \frac{3}{n}\left(q_{n-3,k} + \frac{3-n}{2}\partial q_{n-2,k}\right). \tag{2.42}$$

Then the mKP$_n$ hierarchy is defined by the system

$$\partial_{t_r}\mathcal{M}_n = [(\mathcal{Q}_{n,r})_+, \mathcal{M}_n], \qquad r \in \mathbb{N}. \tag{2.43}$$

In terms of the coefficients $\eta_k, b_{n,j}$, (2.43) yields the mKP$_n$ system

$$\begin{aligned} \mathrm{mKP}_{n,r,j}(\underline{\eta}, \underline{b}_n) &= \partial\eta_{j,t_r} - \mathcal{G}_{n,r,j}(\partial\underline{\eta}, \underline{b}_n) = 0, \quad 1 \leq j \leq n, \quad r \in \mathbb{N}, \\ \mathrm{mKP}_{n,r,j}(\underline{\eta}, \underline{b}_n) &= b_{n,j,t_r} - \mathcal{G}_{n,r,j}(\partial\underline{\eta}, \underline{b}_n) = 0, \quad -j, r \in \mathbb{N}, \\ \underline{\eta} &= (\eta_k)_{1 \leq k \leq n}, \quad \underline{b}_n = (b_{n,-m})_{m \in \mathbb{N}}, \end{aligned} \tag{2.44}$$

where the $\mathcal{G}_{\backslash,\nabla,|}$ are differential polynomials in $\eta_k,\ 1 \le k \le n$ of degree $r+1$ for $j \ge 1$, respectively of order $r+1-j$ for $j \le -1$ and in $b_{n,-m}$ of degree $r-m$ for $j \ge 1$, respectively of order $r-m-j$, for $j \le -1$.

**Remark 2.3.** *The possibility of using a matrix-valued Lax operator $\mathcal{M}_n$ in connection with the mKP hierarchy and an $n^{th}$-order (formal pseudo-differential) operator $L_n$ in connection with the KP hierarchy was first observed in* [21]. *However, our particular factorization of $L_n$ into $n-1$ $1^{st}$-order formal differential operators and one $1^{st}$-order formal pseudo-differential operator and, especially, its use in obtaining KP Bäcklund transformations modeled after our treatment of the GD and DS hierarchies in* [13] *appears to be new.*

**Example 2.4.** *For $r=2$ we get for $\eta_k, b_{n,-1}, b_{n,-m}$*

$$\begin{aligned}
\partial\eta_{k,t_2} &= \partial^3\eta_k - 2\partial\eta_k\partial^2\eta_k - \frac{2}{n}\partial q_{n-2,k}, \qquad 1 \le k \le n-1, && (2.45)\\
\partial\eta_{n,t_2} &= \partial^3\eta_n - 2\partial\eta_n\partial^2\eta_n - \frac{2}{n}\partial q_{n-2,n} + 2\partial b_{n,-1} && (2.46)\\
b_{n,-1,t_2} &= \partial^2 b_{n,-1} - 2b_{n,-1}\partial^2\eta_n + 2\partial b_{n,-2}. && (2.47)\\
b_{n,-m,t_2} &= \partial^2 b_{n,-m} - 2b_{n,-m}\partial^2\eta_n + 2\partial b_{n,-m-1}\\
&\quad -\frac{2}{n}\sum_{p=1}^{m-1}(-1)^p\binom{m-1}{p} b_{n,-m+p}\partial^p q_{n-2,n}, \qquad m \ge 2. && (2.48)
\end{aligned}$$

*This yields the following identities*

$$\begin{aligned}
\frac{2}{n}q_{n-2,k} &= -\eta_{k,t_2} - (\partial\eta_k)^2 + \partial^2\eta_k, \qquad 1 \le k \le n-1,\\
\frac{2}{n}q_{n-2,k} &= -\eta_{k-1,t_2} - (\partial\eta_{k-1})^2 - \partial^2\eta_{k-1}, \qquad 2 \le k \le n-1,\ n \ge 3,\\
\frac{2}{n}q_{n-2,n} &= -\eta_{n,t_2} - (\partial\eta_n)^2 + \partial^2\eta_n + 2b_{n,-1}\\
&= -\eta_{n-1,t_2} - (\partial\eta_{n-1})^2 - \partial^2\eta_{n-1}, && (2.49)
\end{aligned}$$

*and*

$$\begin{aligned}
&\eta_{k,t_2} + (\partial\eta_k)^2 + \partial^2\eta_k = \eta_{k+1,t_2} + (\partial\eta_{k+1})^2 - \partial^2\eta_{k+1}, \quad 1 \le k \le n-2,\ n \ge 3\\
&\eta_{n-1,t_2} + (\partial\eta_{n-1})^2 + \partial^2\eta_{n-1} = \eta_{n,t_2} + (\partial\eta_n)^2 - \partial^2\eta_n - 2b_{n,-1},\\
&\eta_{n,t_2} + (\partial\eta_n)^2 + \partial^2\eta_n = \eta_{1,t_2} + (\partial\eta_1)^2 - \partial^2\eta_1 + 2b_{n,-1}. && (2.50)
\end{aligned}$$

*For $\eta_{k,t_3}$ we get*

$$\begin{aligned}
\partial\eta_{k,t_3} &= \partial^4\eta_k + \frac{3}{n}q_{n-2,k+1}\partial^2\eta_k + \frac{3}{n}(q_{n-3,k+1} - q_{n-3,k})\partial\eta_k\\
&\quad -\frac{3(n-3)}{2n}(\partial q_{n-2,k+1} - \partial q_{n-2,k})\partial\eta_k - \frac{3}{n}\partial q_{n-3,k} + \frac{3(n-3)}{2n}\partial^2 q_{n-2,k}\\
&\quad +3(\partial^2 b_{n,-1} + \partial b_{n,-2} - b_{n,-1}\partial^2\eta_k)\delta_{n,k}, \quad 1 \le k \le n. && (2.51)
\end{aligned}$$

*Eliminating $q_{n-2,k+1}, q_{n-2,k}, q_{n-3,k+1}, q_{n-3,k}, b_{n,-1},$ and $b_{n,-2}$ in (2.51) by (2.18), (2.38) and the identities (2.49)-(2.50), we see that $\eta_k,\ 1 \leq k \leq n-1$ fulfill the mKP equation in standard form*

$$\begin{aligned}
\partial\eta_{k,t_3} &= \frac{1}{4}\partial^4\eta_k - \frac{3}{2}\partial^2\eta_k(\partial\eta_k)^2 - \frac{3}{2}\partial^2\eta_k\,\eta_{k,t_2} + \frac{3}{4}\eta_{k,t_2t_2}, \quad 1 \leq k \leq n-1, \\
\partial\eta_{n,t_3} &= \frac{1}{4}\partial^4\eta_n - \frac{3}{2}\partial^2\eta_n(\partial\eta_n)^2 - \frac{3}{2}\partial^2\eta_n\,\eta_{n,t_2} + \frac{3}{4}\eta_{n,t_2t_2} + 3b_{n,-1}\partial^2\eta_n.
\end{aligned} \tag{2.52}$$

*In the special case $n = 3, b_{n,-m} = 0, m \in \mathbb{N}$ this gives three stationary mKP$_3$ equations ($\partial\eta_{k,t_3} = 0$) which are equivalent to the system of modified Boussinesq equations in* [8].

The identity

$$\frac{\partial(\mathcal{M}_n)^n}{\partial t_r} = [(\mathcal{Q}_{n,r})_+, (\mathcal{M}_n)^n], \quad (\mathcal{M}_n)^n = \operatorname{diag}(L_{n,1}, \ldots, L_{n,n}), \quad r \in \mathbb{N} \tag{2.53}$$

then proves in a trivial way that a solution of the mKP$_n$ hierarchy implies $n$ solutions of the KP$_n$ hierarchy.

Corresponding to our work on the GD and the DS hierarchy we now reverse this process, i.e., given a solution of the KP$_n$ hierarchy we construct a solution of the the modified KP$_n$ hierarchy and obtain $(n-1)$ additional solutions of the KP$_n$ hierarchy. (Note that most of the traditional approaches to the mKP equation use a scalar Lax pair and therefore are restricted to only one further solution of the KP$_n$ hierarchy, see, e.g., [3], [11], [12], [20], [23], [24], [27].)

We introduce the action of formal differential operators $S \in A[\xi]$ on elements $\psi$ of $A$ by

$$\xi\psi = \partial\psi. \tag{2.54}$$

Since $\partial$ is surjective on $A$ there exists an element $x \in A$ such that $\partial x = 1$. Hence we define the action of formal pseudo-differential operators on $x^j$ by

$$\xi^{-1}x^j = (j+1)^{-1}x^{j+1}, \quad x^0 = 1,\ j \in \mathbb{N}_{\nvdash}. \tag{2.55}$$

Our main result then reads as follows.

**Theorem 2.5.** *Given a solution $\underline{q}_1 = (q_{j,1})_{-\infty<j\leq n-2}$, $n \geq 2$ of the KP$_n$ hierarchy (2.17), define the operators $L_{n,1}$ and $K_{n,1}$ as in (2.7) and (2.8) and construct $n$ vectors $\psi_{n,k}$ lying in the kernel of $L_{n,1}$, i.e., $\psi_{n,k} = K_{n,1}\psi_{0,k}$ where $\psi_{0,k} = x^{k-1}, x^0 = 1,\ 1 \leq k \leq n$. Moreover, assume*

$$(\partial_{t_r} - (P_{n,r,1})_+)\,\psi_{n,k} = \sum_{\ell=1}^{n} \alpha_{n,r,k,\ell}\psi_{n,\ell}, \quad r \in \mathbb{N}, \qquad \nVdash \leq \daleth \leq \ltimes, \tag{2.56}$$

*where* $\alpha_{n,r,k,\ell}$ *are possibly* $t_r$*-dependent constants. Define* $\partial\eta_k$ *by*

$$\partial\eta_1 = -\psi_{n,1}^{-1}\partial\psi_{n,1}, \qquad \partial\eta_k = -W(\psi_{n,1},\dots,\psi_{n,k})^{-1}\partial W(\psi_{n,1},\dots,\psi_{n,k})$$
$$+W(\psi_{n,1},\dots,\psi_{n,k-1})^{-1}\partial W(\psi_{n,1},\dots,\psi_{n,k-1}), \qquad 2\le k\le n-1, \ (\textit{if } n\ge 3),$$
$$\partial\eta_n = -\sum_{k=1}^{n-1}\partial\eta_k, \tag{2.57}$$

*where* $W$ *denotes the Wronskian and we assume that* $W(\psi_{n,1},\dots,\psi_{n,k}),\ 1\le k\le n$ *is invertible. Let* $b_{n,-m}, m\in\mathbb{N}$ *be given by (2.36), (2.37). Then*

$$L_{n,1} = \tilde{A}_n A_{n-1}\cdots A_2 A_1, \tag{2.58}$$

*where*

$$A_k = \xi + \partial\eta_k, \quad 1\le k\le n, \qquad \tilde{A}_n = A_n + \sum_{j=-\infty}^{-1} b_{n,j}\xi^j. \tag{2.59}$$

*In addition,* $(\underline{\eta},\underline{b}_n)$ *satisfies the mKP*$_n$ *hierarchy*

$$mKP_{n,r,j}(\underline{\eta},\underline{b}_n) = 0, \qquad -\infty<j\le n,\ j\ne 0, \quad r\in\mathbb{N}, \tag{2.60}$$
$$\underline{\eta} = (\eta_k)_{1\le k\le n}, \quad \underline{b}_n = (b_{n,-m})_{-m\in\mathbb{N}}$$

*iff*

$$\alpha_{n,r,k,\ell} = 0 \quad \textit{for} \quad k+1\le\ell\le n, \quad 1\le k\le n-1 \quad \textit{in (2.56).} \tag{2.61}$$

Proof. We have

$$\Big[(\partial_{t_r}-\mathcal{Q}_{n,r})_+,\mathcal{M}_n\Big]\begin{pmatrix}\psi_{n,1}\\ A_1\psi_{n,2}\\ \vdots\\ A_{n-1}\cdots A_1\psi_{n,n}\end{pmatrix}$$

$$= \begin{pmatrix} 0 & \dots & 0 & d_{1,n}\\ \mathrm{mKP}_{n,r,1}(\underline{\eta},\underline{b}_n) & \ddots & 0 & 0\\ \vdots & \ddots & 0 & 0\\ 0 & \dots & \mathrm{mKP}_{n,r,n-1}(\underline{\eta},\underline{b}_n) & 0\end{pmatrix}\begin{pmatrix}\psi_{n,1}\\ A_1\psi_{n,2}\\ \vdots\\ A_{n-1}\cdots A_1\psi_{n,n}\end{pmatrix} \tag{2.62}$$

with $d_{1,n} = \mathrm{mKP}_{n,r,n}(\underline{\eta},\underline{b}_n) + \sum_{j=-1}^{-\infty}\mathrm{mKP}_{n,r,j}(\underline{\eta},\underline{b}_n)$.

This implies

$$\begin{aligned}\mathrm{mKP}_{n,r,1}(\underline{\eta},\underline{b}_n)\psi_{n,1} &= \left((\partial_{t_r}-(P_{n,r,2})_+)A_1 - A_1(\partial_{t_r}-(P_{n,r,1})_+)\right)\psi_{n,1}\\ &= -A_1(\partial_{t_r}-(P_{n,r,1})_+)\psi_{n,1}\\ &= -A_1\sum_{\ell=1}^{n}\alpha_{n,r,1,\ell}\psi_{n,\ell} = -\sum_{\ell=2}^{n}\alpha_{n,r,1,\ell}A_1\psi_{n,\ell}\end{aligned} \tag{2.63}$$

since

$$A_1\psi_{n,1} = (\xi - \psi_{n,1}^{-1}\partial\psi_{n,1})\psi_{n,1} = 0. \tag{2.64}$$

Thus $\mathrm{mKP}_{n,r,1}(\underline{\eta}, \underline{b}_n) = 0$ iff $\alpha_{n,r,1,\ell} = 0,\ 2 \leq \ell \leq n$.

$$\begin{aligned}
\mathrm{mKP}_{n,r,2}(\underline{\eta}, \underline{b}_n)A_1\psi_{n,2} &= ((\partial_{t_r} - (P_{n,r,3})_+)A_2 - A_2(\partial_{t_r} - (P_{n,r,2})_+))\, A_1\psi_{n,2} \\
&= -\left(A_2(\partial_{t_r} - (P_{n,r,2})_+)\right) A_1\psi_{n,2}, \\
&= -A_2A_1(\partial_{t_r} - (P_{n,r,1})_+)\psi_{n,2}, \\
&= -A_2A_1\sum_{\ell=1}^{n}\alpha_{n,r,2,\ell}\psi_{n,\ell} = -\sum_{\ell=3}^{n}\alpha_{n,r,2,\ell}A_2A_1\psi_{n,\ell}.
\end{aligned} \tag{2.65}$$

Here we used (2.63) and

$$\begin{aligned}
A_2A_1\psi_{n,2} &= (\xi + \partial\eta_2)(\xi + \partial\eta_1)\psi_2 = \left(\xi^2 + \partial(\eta_1 + \eta_2)\xi + \partial\eta_1\partial\eta_2 + \partial^2\eta_1\right)\psi_{n,2} \\
&= \partial^2\psi_{n,2} - \partial(\eta_1 + \eta_2)\partial\psi_{n,2} + \partial\eta_1\partial\eta_2\psi_{n,2} + \partial^2\eta_1\psi_{n,2} = 0.
\end{aligned} \tag{2.66}$$

Therefore $\mathrm{mKP}_{n,r,2}(\underline{\eta}, \underline{b}_n) = 0$ iff $\alpha_{n,r,2,\ell} = 0,\ 3 \leq \ell \leq n$.

Iterating this process we finally get

$$\begin{aligned}
&\left(\mathrm{mKP}_{n,r,n}(\underline{\eta}, \underline{b}_n) + \sum_{j=-1}^{-\infty}\mathrm{mKP}_{n,r,j}(\underline{\eta}, \underline{b}_n)\xi^j\right) \\
&\quad = \left((\partial_{t_r} - (P_{n,r,1})_+)\tilde{A}_n - \tilde{A}_n(\partial_{t_r} - (P_{n,r,n})_+)\right)(A_{n-1}\cdots A_1)\,(A_{n-1}\cdots A_1)^{-1} \\
&\quad = \left((\partial_{t_r} - (P_{n,r,1})_+)\tilde{A}_nA_{n-1}\cdots A_1 - \tilde{A}_nA_{n-1}\cdots A_1(\partial_{t_r} - (P_{n,r,1})_+)\right)\cdot \\
&\qquad\qquad \cdot (A_{n-1}\cdots A_1)^{-1} \\
&\quad = \left[(\partial_{t_r} - (P_{r,1})_+),\, L_{n,1}\right](A_{n-1}\cdots A_1)^{-1} = 0
\end{aligned} \tag{2.67}$$

and therefore $\mathrm{mKP}_{n,r,n}(\underline{\eta}, \underline{b}_n)$ and $\mathrm{mKP}_{n,r,j}(\underline{\eta}, \underline{b}_n)$ for all $j \leq -1$ must vanish. Hence (2.60) holds iff (2.61) is valid. ∎

The auto-Bäcklund transformations of the $\mathrm{KP}_n$ hierarchy are then described in

**Corollary 2.6.** *In addition to the hypotheses in Theorem 2.5 assume that*

$$(\partial_{t_r} - (P_{n,r,1})_+)\,\psi_{n,k} = \sum_{\ell=1}^{k}\alpha_{n,r,k,\ell}\psi_{n,\ell}, \qquad 1 \leq k \leq n,\ n \geq 2 \tag{2.68}$$

*instead of (2.56). Then by (2.35), the solution $(\underline{\eta}, \underline{b}_n)$ constructed in Theorem 2.5 of the $mKP_n$ equations (2.60) yields $(n-1)$ further solutions $\underline{q}_k$, $2 \leq k \leq n$ of the $KP_n$ equations (2.17), i.e., $\underline{q}_k$ satisfy*

$$KP_{n,r,j}(\underline{q}_k) = 0, \quad -\infty \leq j \leq n, \quad r \in \mathbb{N}, \quad \underline{q}_k = (q_{k,j})_{-\infty<j\leq n-1},\ 2 \leq k \leq n. \tag{2.69}$$

In the case that we restrict to solutions of the KP hierarchy which are characterized by $(L_{1,1})^n = L_{n,1} = (L_{n,1})_+$ (i.e., $L_{n,1}$ is a formal differential operator) we can improve Theorem 2.5 in the following way. (We note that since $L_{n,1} = (L_{n,1})_+$ in this case, $P_{n\ell,1} = (P_{n\ell,1})_+$, $\ell \in \mathbb{N}$ and then $[P_{n\ell,1}, L_{n,1}] = 0$ implies that $q_j$, $0 \leq j \leq n-2$ are $t_{n\ell}$ independent.)

**Theorem 2.7.** *Assume* $\underline{q}_1 = (q_{j,1})_{-\infty \leq j \leq n-2}$, $n \geq 2$ *is such that* $q_{j,1} = 0$ *for* $-j \in \mathbb{N}$ *and define the operator* $L_{n,1}$ *as in (2.7). Let* $\psi_{n,k}$, $1 \leq k \leq n$ *be a basis of the kernel of* $L_{n,1}$*, i.e.,* $L_{n,1}\psi_{n,k} = 0$, $1 \leq k \leq n$ *and assume that the Wronskian* $W(\psi_{n,1}, \dots, \psi_{n,k})$ *is invertible for all* $1 \leq k \leq n$*. Define* $\partial\eta_k$ *by*

$$\begin{aligned}
&\partial\eta_1 = -\psi_{n,1}^{-1}\partial\psi_{n,1}, \quad \partial\eta_k = -W(\psi_{n,1}, \dots, \psi_{n,k})^{-1}\partial W(\psi_{n,1}, \dots, \psi_{n,k}) \\
&\quad +W(\psi_{n,1}, \dots, \psi_{n,k-1})^{-1}\partial W(\psi_{n,1}, \dots, \psi_{n,k-1}), \qquad 2 \leq k \leq n-1, \ (\textit{if } n \geq 3), \\
&\partial\eta_n = -\sum_{k=1}^{n-1} \partial\eta_k.
\end{aligned} \tag{2.70}$$

*Then*

$$L_{n,1} = A_n \cdots A_2 A_1, \tag{2.71}$$

*where*

$$A_k = \xi + \partial\eta_k, \quad 1 \leq k \leq n. \tag{2.72}$$

*Moreover,* $\underline{q}_1$ *satisfies the KP*$_n$ *hierarchy iff*

$$L_{n,1}(\partial_{t_r} - (P_{n,r,1})_+)\psi_k = 0, \quad 1 \leq k \leq n-1, \quad r \in \mathbb{N}_{\ltimes}, \tag{2.73}$$

$$\mathbb{N}_{\ltimes} = \mathbb{N} \setminus \{\ltimes\ell\}, \ \ell \in \mathbb{N} \tag{2.74}$$

*or equivalently, iff*

$$(\partial_{t_r} - (P_{n,r,1})_+)\,\psi_{n,k} = \sum_{\ell=1}^{n} \alpha_{n,r,k,\ell}\psi_{n,\ell}, \quad r \in \mathbb{N}_{\ltimes}, \qquad \nVdash \leq \daleth \leq \ltimes - \nVdash, \tag{2.75}$$

*where* $\alpha_{n,r,k,\ell}$ *are possibly* $t_r$*-dependent constants. Finally, assuming* $KP_{n,r,j}(\underline{q}_1) = 0$, $0 \leq j \leq n-2$, $r \in \mathbb{N}_{\ltimes}$*, we find that* $\underline{\eta}$ *satisfies the mKP*$_n$ *hierarchy*

$$\begin{aligned}
&mKP_{n,r,j}(\underline{\eta}, 0) = 0, \qquad 1 < j \leq n, \quad r \in \mathbb{N}_{\ltimes}, \\
&\underline{\eta} = (\eta_k)_{1 \leq k \leq n}, \quad \underline{b}_n = (b_{n,-m})_{-m \in \mathbb{N}} = 0
\end{aligned} \tag{2.76}$$

*iff*

$$\alpha_{n,r,k,\ell} = 0 \quad \textit{for} \quad k+1 \leq \ell \leq n, \quad 1 \leq k \leq n-1 \quad \textit{in (2.75)}. \tag{2.77}$$

Proof. We have

$$[\partial_{t_r} - (P_{n,r,1})_+, L_{n,1}]\psi_{n,k} = \sum_{j=0}^{n-2} \mathrm{KP}_{n,r,j}(\underline{q}_1)\xi^j\psi_{n,k} \tag{2.78}$$

$$= (\partial_{t_r} - (P_{n,r,1})_+)\underbrace{L_{n,1}\psi_{n,k}}_{=0} - L_{n,1}(\partial_{t_r} - (P_{n,r,1})_+)\,\psi_{n,k}, \quad 1 \leq k \leq n, \quad r \in \mathbb{N}_{\ltimes}.$$

Hence $\mathrm{KP}_{n,r,j}(\underline{q}_1) = 0$, $0 \le j \le n-2$ iff (2.75) holds. The rest of the proof is analogous to that of Theorem 2.5. ∎

We conclude with two examples illustrating our approach.

**Example 2.8.** *(Examples (2.1) and (2.4) revisited). Let $n \ge 4$,*

$$L_{n,1} = \prod_{j=1}^{n} (\xi - k_j), \quad \sum_{j=1}^{n} k_j = 0, \quad k_j = \text{ const. }, \quad k_j \neq k_\ell, \quad j \neq \ell, \tag{2.79}$$

*i.e.,*

$$\begin{aligned} q_{n-2,1} &= \sum_{\substack{j=1 \\ j<\ell}}^{n} \sum_{\ell=1}^{n} k_j k_\ell = d_{n-2} = \text{ const. }, \\ q_{n-3,1} &= -\sum_{j=1}^{n} \sum_{\substack{\ell=1 \\ j<\ell<m}}^{n} \sum_{m=1}^{n} k_j k_\ell k_m = d_{n-3} = \text{ const. }, \\ &\vdots \\ q_{0,1} &= (-1)^n \prod_{j=1}^{n} k_j = d_0 = \text{ const. }, \quad q_{-j,1} = 0, \quad j \in \mathbb{N}. \end{aligned} \tag{2.80}$$

*The constant solution $\tilde{q}_{n-2,1} = \frac{2}{n} q_{n-2,1}$ trivially fulfills the KP equation (2.21). Solutions of*

$$\begin{aligned} &L_{n,1}\psi = 0, \qquad \psi_{t_2} = (P_{n,2,1})_+ \psi, \qquad \psi_{t_3} = (P_{n,3,1})_+ \psi, \\ &(P_{n,2,1})_+ = \xi^2 + \frac{2}{n} d_{n-2}, \quad (P_{n,3,1})_+ = \xi^3 + \frac{3}{n} d_{n-2} \xi + \frac{3}{n} d_{n-3} \end{aligned} \tag{2.81}$$

*are then given by*

$$\psi_{n,j}(x,t_2,t_3) = e^{k_j x + (k_j^2 + \frac{2}{n} d_{n-2}) t_2 + (k_j^3 + \frac{3}{n} d_{n-2} k_j + \frac{3}{n} d_{n-3}) t_3}, \tag{2.82}$$

*respectively by,*

$$\tilde{\psi}_{n,j} = \sum_{k=1}^{n} c_{j,k} \psi_{n,k}, \qquad 1 \le j \le n, \quad c_{j,k} \in \mathbb{C}, \tag{2.83}$$

*where we assume $(c_{j,k})_{1 \le j,k \le n}$ to be invertible. Define*

$$\begin{aligned} \partial\eta_1 &= -\tilde{\psi}_{n,1}^{-1} \partial \tilde{\psi}_{n,1}, \quad \partial\eta_k = -W(\tilde{\psi}_{n,1}, \dots, \tilde{\psi}_{n,k})^{-1} \partial W(\tilde{\psi}_{n,1}, \dots, \tilde{\psi}_{n,k}) \\ &\quad + W(\tilde{\psi}_{n,1}, \dots, \tilde{\psi}_{n,k-1})^{-1} \partial W(\tilde{\psi}_{n,1}, \dots, \tilde{\psi}_{n,k-1}), \quad 2 \le k \le n-1, \\ \partial\eta_n &= -\sum_{k=1}^{n-1} \partial\eta_k, \qquad b_{n,-m} = 0, \quad m \in \mathbb{N}. \end{aligned} \tag{2.84}$$

*Then $\eta_k$ fulfill the mKP equation (2.52), i.e.,*

$$\partial\eta_{k,t_3} - \frac{1}{4}\partial^4\eta_k + \frac{3}{2}\partial^2\eta_k (\partial\eta_k)^2 + \frac{3}{2}\partial^2\eta_k \eta_{k,t_2} - \frac{3}{4}\eta_{k,t_2 t_2} = 0, \quad 1 \le k \le n. \tag{2.85}$$

*Moreover, define (indices are taken mod n)*

$$\begin{aligned}\tilde{q}_{n-2,k} &= \frac{2}{n}q_{n-2,k} = \frac{2}{n}(\partial^2\left((n-1)\eta_k + (n-2)\eta_{k+1} + \ldots + \eta_{k+n-2}\right) \\ &\quad + (\partial\eta_1\partial\eta_2 + \partial\eta_1\partial\eta_3 + \ldots + \partial\eta_{n-1}\partial\eta_n)) \qquad (2.86)\\ &= \frac{2}{n}q_{n-2,1} + 2\partial\left(W(\tilde{\psi}_{n,1},\ldots,\tilde{\psi}_{n,k-1})^{-1}\partial W(\tilde{\psi}_{n,1},\ldots,\tilde{\psi}_{n,k-1})\right), \quad 2 \le k \le n.\end{aligned}$$

*Then $\tilde{q}_{n-2,k}$ fulfill the KP equation (2.21) , i.e.,*

$$\partial\tilde{q}_{n-2,k,t_3} - \frac{1}{4}\partial^4\tilde{q}_{n-2,k} - \frac{3}{2}\partial(\tilde{q}_{n-2,k}\partial\tilde{q}_{n-2,k}) - \frac{3}{4}\tilde{q}_{n-2,k,t_2t_2} = 0, \quad 2 \le k \le n. \qquad (2.87)$$

**Example 2.9.** *In order to derive the standard soliton solutions of the KP and mKP equation we consider a variant of Example 2.8. For $A$ we choose the algebra of smooth functions in $x, t_2, t_3, \ldots,$ where we identify $x$ with $t_1$ and $\partial$ with $\partial_x$. We choose $L_n$ with $n = 2N+2,\ N \in \mathbb{N}$,*

$$\begin{aligned}&L_{n,1} = \prod_{j=1}^{n}(\xi - k_j), \qquad k_n = -\sum_{j=1}^{n-1}k_j, \quad k_j = \text{ const. }, \quad k_j \ne k_\ell, \quad j \ne \ell,\\ &k_{n-1} = -\frac{1}{2}\sum_{j=1}^{n-2}k_j + \frac{1}{2}\sqrt{-3\sum_{j=1}^{n-2}k_j^2 - 2\sum_{\substack{j=1\\ j<\ell}}^{n-2}\sum_{\ell=1}^{n-2}k_jk_\ell}, \qquad (2.88)\end{aligned}$$

*i.e.,*

$$\begin{aligned}q_{n-2,1} &= \sum_{\substack{j=1\\ j<\ell}}^{n}\sum_{\ell=1}^{n}k_jk_\ell = d_{n-2} = 0,\\ q_{n-3,1} &= -\sum_{\substack{j=1\\ j<\ell<m}}^{n}\sum_{\ell=1}^{n}\sum_{m=1}^{n}k_jk_\ell k_m = d_{n-3} = \text{ const. },\\ &\vdots\\ q_{0,1} &= (-1)^n\prod_{j=1}^{n}k_j = d_0 = \text{ const. }, \quad q_{-j,1} = 0, \quad j \in \mathbb{N}. \qquad (2.89)\end{aligned}$$

*The solution $\tilde{q}_{n-2,1} = \frac{2}{n}q_{n-2,1} = 0$ trivially fulfills the KP equation (2.21). Solutions of*

$$\begin{aligned}&L_{n,1}\psi_{n,j} = 0, \qquad (\partial_{t_2} - (P_{n,2,1})_+)\psi_{n,j} = 0,\\ &(\partial_{t_3} - (P_{n,3,1})_+)\psi_{n,j} = -\frac{3}{n}d_{n-3}\psi_{n,j}, \qquad 1 \le j \le n,\\ &(P_{n,2,1})_+ = \xi^2, \quad (P_{n,3,1})_+ = \xi^3 + \frac{3}{n}d_{n-3} \qquad (2.90)\end{aligned}$$

*are then given by*

$$\psi_{n,j}(x,t_2,t_3) = e^{k_jx + k_j^2t_2 + k_j^3t_3}. \qquad (2.91)$$

*Define*

$$
\begin{aligned}
a_j = \tilde{\psi}_{n,j} &= (-1)^{j+1}\psi_{n,j} + \alpha_j \psi_{n,N+j}, \quad \alpha_j \in \mathbb{C} \setminus \{\nvdash\},\ \nVdash \leq \gimel \leq \mathbb{N} = \frac{\ltimes - \nvDash}{\nvDash}, \\
\tilde{\psi}_{n,j} &= \psi_{n,j}, \quad N+1 \leq j \leq n. \qquad (2.92)
\end{aligned}
$$

*Then the* $\tilde{\psi}_{n,j},\ 1 \leq j \leq n$ *fulfill (2.90) too. Define* $\partial_x \eta_k$ *by*

$$
\begin{aligned}
&\partial_x \eta_1 = -\partial_x \ln \tilde{\psi}_{n,1}, \qquad \partial_x \eta_k = -\partial_x \ln \left[ \frac{W(\tilde{\psi}_{n,1}, \ldots, \tilde{\psi}_{n,k})}{W(\tilde{\psi}_{n,1}, \ldots, \tilde{\psi}_{n,k-1})} \right], \quad 2 \leq k \leq n-1, \\
&\partial_x \eta_n = -\sum_{k=1}^{n} \partial_x \eta_k. \qquad (2.93)
\end{aligned}
$$

*Then* $\partial_x \eta_k$ *is a* $(2k-1)$*-soliton solution of the mKP equation (2.52),* $1 \leq k \leq N$ *(see Remark 2.10). Moreover, define (indices are taken mod n)*

$$
\begin{aligned}
q_{n-2,k} &= \partial_x^2 \left((n-1)\eta_k + (n-2)\eta_{k+1} + \ldots + \eta_{k+n-2}\right) \\
&\quad + \left(\partial_x \eta_1 \partial_x \eta_2 + \partial_x \eta_1 \partial_x \eta_3 + \ldots + \partial_x \eta_{n-1} \partial_x \eta_n\right), \\
\tilde{q}_{n-2,k} &= \frac{2}{n} q_{n-2,k}, \quad 2 \leq k \leq n. \qquad (2.94)
\end{aligned}
$$

*Then (using (2.38))*

$$
\tilde{q}_{n-2,k+1} = 2\partial_x^2 \ln W(a_1, \ldots, a_k), \qquad 1 \leq k \leq N \qquad (2.95)
$$

*turns out to be the k-soliton solution of the KP equation (2.21) (see Remark 2.10).*

**Remark 2.10.** *(i) In order to identify our soliton solutions with the one in* [11] *we use the following dictionary:*

$$
\begin{aligned}
\epsilon_{[11]} &= -1, \\
V_{[11]} &= -\tilde{q}_{n-2}, \quad \phi_{[11]} = \partial_x \eta, \\
-4t_{[11]} &= t_3, \quad y_{[11]} = t_2, \\
p_{j,[11]} &= k_j, \quad 1 \leq j \leq N, \\
-q_{j,[11]} &= k_{j+N}, \quad 1 \leq j \leq N,\ \textit{if}\ k_{j+N} \neq k_{\ell+N},\ 1 \leq \ell \leq N, \\
\textit{if}\ q_{j_0,[11]} &= q_{j_1,[11]}, \ldots, q_{j_s,[11]}, \quad j_0 < j_s,\ 1 \leq s \leq N\ \textit{we modify}\ a_{j_r}\ \textit{by} \\
a_{j_r} &= (-1)^{j_r+1}\psi_{j_r} + \alpha_{j_0}\psi_{N+j_0}, \quad 1 \leq r \leq s. \qquad (2.96)
\end{aligned}
$$

*(ii) Due to the simple structure of* $\tilde{\psi}_{N+j},\ 1 \leq j \leq N+2,\ \partial_x \eta_{N+j}, 1 \leq j \leq N$*, is a* $(2(N-j)+1)$*-soliton solution of the mKP equation and* $\partial_x \eta_{2N+1}, \partial_x \eta_{2N+2}$ *are constant. Similarly,* $\tilde{q}_{n-2,N+j},\ 2 \leq j \leq N$*, is a* $(N+1-j)$*-soliton solution of the KP equation and* $\tilde{q}_{n-2,2N+1} = \tilde{q}_{n-2,2N+2} = 0$.

For simplicity we only presented the soliton solutions of the (m)KP equations (2.21) respectively (2.52) but the result obviously extends to the entire m(KP) hierarchy in a straight-forward manner.

**Remark 2.11.** *On the algebra $A$ of smooth functions we derive the following connection of our solutions with the $\tau$-function formalism (see, e.g.,* [4]*). Since*

$$\tilde{q}_{n-2,k}(x, t_2, \dots) = 2\,\partial_x^2 \ln \tau_k(x, t_2, \dots), \tag{2.97}$$

*(2.38) yields*

$$\begin{aligned} \tilde{q}_{n-2,k}(x, t_2, \dots) &= \tilde{q}_{n-2,1}(x, t_2, \dots) - 2\,\partial_x^2 \sum_{\ell=1}^{k-1} \eta_\ell(x, t_2, \dots) \\ &= \tilde{q}_{n-2,1}(x, t_2, \dots) + 2\,\partial_x^2 \ln W(\psi_1(x, t_2, \dots), \dots, \psi_{k-1}(x, t_2, \dots)) \end{aligned} \tag{2.98}$$

*and hence*

$$\tau_k(x, t_2, \dots) = \tau_1(x, t_2, \dots) W(\psi_1(x, t_2, \dots), \dots, \psi_{k-1}(x, t_2, \dots)), \quad 2 \le k \le n. \tag{2.99}$$

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