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ON THE SOLUTIONS OF HALPHEN'S EQUATION

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Abstract. We study Halphen's equation and provide solutions in terms of elliptic functions of the second kind. The connection between Halphen's equation and algebro-geometric solutions of the Boussinesq hierarchy is discussed.

1. Introduction

We consider the third-order differential equation

$$\psi'''(z,x) + q_1(x)\,\psi'(z,x) + \big(\tfrac{1}{2}\,q_1'(x) - z\big)\psi(z,x) = 0, \quad z \in \mathbb{C} \tag{1.1}$$

where $\psi(z,\cdot) : \mathbb{C} \to \mathbb{C}\cup\{\infty\}, x \mapsto \psi(z,x)$ and the coefficient $q_1 : \mathbb{C} \to \mathbb{C}\cup\{\infty\}$ has the form

$$q_1(x) = h_g - g(g+2)\wp(x), \quad h_g \in \mathbb{C},\ g \in \mathbb{N}, \quad g \not\equiv 2 \pmod 3. \tag{1.2}$$

Here $\wp(x) = \wp(x,\omega_1,\omega_3)$ denotes the elliptic Weierstrass function with fundamental periods $2\,\omega_1, 2\,\omega_3$, and invariants g_2, g_3 (see, e.g., [1]). The potentials (1.2) were introduced by Halphen [21, Ch. IV, p. 179] in the case $h_g = 0$, $g_2 = 0$ $(g = n-1)$ and are associated with the third-order differential expression

$$L_3 = \frac{d^3}{dx^3} + q_1(x)\frac{d}{dx} + \frac{1}{2}q_{1,x}(x) + q_0(x). \tag{1.3}$$

The nonlinear evolution equations of the Boussinesq (Bsq) hierarchy are generated by L_3 and certain differential expressions P_m of order m ($m \not\equiv 0$ (mod 3)) such that the commutator $[L_3, P_m]$ is a first-order differential expression. Then $[L_3, P_m] = 0$ yields the stationary equations of the Bsq hierarchy. Next introduce a time parameter t_m replacing $q_1(x)$ and $q_0(x)$ by $q_1(t_m, x)$ and $q_0(t_m, x)$, $t_m \in \mathbb{C}$. Then $\frac{d}{dt_m}L_3 - [L_3, P_m] = 0$ gives rise to the nonlinear evolution equations of the Bsq hierarchy. By definition,

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solutions of any of the stationary Bsq equations $[L_3, P_m] = 0$ are called algebro-geometric Bsq solutions or algebro-geometric Bsq potentials. More generally, this concept can be extended to pairs of differential expressions (L_n, P_r) of order n, respectively r (n, r relatively prime), generating the nonlinear evolution equation of the Gelfand–Dickii hierarchy. From the work of Segal and Wilson [37] one may obtain that solutions of n^{th}-order differential equations $L_n\psi = z\psi$ are necessarily meromorphic if the coefficients of L_n are algebro-geometric potentials. Note that solutions of stationary equations yield commuting pairs of differential expressions (L_n, P_r). By appealing to a result of Burchnall and Chaundy [3], [4] this results in an algebraic relationship between L_n and P_r and naturally leads to a plane algebraic curve $\mathcal{K}_{r-1}$.

Recent work by Gesztesy and Weikard [16] revealed that an equation $y''(x) + q(x)\,y(x) = \tilde{z}\,y(x)$, $q(x)$ elliptic, which has a meromorphic fundamental system of solutions with respect to x for all values of the spectral parameter $\tilde{z} \in \mathbb{C}$ necessarily yields elliptic algebro-geometric solutions of the Korteweg–de Vries (KdV) hierarchy.

Weikard [39] proved an analogous theorem for the entire Gelfand–Dickii hierarchy (this includes for $n = 2$ the KdV and for $n = 3$ the Bsq hierarchy) for rational and simply periodic algebro-geometric potentials. It is assumed that this is also true for elliptic algebro-geometric potentials.

Within the Bsq hierarchy Halphen potentials play the same role as the Lamé–Ince potentials [14] in connection with the Schrödinger operator and the KdV hierarchy.

The usual approach for solving Halphen's equation is by an argument already used by Halphen himself (cf. Hermite [22], p. 372) of the form

$$\psi(z,x) = e^{vx} \sum_{j=0}^{g-1} \alpha_j(z,\tilde{z},v) \frac{d^j \phi(\tilde{z},x)}{dx^j} \tag{1.4}$$

where $\phi(\tilde{z},x)$ is a solution of $\phi'' - (2\,\wp(x) + \tilde{z})\phi = 0$. An extended version of this argument was used by Eilbeck and Enol'skii [7] to compute a solution in the case $g = 3 (g_2 = 0, h_3 = 0)$ and by Enol'skii and Kostov [9] in the case $g = 4 (g_2 = 0, h_4 = 0)$ (cf. [12]).

Our approach relies on a powerful theorem of Picard which guarantees the existence of an elliptic solution of the second kind, when the fundamental system of (1.1) is meromorphic. As pointed out in Remark 3.9 of [18], these two approaches are intimately connected.

Section 2 recalls some basic facts about the Bsq hierarchy. In Section 3 we prove the existence of a meromorphic fundamental system for Halphen's equation, if $g_2 = 0$. Using a theorem of Picard we express the solutions of Halphen's equation in terms of elliptic functions of the second kind. If the invariant $g_2 \neq 0$, there exists only a finite number of cases such that the fundamental system of Halphen's equation is meromorphic. However, as pointed out in Remark 10 there exist algebro-geometric Bsq potentials in that case too if one chooses q_0 and q_1 in (1.3) appropriately. Remark 5, respectively, the results from Subsection 3.2 discuss some exceptional cases of the spectral parameter z. In Section 4 we present a couple of examples including the corresponding algebraic curves. Appendix A demonstrates the rational limit where the half-periods $\omega_1 \to \infty$, $\omega_3 \to \infty$ and $q_1(x)$ becomes $-g(g+2)/x^2$. And finally Appendix B summarizes some basic representation theorems for arbitrary elliptic functions.

2. The stationary Boussinesq hierarchy and commuting operators

In this section we briefly recall some basic facts about the Bsq hierarchy, [5], [6]. Suppose q_0 and q_1 are meromorphic on $\mathbb{C}$ and introduce the third-order differential expression

$$L_3 = \frac{d^3}{dx^3} + q_1 \frac{d}{dx} + \frac{1}{2}\, q_{1,x} + q_0, \quad x \in \mathbb{C}. \tag{2.1}$$

For each fixed $m \in \mathbb{N}_0 (= \mathbb{N} \cup \{0\})$ with $m \not\equiv 0 \pmod 3$ we write

$$m = 3n + \epsilon, \quad \epsilon \in \{1, 2\}, \tag{2.2}$$

and then construct two distinct differential expressions of order $3n+1$ and $3n+2$, respectively, denoted by P_m, where $m = 3n+1$ or $m = 3n+2$. In order for these differential expressions P_m to commute with L_3, one proceeds as follows (cf. [5] for more details).

In order to simplify notation we have dropped the ϵ-dependence; i.e., instead of $f_\ell^{(\epsilon)}, g_\ell^{(\epsilon)}$ we simply write f_ℓ, g_ℓ further on.

Pick $n \in \mathbb{N}_0$, $\epsilon \in \{1, 2\}$, and define the sequences $\{f_\ell(x)\}_{\ell=0,\dots,n+1}$, $\{g_\ell(x)\}_{\ell=0,\dots,n+1}$ recursively by

$$(f_0, g_0) = (c_0, d_0) = \begin{cases} (0,1) & \text{for } \epsilon = 1, \\ (1, d_0) & \text{for } \epsilon = 2, \end{cases} \qquad d_0 \in \mathbb{C},$$

$$3f_{\ell,x} = 2g_{\ell-1,xxx} + 2q_1 g_{\ell-1,x} + q_{1,x} g_{\ell-1} + 3q_0 f_{\ell-1,x} + 2q_{0,x} f_{\ell-1}, \tag{2.3}$$

$$3g_{\ell,x} = 3q_0 g_{\ell-1,x} + q_{0,x} g_{\ell-1} - \tfrac{1}{6} f_{\ell-1,xxxxx} - \tfrac{5}{6} q_1 f_{\ell-1,xxx} - \tfrac{5}{4} q_{1,x} f_{\ell-1,xx}$$
$$- \big(\tfrac{3}{4} q_{1,xx} + \tfrac{2}{3} q_1^2\big) f_{\ell-1,x} - \big(\tfrac{1}{6} q_{1,xxx} + \tfrac{2}{3} q_1 q_{1,x}\big) f_{\ell-1}, \quad \ell = 1, \dots, n+1.$$

Explicitly, one computes
(i) Let $m \equiv 1 \pmod 3$ (i.e., $\epsilon = 1$):

$$f_0 = 0, \quad g_0 = 1, \quad 3f_1 = q_1 + 3c_1, \tag{2.4}$$
$$3g_1 = q_0 + 3d_1, \quad 3f_2 = \tfrac{2}{3} q_{0,xx} + \tfrac{4}{3} q_0 q_1 + c_1 2q_0 + d_1, q_1 + 3c_2, \quad \text{etc.}$$

(ii) Let $m \equiv 2 \pmod 3$ (i.e., $\epsilon = 2$):

$$f_0 = 1, \quad g_0 = d_0, \quad 3f_1 = 2q_0 + d_0 q_1 + 3c_1, \tag{2.5}$$
$$3g_1 = -\tfrac{1}{6} q_{1,xx} - \tfrac{1}{3} q_1^2 + d_0 q_0 + 3d_1, \quad \text{etc.},$$

where $\{c_\ell\}_{\ell=1,\dots,n}$ and $\{d_\ell\}_{\ell=0,\dots,n}$ are integration constants which arise when solving (2.3).

Given (2.3) one defines the differential expression P_m of order m by

$$P_m = \sum_{\ell=0}^{n} \Big(f_{n-\ell} \frac{d^2}{dx^2} + \big(g_{n-\ell} - \tfrac{1}{2} f_{n-\ell,x}\big) \frac{d}{dx}$$
$$+ \big(\tfrac{1}{6} f_{n-\ell,xx} - g_{n-\ell,x} + \tfrac{2}{3} q_1 f_{n-\ell}\big)\Big) L_3^\ell, \tag{2.6}$$
$$\ell = 0, \dots, n, \quad m = 3n + \epsilon, \ \epsilon \in \{1, 2\}, \ n \in \mathbb{N}_0,$$

and verifies that

$$[P_m, L_3] = 3\, f_{n+1,x} \frac{d}{dx} + \tfrac{3}{2} f_{n+1,xx} + 3\, g_{n+1,x},$$
$$m = 3n + \epsilon, \ \epsilon \in \{1, 2\}, \ n \in \mathbb{N}_0 \tag{2.7}$$

(where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator symbol). The pair (L_3, P_m) represents the Lax pair for the Bsq hierarchy. Varying $n \in \mathbb{N}_0$ and $\epsilon \in \{1, 2\}$, the stationary Bsq hierarchy is then defined by the vanishing of the commutator of P_m and L_3 in (2.7), that is, by

$$[P_m, L_3] = 0, \qquad m = 3n + \epsilon, \ \epsilon \in \{1, 2\}, \ n \in \mathbb{N}_0, \tag{2.8}$$

or equivalently, by

$$f_{n+1,x} = 0, \quad g_{n+1,x} = 0, \qquad \epsilon \in \{1, 2\}, \ n \in \mathbb{N}_0. \tag{2.9}$$

By definition, solutions (q_0, q_1) of any of the stationary Bsq equations (2.9) are called stationary algebro-geometric Bsq solutions or simply **algebro-geometric** Bsq potentials.

Next, we introduce two polynomials F_m and G_m, both of degree at most n with respect to the variable $z \in \mathbb{C}$:

$$F_m(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}(x) z^{\ell}, \tag{2.10}$$

$$G_m(z,x) = \sum_{\ell=0}^{n} g_{n-\ell}(x) z^{\ell}, \quad m = 3n+\epsilon,\ \epsilon \in \{1,2\},\ n \in \mathbb{N}_0. \tag{2.11}$$

Given (2.10) and (2.11), (2.8) (or equivalently, (2.9)) becomes

$$2\,G_{m,xxx} + 2\,q_1 G_{m,x} + q_{1,x} G_m - 3\,(z-q_0) F_{m,x} + 2\,q_{0,x} F_m = 0, \tag{2.12}$$

$$\begin{aligned} &\tfrac{1}{6}\,F_{m,xxxxx} + \tfrac{5}{6}\,q_1 F_{m,xxx} + \tfrac{5}{4}\,q_{1,x} F_{m,xx} + \big(\tfrac{3}{4}\,q_{1,xx} + \tfrac{2}{3}\,q_1^2\big) F_{m,x} \\ &\quad + \big(\tfrac{1}{6}\,q_{1,xxx} + \tfrac{2}{3}\,q_1 q_{1,x}\big) F_m + 3(z-q_0) G_{m,x} - q_{0,x} G_m = 0. \end{aligned} \tag{2.13}$$

Both equations can be integrated (cf. [5]) to get

$$\begin{aligned} S_m(z) = {} & -\tfrac{1}{6}\,F_{m,xxxx} F_m + \tfrac{1}{6}\,F_{m,xxx} F_{m,x} - \tfrac{1}{12}\,F_{m,xx}^2 - \tfrac{5}{6}\,q_1 F_{m,xx} F_m \\ & - \tfrac{5}{12}\,q_{1,x} F_{m,x} F_m + \tfrac{5}{12}\,q_1 F_{m,x}^2 - \tfrac{1}{3}\,\big(\tfrac{1}{2}\,q_{1,xx} + q_1^2\big) F_m^2 + 2\,G_{m,xx} G_m \\ & - G_{m,x}^2 + q_1 G_m^2 - 3(z-q_0) F_m G_m, \end{aligned} \tag{2.14}$$

where the integration constant $S_m(z)$ is a polynomial in z of degree at most $2n-1+\epsilon$, $m = 3n+\epsilon$, $\epsilon \in \{1,2\}$, $n \in \mathbb{N}_0$,

$$S_m(z) = \sum_{p=0}^{2\,n-1+\epsilon} s_{m,p} z^p, \quad m = 3n+\epsilon,\ \epsilon \in \{1,2\},\ n \in \mathbb{N}_0, \tag{2.15}$$

and

$$\begin{aligned} T_m(z) = {} & \tfrac{1}{18} F_{m,xxxx} F_{m,xx} F_m - \tfrac{1}{24} F_{m,xxxx} F_{m,x}^2 \\ & + \tfrac{1}{36} F_{m,xxx} F_{m,xx} F_{m,x} - \tfrac{1}{108} F_{m,xx}^3 - \tfrac{1}{36} F_m F_{m,xxx}^2 + \tfrac{1}{18} q_1 F_{m,xxxx} F_m^2 \\ & - \tfrac{1}{18} q_{1,x} F_{m,xxx} F_m^2 - \tfrac{1}{9} q_1 F_{m,xxx} F_{m,x} F_m + \tfrac{1}{18} q_{1,xx} F_{m,xx} F_m^2 \\ & + \tfrac{2}{9} q_{1,x} F_{m,xx} F_{m,x} F_m - \tfrac{7}{72} q_1 F_{m,xx} F_{m,x}^2 + \tfrac{7}{36} q_1 F_{m,xx}^2 F_m \\ & + \tfrac{5}{18} q_1^2 F_{m,xx} F_m^2 - \tfrac{1}{24} q_{1,xx} F_{m,x}^2 F_m - \tfrac{7}{48} q_{1,x} F_{m,x}^3 + \tfrac{1}{12} q_{1,x} q_1 F_{m,x} F_m^2 \\ & - \tfrac{1}{6} q_1^2 F_{m,x}^2 F_m + \big(\tfrac{2}{27} q_1^3 - \tfrac{1}{36}\, q_{1,x}^2 + \tfrac{1}{18} q_{1,xx} q_1 + (z-q_0)^2\big) F_m^3 \\ & + (z-q_0) G_m^3 + \tfrac{1}{6} F_{m,xxxx} G_m^2 - \tfrac{1}{3} F_{m,xxx} G_{m,x} G_m + F_m G_{m,xx}^2 \\ & + \tfrac{1}{3} F_{m,xx} \big(G_{m,x}^2 + G_{m,xx} G_m\big) - F_{m,x} G_{m,xx} G_{m,x} - q_1 (z-q_0) F_m^2 G_m \\ & + \tfrac{2}{3} q_1^2 F_m G_m^2 + \tfrac{5}{6} q_1 F_{m,xx} G_m^2 - \tfrac{4}{3} q_1 F_{m,x} G_{m,x} G_m + \tfrac{7}{12} q_{1,x} F_{m,x} G_m^2 \end{aligned} \tag{2.16}$$

$$+\tfrac{1}{3}q_1F_mG_{m,x}^2+\tfrac{4}{3}q_1F_mG_{m,xx}G_m+\tfrac{1}{6}q_{1,xx}F_mG_m^2-\tfrac{1}{3}q_{1,x}F_mG_{m,x}G_m$$
$$+(z-q_0)F_{m,x}F_mG_{m,x}-\tfrac{1}{4}(z-q_0)F_{m,x}^2G_m-2(z-q_0)F_m^2G_{m,xx},$$

where the integration constant $T_m(z)$ is a monic polynomial of degree m,

$$T_m(z)=z^m+\sum_{q=0}^{m-1}t_{m,q}z^q,\quad m=3n+\epsilon,\ \epsilon\in\{1,2\},\, n\in\mathbb{N}_0. \tag{2.17}$$

Next, we consider the algebraic kernel of (L_3-z), $z\in\mathbb{C}$ (i.e., the formal nullspace in a purely algebraic sense),

$$\ker(L_3-z)=\{\psi:\mathbb{C}\to\mathbb{C}\cup\{\infty\}\text{ meromorphic }:(L_3-z)\psi=0\},\quad z\in\mathbb{C}. \tag{2.18}$$

Taking into account (2.8), that is, $[P_m,L_3]=0$, computing the restriction of P_m to $\ker(L_3$-$z)$, and using

$$\psi_{xxx}=-q_1\,\psi_x+\big(z-\tfrac{1}{2}\,q_{1,x}-q_0\big)\psi,\quad\text{etc.}, \tag{2.19}$$

to eliminate higher-order derivatives of ψ, one obtains from (2.3), (2.6), (2.9), (2.10), (2.11), (2.12), and (2.13)

$$P_m\big|_{\ker(L_3-z)}=\Big(F_m\frac{d^2}{dx^2}+\big(G_m-\tfrac{1}{2}F_{m,x}\big)\frac{d}{dx}+H_m\Big)\Big|_{\ker(L_3-z)}. \tag{2.20}$$

Here

$$H_m(z,x)=\tfrac{1}{6}\,F_{m,xx}(z,x)+\tfrac{2}{3}\,q_1(x)F_m(z,x)-G_{m,x}(z,x), \tag{2.21}$$

where we suppressed an integration constant which can be trivially implemented.

Still assuming $f_{n+1,x}=g_{n+1,x}=0$ as in (2.9), $[P_m,L_3]=0$ in (2.6) yields an algebraic relationship between P_m and L_3 by appealing to a result of Burchnall and Chaundy [3], [4] (see also [11], [19], [35], [41]). In fact, one can prove

Theorem 1 ([5]). *Assume $f_{n+1,x}=g_{n+1,x}=0$; that is, $[P_m,L_3]=0$, $m=3n+\epsilon$, $\epsilon\in\{1,2\}$, $n\in\mathbb{N}_0$. Then the Burchnall–Chaundy polynomial $\mathcal{F}_{m-1}(L_3,P_m)$ of the pair (L_3,P_m) explicitly reads (cf.* (2.15) *and* (2.17)*)*

$$\mathcal{F}_{m-1}(L_3,P_m)=P_m^3+P_m\,S_m(L_3)-T_m(L_3)=0,$$
$$S_m(z)=\sum_{p=0}^{2n-1+\epsilon}s_{m,p}z^p,\quad T_m(z)=z^m+\sum_{q=0}^{m-1}t_{m,q}z^q, \tag{2.22}$$
$$m=3n+\epsilon,\ \epsilon\in\{1,2\},\, n\in\mathbb{N}_0.$$

Remark 2. $\mathcal{F}_{m-1}(L_3, P_m) = 0$ naturally leads to the plane algebraic curve $\mathcal{K}_{m-1}$,

$$\mathcal{K}_{m-1}: \ \mathcal{F}_{m-1}(z,y) = y^3 + y\, S_m(z) - T_m(z) = 0 \tag{2.23}$$

of (arithmetic) genus $g = m-1$. For $m \geq 4$ these curves are nonhyperelliptic. (We denote points P on the curve $\mathcal{K}_{m-1}$ by $P = (z,y) \in \mathcal{K}_{m-1} \setminus \{P_\infty\}$.)

When $m = 1$, corresponding to $g = 0$, there are no nonzero holomorphic differentials on $\mathcal{K}_g$. When $m = 2$, corresponding to $g = 1$, the only holomorphic differential on $\mathcal{K}_g$ is $dz/(3y(P)^2 + S_m(z))$. Recall also that $m \not\equiv 0 \pmod 3$, so we need not consider holomorphic differentials for the case $m = 3$. One verifies that $dz/(3y(P)^2 + S_m(z))$ and $y(P)dz/(3y(P)^2 + S_m(z))$ are holomorphic differentials $\mathcal{K}_g$ with zeros at P_∞ of order $2(m-2)$ and $(m-4)$, respectively, for $m \geq 4$. It follows that the differentials ($m = 3n + \varepsilon$, $\varepsilon \in \{1,2\}$)

$$\eta_\ell(P) = \tfrac{1}{3y(P)^2 + S_m(z)} \begin{cases} z^{\ell-1}dz & \text{for } 1 \leq \ell \leq g-n, \\ y(P) z^{\ell+n-g-1} dz & \text{for } g-n+1 \leq \ell \leq g, \end{cases} \tag{2.24}$$

form a basis in the space of holomorphic differentials $\mathcal{H}^1(\mathcal{K}_g)$ (cf. [6]).

Curves of the form above are trigonal curves and have been studied by, e.g., Matveev and Smirnov in [27], [28], [29], Previato and Verdier in [36], Previato in [34], and Eilbeck, Enol'skii, and Leykin in [8]. Finally, introducing a deformation parameter $t_m \in \mathbb{C}$ into the pair (q_0, q_1) (i.e., $q_\ell(x) \to q_\ell(x, t_m)$, $\ell = 0, 1$), the time-dependent Bsq hierarchy is defined as a collection of evolution equations (varying $m = 3n + \epsilon$, $\epsilon \in \{1,2\}$, $n \in \mathbb{N}_0$)

$$\frac{d}{d\,t_m} L_3(t_m) - [P_m(t_m), L_3(t_m)] = 0,$$
$$(x, t_m) \in \mathbb{C}^2,\ m = 3n + \epsilon,\ \epsilon \in \{1,2\},\ n \in \mathbb{N}_0, \tag{2.25}$$

or equivalently, by

$$Bsq_m(q_0, q_1) = \begin{cases} q_{0,t_m}(x, t_m) - 3\, g_{n+1,x}(x, t_m) = 0, \\ q_{1,t_m}(x, t_m) - 3\, f_{n+1,x}(x, t_m) = 0, \end{cases}$$
$$(x, t_m) \in \mathbb{C}^2,\ m = 3n + \epsilon,\ \epsilon \in \{1,2\},\ n \in \mathbb{N}_0. \tag{2.26}$$

Explicitly, one obtains for the first few equations in (2.26)

$$Bsq_1(q_0, q_1) = \begin{cases} q_{0,t_1} - q_{0,x} = 0, \\ q_{1,t_1} - q_{1,x} = 0, \end{cases}$$

$$Bsq_2(q_0, q_1) = \begin{cases} q_{0,t_2} + \frac{1}{6}\, q_{1,xxx} + \frac{2}{3}\, q_1 q_{1,x} - d_0 q_{0,x} = 0, \\ q_{1,t_2} - 2\, q_{0,x} - d_0 q_{1,x} = 0, \end{cases}$$

$$Bsq_4(q_0,q_1)=\begin{cases} q_{0,t_4}+\frac{1}{18}\,q_{1,xxxxx}+\frac{1}{3}\,q_1q_{1,xxx}+\frac{2}{3}\,q_{1,x}q_{1,xx}+\frac{4}{9}\,q_1^2q_{1,x} \\ -\frac{4}{3}\,q_0q_{0,x}+c_1\big(\frac{1}{6}\,q_{1,xxx}+\frac{2}{3}\,q_1q_{1,x}\big)-d_1q_{0,x}=0, \\ q_{1,t_4}-\frac{2}{3}\,q_{0,xxx}-\frac{4}{3}\,q_1q_{0,x}-\frac{4}{3}\,q_{1,x}q_0-c_12q_{0,x}-d_1q_{1,x}=0, \end{cases}$$
$$\text{etc.} \tag{2.27}$$

From the work of Segal and Wilson [37] one may obtain that solutions of $L_3\psi = z\psi$ are necessarily meromorphic if the coefficients of L_3 are algebro-geometric potentials. That this condition is also sufficient for elliptic algebro-geometric solutions of the KdV hierarchy was recently proven by Gesztesy and Weikard in [16] (see also [15], [17]).

Theorem 3. *Let q be an elliptic function. Then q is an elliptic algebro-geometric KdV potential if and only if the equation $y''(x)+q(x)y(x)=\tilde{z}y(x)$ has a meromorphic fundamental system of solutions with respect to x for all values of the spectral parameter $\tilde{z}\in\mathbb{C}$.*

Recently Weikard [39] (cf. [38]) proved an analogous theorem for the entire Gelfand–Dickii hierarchy for rational and simply periodic algebro-geometric potentials. It is assumed that this is also true for elliptic algebro-geometric potentials.

3. Halphen potentials associated with the Bsq hierarchy

In this section we study in detail Halphen potentials

$$q_1(x)=h_g-g(g+2)\wp(x),\quad h_g\in\mathbb{C},\quad g\in\mathbb{N},\quad g\not\equiv 2 \pmod 3 \tag{3.1}$$

and the associated linear third-order differential equation

$$\psi'''(z,x)+(h_g-g(g+2)\,\wp(x))\,\psi'(z,x)-\big(\tfrac{1}{2}\,g(g+2)\wp'(x)+z\big)\psi(z,x)=0,$$
$$z\in\mathbb{C},\quad h_g\in\mathbb{C},\quad g\not\equiv 2 \pmod 3. \tag{3.2}$$

Since we expect that (3.2) will lead to algebro-geometric Bsq potentials only when the fundamental system is meromorphic, we investigate when (3.2) possesses a meromorphic fundamental system around $x=0$. We distinguish two cases.

(i) $g_2=0, h_g=0$. If $g_2=0$ the Laurent series ([1], p. 656) for $\wp(x)$ reduces to

$$\wp(x)=\tfrac{1}{x^2}\Big(1+\sum_{m=1}^{\infty}c_{3m}x^{6m}\Big). \tag{3.3}$$

According to the theory of Fuchs, $x=0$ is a regular singular point of (3.2). By the method of Frobenius we set (see, e.g., [25])

$$\psi(z,x) = x^\rho \sum_{\ell=0}^{\infty} r_\ell x^\ell, \quad r_0 \in \mathbb{C} \setminus \{0\} \tag{3.4}$$

which yields from the indicial equation $\rho = -g, 1, (g+2)$. This directly leads to the following three linearly independent meromorphic solutions

$$\psi_1(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell+g+2}, \quad r_{3\ell+1} = r_{3\ell+2} = 0, \tag{3.5}$$

$$r_{3\ell+3} = \frac{z r_{3\ell} + g(g+2) \sum_{m=1}^{[(\ell+1)/2]} (g+3\ell+4-3m) c_{3m} r_{3\ell+3-6m}}{(3\ell+3)(3\ell+g+4)(3\ell+2g+5)}, \quad \ell \in \mathbb{N}_0,$$

$$\psi_2(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell+1}, \quad r_{3\ell+1} = r_{3\ell+2} = 0, \tag{3.6}$$

$$r_{3\ell+3} = \frac{z r_{3\ell} + g(g+2) \sum_{m=1}^{[(\ell+1)/2]} 3\,(\ell+1-m) c_{3m} r_{3\ell+3-6m}}{(3\ell+3)(3\ell+g+4)(3\ell-g+2)}, \quad \ell \in \mathbb{N}_0,$$

$$\psi_3(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell-g}, \quad r_{3\ell+1} = r_{3\ell+2} = 0, \tag{3.7}$$

$$r_{3\ell+3} = \frac{z r_{3\ell} + g(g+2) \sum_{m=1}^{[(\ell+1)/2]} (3\ell+2-g-3m) c_{3m} r_{3\ell+3-6m}}{(3\ell+3)(3\ell-g+2)(3\ell-2g+1)}, \quad \ell \in \mathbb{N}_0$$

where $[s]$ denotes the integer part of $s \in \mathbb{R}$. Note that the denominators in the coefficients $r_{3\ell+3}$ in (3.6) and (3.7) can not become zero since $g \not\equiv 2$ (mod 3). Thus we have proven that (3.2) possesses a meromorphic fundamental system, whenever $g_2 = 0$ and $h_g = 0$.

Remark 4. Halphen studied invariants of $X^m + Y^n = Z^p$, $m, n, p \in \mathbb{N}$ and applied this to differential equations to prove the meromorphy of their fundamental systems. In the case of equation (3.2) this polynomial reads $h^3 = A\ell^2 + B$ with $h^3 = \frac{(-g(g+2))^3}{4z^2} \wp(x)^3$, $\ell = \frac{-g(g+2)}{2z} \wp'(x)$, $A = \frac{-g(g+2)}{4}$, $B = \frac{(-g(g+2))^3}{16z^2} g_3$.

(ii) If $g_2 \neq 0$, direct computations show that meromorphic fundamental systems exist for the following six cases (cf. Example 1–4):

$$\begin{aligned} &g = 1, \quad h_1 = 0, \quad g = 3, \quad h_3 = \pm 2\sqrt{3g_2}, \\ &g = 4, \quad h_4 = 0, \quad g = 6, \;\; h_6 = \pm \frac{30}{7}\sqrt{3g_2}. \end{aligned} \tag{3.8}$$

In general, however, if $g \geq 7$ and $g_2 \neq 0$, a constant $h_g \in \mathbb{C}$ does not exist such that the fundamental system is meromorphic for arbitrary spectral parameters $z \in \mathbb{C}$.

Setting $\psi_2(z,x) = \sum_{\ell=0}^{\infty} r_{\ell,2} x^{\ell+1}$ yields, for $r_{g+1,2}$ finite, the condition

$$0 = z\, r_{g-2,2} - g\, h_g\, r_{g-1,2} + g(g+2) \sum_{m=2}^{[(g+1)/2]} (g+1-m)\, c_m r_{g+1-2m,2}. \quad (3.9)$$

The solution of this equation for h_g will in general always contain a term dependent of z if $g \geq 7$.

Remark 5. Let $\psi_1(z,x)$ be a solution of

$$\psi'''(z,x) + q_1(x)\, \psi'(z,x) + \left(\tfrac{1}{2}\, q_{1,x}(x) - z\right)\psi(z,x) = 0, \quad z \in \mathbb{C}, \quad (3.10)$$

and define $\hat{\psi}_1(z,x)$ by $\hat{\psi}_1(z,x) = \psi_1(-z,x)$. Then $\psi_2(z,x)$ given by

$$\psi_2(z,x) = \psi_1(z,x) \int^x \frac{\hat{\psi}_1(z,x')}{\psi_1^2(z,x')}\, dx' \quad (3.11)$$

yields a second linearly independent solution of (3.10). It is well known (cf. Ince [25], p. 122, or [13]) that the third linearly independent solution can be represented as

$$\psi_3(z,x) = -\psi_1(z,x) \int^x \frac{\psi_2(z,x')}{W(\psi_1,\psi_1;x')^2}\, dx' + \psi_2(z,x) \int^x \frac{\psi_1(z,x')}{W(\psi_1,\psi_1;x')^2}\, dx' \quad (3.12)$$

where $W(f,g;x) = f g_x - f_x g$ denotes the Wronskian of f and g.

Note that if $z = 0$ (3.10) reduces to the well-known third-order differential equation which is fulfilled by the product of two solutions of a second-order differential equation of the type $y''(x) + q(x)\, y(x) = \tilde{z}\, y(x)$ (see, e.g., [10], Part III, Chapter V, Section 71, Ex. 1, or [23], p. 511, equation (3.15)).

According to a theorem of Picard ([31], [32], [33]; see also [24], [2], pp. 182–187, [25], pp. 375–376) a differential equation with doubly periodic coefficients and a meromorphic fundamental system possesses solutions which are in general elliptic of the second kind. Since there exists at least one solution which is elliptic of the second kind and every elliptic function can be expressed in terms of σ functions, we set (see Appendix B, Theorem 14)

$$\psi_a(z,x) = e^{\lambda_a(z)x} \prod_{j=1}^{g} \frac{\sigma(x - a_j(z))}{\sigma(x)\sigma(a_j(z))}, \quad a(z) = (a_1(z), \ldots, a_g(z)), \quad (3.13)$$

which yields

$$\begin{aligned}
&\tfrac{1}{\psi_a}\Big(\psi_a''' + (h_g - g(g+2)\wp(x))\ \psi_a' - \frac{g(g+2)}{2}\wp'(x)\psi_a\Big) \\
&= (2g^2+g)\wp(x)\Big(\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big) \\
&\quad + g\zeta(x)\Big((2-2g)\sum_{j=1}^{g}\wp(a_j) - h_g - 3\Big(\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big)^2\Big) \\
&\quad + \sum_{j=1}^{g}\zeta(x-a_j)\Big(3\Big(\sum_{\ell=1,\ \ell\neq j}^{g}\zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big)^2 + \wp(a_j)(g-g^2) \\
&\quad + h_g + 3\sum_{\ell=1,\ \ell\neq j}^{g}\wp(a_\ell - a_j)\Big) + c_1 = z, \qquad c_1 \in \mathbb{C}
\end{aligned} \tag{3.14}$$

if and only if

$$z = (g^2 - \frac{5}{2}g + 1)\sum_{j=1}^{g}\wp'(a_j), \tag{3.15}$$

$$\lambda_a = \sum_{j=1}^{g}\zeta(a_j), \tag{3.16}$$

$$h_g = (2-2g)\sum_{j=1}^{g}\wp(a_j), \tag{3.17}$$

$$\begin{aligned}
0 &= 3\Big(\sum_{\ell=1,\ \ell\neq j}^{g}\zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big)^2 + (2-2g)\sum_{\ell=1}^{g}\wp(a_\ell) \\
&\quad + (g-g^2)\wp(a_j) + 3\sum_{\ell=1,\ \ell\neq j}^{g}\wp(a_\ell - a_j) \\
&= -(2g+1)\sum_{\ell=1}^{g}\wp(a_\ell) - (g^2+2g-6)\wp(a_j) + \tfrac{3}{4}\sum_{\ell=1,\ \ell\neq j}^{g}\Big(\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\Big)^2 \\
&\quad + \tfrac{3}{4}\Big(\sum_{\ell=1,\ \ell\neq j}^{g}\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\Big)^2, \quad 1 \le j \le g.
\end{aligned} \tag{3.18}$$

In order to derive (3.14) we used

$$\frac{\psi_a'''}{\psi_a} = \big(\frac{\psi_a'}{\psi_a}\big)'' + \big(\frac{\psi_a'}{\psi_a}\big)^3 + 3\big(\frac{\psi_a'}{\psi_a}\big)\big(\frac{\psi_a'}{\psi_a}\big)', \tag{3.19}$$

$$\frac{\psi_a'}{\psi_a} = \lambda_a + \sum_{j=1}^{g} \zeta(x-a_j) - g\zeta(x), \tag{3.20}$$

$$\big(\frac{\psi_a'}{\psi_a}\big)' = g\wp(x) - \sum_{j=1}^{g} \wp(x-a_j), \tag{3.21}$$

$$\begin{aligned}\frac{\psi_a'}{\psi_a}\big(\frac{\psi_a'}{\psi_a}\big)' = &-\sum_{j=1}^{g} \wp(a_j)\Big(\Big[\sum_{\ell=1,\ \ell\neq j}^{g} \zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big] \\ &+ \Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big]\Big) + \frac{1-g}{2}\sum_{j=1}^{g}\wp'(a_j) + \frac{g^2}{2}\wp'(x) + g\wp(x)\Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big] \\ &+ \tfrac{1}{2}\sum_{j=1}^{g}\wp'(x-a_j) + \sum_{j=1}^{g}\wp(x-a_j)\Big[\sum_{\ell=1,\ \ell\neq j}^{g}\zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big],\end{aligned} \tag{3.22}$$

$$\begin{aligned}\wp(x)\big(\frac{\psi_a'}{\psi_a}\big) = &\tfrac{1}{2}\sum_{j=1}^{g}\wp'(a_j) + \sum_{j=1}^{g}\wp(a_j)\zeta(a_j) + \frac{g}{2}\wp'(x) \\ &+ \wp(x)\Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big] - \zeta(x)\sum_{j=1}^{g}\wp(a_j) + \sum_{j=1}^{g}\zeta(x-a_j)\wp(a_j),\end{aligned} \tag{3.23}$$

$$\begin{aligned}\big(\frac{\psi_a'}{\psi_a}\big)^3 = &\frac{g^3}{2}\wp'(x) + 3g^2\wp(x)\Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big] \\ &- 3g\zeta(x)\Big(\Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big]^2 + g\sum_{j=1}^{g}\wp(a_j)\Big) \\ &- \tfrac{1}{2}\sum_{j=1}^{g}\wp'(x-a_j) - 3\sum_{j=1}^{g}\wp(x-a_j)\Big[\sum_{\ell=1,\ \ell\neq j}^{g}\zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big] \\ &+ 3\sum_{j=1}^{g}\zeta(x-a_j)\Big(\Big[\sum_{\ell=1,\ \ell\neq j}^{g}\zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big]^2 + g\wp(a_j) \\ &+ \sum_{\ell=1,\ \ell\neq j}^{g}\wp(a_\ell - a_j)\Big) + \frac{3g^2-1}{2}\sum_{j=1}^{g}\wp'(a_j) + \Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big]^3\end{aligned} \tag{3.24}$$

$$+ 6g \sum_{j=1}^{g} \wp(a_j) \Big[\lambda_a - \sum_{j=1}^{g} \zeta(a_j)\Big]$$
$$+ 3 \sum_{j=1}^{g} \zeta(a_j) \Big(\Big[\sum_{\ell=1,\ \ell\neq j}^{g} \zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a \Big]^2$$
$$+ g\wp(a_j) + \sum_{\ell=1,\ \ell\neq j}^{g} \wp(a_\ell - a_j) \Big)$$
$$+ 3 \sum_{j=1}^{g} \wp(a_j) \Big[\sum_{\ell=1,\ \ell\neq j}^{g} \zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a \Big].$$

Remark 6. The transformation $a \to -a$ (i.e., $a_j \to -a_j,\ 1 \le j \le g$) in (3.13) yields a solution of $L_3\psi_{-a} = -z\,\psi_{-a}$. By Remark 5 this yields two further solutions of (3.2).

3.1. **The equiharmonic case** $g_2 = 0, h_g = 0$**.** In the equiharmonic case where $g_2 = 0$ and $h_g = 0$, the two other solutions of (3.2) can be obtained in the following way. We start with

Remark 7. Given $\wp'(v) = z$, $z \neq 0$, there exist three different points v_j, $j = 1,2,3$, with $\wp'(v_j) = z$ and $v_1 + v_2 + v_3 = 0$. Assume $\wp'(v_j) = \wp'(v_k)$, $v_j \neq v_k$, $j,k = 1,2,3$. Then

$$\wp(v_2) = \wp(-v_3 - v_1) = -\wp(v_3) - \wp(v_1) + \tfrac{1}{4}\Big(\frac{\wp'(v_3) - \wp'(v_1)}{\wp(v_3) - \wp(v_1)}\Big)^2. \tag{3.25}$$

This implies

$$\wp(v_1) + \wp(v_2) + \wp(v_3) = 0 \text{ and } \ \zeta(v_1) + \zeta(v_2) + \zeta(v_3) = 0. \tag{3.26}$$

Now

$$\wp'^2(v_j) = 4\wp^3(v_j) - g_2\wp(v_j) - g_3, \quad j = 1,2,3 \tag{3.27}$$

yields

$$\frac{g_2}{4} = \wp^2(v_j) + \wp(v_j)\wp(v_k) + \wp^2(v_k), \qquad j,k = 1,2,3, \quad j \neq k. \tag{3.28}$$

From that we conclude that for $g_2 = 0$

$$\wp(v_2) = \wp(v_1)\alpha_3, \quad \wp(v_3) = \wp(v_1)\alpha_3^2, \qquad \alpha_3 = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \tag{3.29}$$

It follows that $v_2 = \alpha_3\, v_1$ and $v_3 = \alpha_3^2\, v_1$.

We define

$$\psi_{a,1}(z,x) = \psi_a(a_{1,1},\dots,a_{g,1},z,x), \quad \psi_{a,2}(z,x) = \psi_a(a_{1,2},\dots,a_{g,2},z,x),$$
$$\psi_{a,3}(z,x) = \psi_a(a_{1,3},\dots,a_{g,3},z,x), \tag{3.30}$$

where $\wp'(a_{j,1}) = \wp'(a_{j,2}) = \wp'(a_{j,3}),\ a_{j,\ell} = \alpha_3^{\ell-1}a_j,\ \ell = 1,2,3,\ 1 \le j \le g.$

One immediately recognizes that the conditions (3.15)–(3.17) are fulfilled if $g_2 = 0, h_g = 0$, and hence $\psi_{a,k}(z,x)$, $k = 1,2,3$, are solutions of (3.2).

The product $D_g(z,x) = \psi_{a,1}(z,x)\,\psi_{a,2}(z,x)\,\psi_{a,3}(z,x)$ of all three solutions then reads

$$\begin{aligned} D_g(z,x) &= \prod_{j=1}^{g}\frac{\sigma(x-a_{j,1}(z))}{\sigma(x)\sigma(a_{j,1}(z))}\prod_{j=1}^{g}\frac{\sigma(x-a_{j,2}(z))}{\sigma(x)\sigma(a_{j,2}(z))}\prod_{j=1}^{g}\frac{\sigma(x-a_{j,3}(z))}{\sigma(x)\sigma(a_{j,3}(z))} \\ &= \prod_{j=1}^{g}\frac{\sigma(x-a_{j,1}(z))}{\sigma(x)\sigma(a_{j,1}(z))}\frac{\sigma(x-a_{j,2}(z))}{\sigma(x)\sigma(a_{j,2}(z))}\frac{\sigma(x-a_{j,3}(z))}{\sigma(x)\sigma(a_{j,3}(z))} \\ &= \prod_{j=1}^{g}\tfrac{1}{2}\left(\wp'(x) - \wp'(a_{j,1})\right). \end{aligned} \tag{3.31}$$

The Wronskian $W(\psi_{a,1},\psi_{a,2},\psi_{a,3})$ is given by

$$\begin{aligned} W(\psi_{a,1},\psi_{a,2},\psi_{a,3}) = D_g(z,x)\Big(&\frac{\psi'_{a,2}}{\psi_{a,2}}\frac{\psi''_{a,3}}{\psi_{a,3}} + \frac{\psi'_{a,3}}{\psi_{a,3}}\frac{\psi''_{a,1}}{\psi_{a,1}} + \frac{\psi'_{a,1}}{\psi_{a,1}}\frac{\psi''_{a,2}}{\psi_{a,2}} \\ &- \frac{\psi'_{a,2}}{\psi_{a,2}}\frac{\psi''_{a,1}}{\psi_{a,1}} - \frac{\psi'_{a,1}}{\psi_{a,1}}\frac{\psi''_{a,3}}{\psi_{a,3}} - \frac{\psi'_{a,3}}{\psi_{a,3}}\frac{\psi''_{a,2}}{\psi_{a,2}}\Big). \end{aligned} \tag{3.32}$$

With

$$\frac{\psi'_{a,j}}{\psi_{a,j}} = \tfrac{1}{2}\sum_{\ell=1}^{g}\frac{\wp'(x)+\wp'(a_{\ell,j})}{\wp(x)-\wp(a_{\ell,j})}, \quad j = 1,2,3, \tag{3.33}$$

and

$$\frac{\psi''_{a,k}}{\psi_{a,k}} = 2\,g\,\wp(x) + \tfrac{1}{2}\sum_{\ell,s=1,\ \ell<s}^{g}\frac{\wp'(x)+\wp'(a_{\ell,k})}{\wp(x)-\wp(a_{\ell,k})}\frac{\wp'(x)+\wp'(a_{s,k})}{\wp(x)-\wp(a_{s,k})}, \tag{3.34}$$

$k = 1,2,3$, we may evaluate $W(\psi_{a,1},\psi_{a,2},\psi_{a,3})$ at $x = a_{j,1}$ since it is independent of x. This yields

$$W(\psi_{a,1},\psi_{a,2},\psi_{a,3}) = \frac{3}{2^g}\wp(a_{j,1})^2\prod_{\ell=1,\ \ell\neq j}^{g}(\wp'(a_{j,1}) - \wp'(a_{\ell,1}))$$

$$\Big[\sum_{\ell,s=1,\ \ell<s}^{g}\Big(\frac{\wp'(a_{j,1})+\wp'(a_{\ell,2})}{\wp(a_{j,1})-\wp(a_{\ell,2})}\frac{\wp'(a_{j,1})+\wp'(a_{s,2})}{\wp(a_{j,1})-\wp(a_{s,2})} - \frac{\wp'(a_{j,1})+\wp'(a_{\ell,3})}{\wp(a_{j,1})-\wp(a_{\ell,3})}\frac{\wp'(a_{j,1})+\wp'(a_{s,3})}{\wp(a_{j,1})-\wp(a_{s,3})}\Big) \tag{3.35}$$
$$+\Big(\sum_{\ell=1,\ \ell\neq j}^{g}\frac{\wp'(a_{j,1})+\wp'(a_{\ell,1})}{\wp(a_{j,1})-\wp(a_{\ell,1})}\Big)\Big(\sum_{\ell=1}^{g}\frac{\wp'(a_{j,1})+\wp'(a_{\ell,3})}{\wp(a_{j,1})-\wp(a_{\ell,3})}-\frac{\wp'(a_{j,1})+\wp'(a_{\ell,2})}{\wp(a_{j,1})-\wp(a_{\ell,2})}\Big)\Big].$$

Note that

$$\frac{\wp'(x)^2-\wp'(v)^2}{\wp(x)-\wp(v)} = 4(\wp(x)^2+\wp(x)\wp(v)+\wp(v)^2) \qquad (g_2=0), \tag{3.36}$$

and hence all remaining fractions in (3.35) will cancel out. Thus $\psi_{a,1}(z,x)$ and $\psi_{a,2}(z,x)$, $\psi_{a,3}(z,x)$ will not form a fundamental system when one of the values $\wp(a_{j,1})=0,\ 1\le j\le g$. In this case we can apply either Remark 5 or the results from Subsection 3.2 to obtain a fundamental system.

Remark 8. Halphen used the following

$$\psi(z,x) = e^{vx}\sum_{j=0}^{g-1}\alpha_j(z,\tilde z,v)\frac{d^j\phi(\tilde z,x)}{dx^j}, \tag{3.37}$$

to solve (3.2) where $\phi(\tilde z,x)$, which he called “élément simple”, is a solution of

$$\phi'' - (2\,\wp(x)+\tilde z)\phi = 0. \tag{3.38}$$

3.2. **Reduction of the order of the differential equation.** Here we briefly discuss the well-known process of the reduction of the order of a differential equation when one solution is known and apply it to Halphen's equation.

Having determined the solution $\psi_a(z,x)=\psi_1(z,x)$, we now consider the reduced equation (d'Alembert's method) and write

$$\psi_2(z,x) = \psi_1(z,x)\int^x u(z,x')\,dx'. \tag{3.39}$$

This yields

$$u'' + 3\,\frac{\psi_a'}{\psi_a}\,u' + (3\,\frac{\psi_a''}{\psi_a}+q_1)u = 0. \tag{3.40}$$

Picard's theorem applies again, and hence we set

$$u_b(z,x) = e^{\lambda_b(z)x} \prod_{j=1}^{g} \frac{\sigma(x-b_j(z))\sigma(x)\sigma(a_j(z))^2}{\sigma(x-a_j(z))^2\sigma(-b_j(z))}, \tag{3.41}$$

$b(z) = (b_1(z), \ldots, b_g(z))$. A similar analysis as before yields, for $c \in \mathbb{C}$,

$$\begin{aligned}
&\tfrac{1}{u_b}\Big(u_b'' + 3\,\frac{\psi_a'}{\psi_a}u_b' + (3\,\frac{\psi_a''}{\psi_a} + q_1)u_b\Big) \qquad (3.42)\\
&= \ c + g\zeta(x)\Big(-\lambda_b + \sum_{j=1}^{g}\zeta(b_j) - 2\sum_{j=1}^{g}\zeta(a_j)\Big)\\
&+ \sum_{j=1}^{g}\zeta(x-a_j)\Big(2\sum_{\ell=1,\ \ell\neq j}^{g}\zeta(a_j-a_\ell) - g\zeta(a_j) - \lambda_b - \sum_{\ell=1}^{g}\zeta(a_j-b_\ell)\Big)\\
&+ \sum_{j=1}^{g}\zeta(x-b_j)\Big(2\sum_{\ell=1,\ \ell\neq j}^{g}\zeta(b_j-b_\ell) - g\zeta(b_j) + 2\lambda_b + 3\lambda_a - \sum_{\ell=1}^{g}\zeta(b_j-a_\ell)\Big) = 0.
\end{aligned}$$

Equation (3.42) is fulfilled if and only if the following conditions hold:

$$\lambda_b = \sum_{j=1}^{g}\zeta(b_j) - 2\sum_{j=1}^{g}\zeta(a_j), \tag{3.43}$$

$$0 = 2\sum_{\ell=1,\ \ell\neq j}^{g}\zeta(a_j-a_\ell) - g\zeta(a_j) - \lambda_b - \sum_{\ell=1}^{g}\zeta(a_j-b_\ell), \quad 1 \le j \le g, \tag{3.44}$$

$$0 = 2\sum_{\ell=1,\ \ell\neq j}^{g}\zeta(b_j-b_\ell) - g\zeta(b_j) + 2\lambda_b + 3\lambda_a - \sum_{\ell=1}^{g}\zeta(b_j-a_\ell), \quad 1 \le j \le g, \tag{3.45}$$

$$0 = (1-g)(\sum_{\ell=1}^{g}\wp(a_\ell) - \sum_{\ell=1}^{g}\wp(b_\ell)). \tag{3.46}$$

The second solution u_2 of (3.40) can be obtained either by the transformation $a \to -a$, $b \to -b$ (i.e., $a_j \to -a_j$, $b_j \to -b_j$, $1 \le j \le g$) or by

$$u_2(z,x) = u_b(z,x)\int^x \tfrac{1}{u_b^2(z,x')\psi_a^3(z,x')}\,dx'. \tag{3.47}$$

4. Examples

4.1. **Example 1.** $g = 1$. Differential expressions

$$L_3 = \frac{d^3}{d\,x^3} - 3\,\wp(x)\,\frac{d}{d\,x} - \frac{3}{2}\,\wp'(x),$$
$$P_2 = \frac{d^2}{d\,x^2} - 2\wp(x).$$

Curve

$$\mathcal{F}_1(z,y) = y^3 - \frac{g_2}{4}\,y - z^2 - \frac{g_3}{4} = 0. \tag{4.1}$$

Elliptic solutions of the second kind

$$\psi_{a,j}(z,x) = \frac{\sigma(x - a_{1,j}(z))}{\sigma(x)\sigma(a_{1,j}(z))}\,e^{x\zeta(a_{1,j})}, \quad z = -\tfrac{1}{2}\wp'(a_{1,j}) \quad z \neq 0, \quad j = 1,2,3.$$

Product of solutions

$$D_1(z,x) = -\tfrac{1}{2}\wp'(a_{1,\ell}) + \tfrac{1}{2}\wp'(x) = z + \tfrac{1}{2}\wp'(x), \qquad \ell = 1,2,3.$$

4.2. **Example 2.** $g = 3$. Differential expressions

$$L_3 = \frac{d^3}{d\,x^3} + \big(2\,\sqrt{3g_2} - 15\,\wp(x)\big)\,\frac{d}{d\,x} - \frac{15}{2}\,\wp'(x),$$
$$P_4 = \frac{d^4}{d\,x^4} + \big(\frac{\sqrt{3g_2}}{3} - 20\,\wp(x)\big)\,\frac{d^2}{d\,x^2} - 20\,\wp'(x)\frac{d}{d\,x} + \big(10\,\sqrt{3g_2}\,\wp(x) - \frac{5}{2}\,g_2\big).$$

Curve

$$\begin{aligned}\mathcal{F}_3(z,y) = {}& y^3 + y\Big(-\frac{375}{16}\,{g_2}^2 - \frac{225}{4}\,\sqrt{3g_2}\,g_3 + 7\,\sqrt{3g_2}\,z^2\Big) + \frac{1375}{32}\,{g_2}^3 \\ &+ \frac{2625}{16}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,g_3 + \frac{3375}{16}\,g_3^2 + \frac{1505}{36}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,z^2 + \frac{55}{2}\,g_3\,z^2 - z^4 = 0.\end{aligned} \tag{4.2}$$

Elliptic solution of the second kind

$$\psi_a(z,x) = e^{\lambda_a(z)x}\prod_{j=1}^{3}\frac{\sigma(x - a_j(z))}{\sigma(x)\sigma(a_j(z))}, \quad a(z) = (a_1(z),\dots,a_3(z)),$$
$$z = \frac{5}{2}(\wp'(a_1) + \wp'(a_2) + \wp'(a_3)), \quad \lambda_a = \zeta(a_1) + \zeta(a_2) + \zeta(a_3),$$
$$2\sqrt{3g_2} = -4(\wp(a_1) + \wp(a_2) + \wp(a_3)),$$
$$0 = -7\sum_{\ell=1}^{3}\wp(a_\ell) - 9\wp(a_j)$$

$$+\frac{3}{4}\sum_{\ell=1,\ \ell\neq j}^{3}\Big(\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\Big)^2+\frac{3}{4}\Big(\sum_{\ell=1,\ \ell\neq j}^{3}\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\Big)^2,\ 1\leq j\leq 3.$$

Series solutions

$$\psi_1=x^5-\frac{\sqrt{3\,g_2}}{12}x^7+O(x^8),$$
$$\psi_2=x+\frac{\sqrt{3\,g_2}}{12}\,x^3-\frac{z}{21}x^4+r_{4,2}\,x^5+\frac{13}{1260}\,\sqrt{3\,g_2}\,z\,x^6+O(x^7),\ r_{4,2}\in\mathbb{C},$$
$$\psi_3=\tfrac{1}{x^3}+\frac{\sqrt{3\,g_2}}{4}\,\tfrac{1}{x}+\frac{z}{15}+r_{4,3}\,x-\frac{\sqrt{3\,g_2}}{60}\,z\,x^2$$
$$+\Big(\frac{\sqrt{3\,g_2}}{12}\,r_{4,3}+\frac{5\,g_3}{224}-\frac{z^2}{360}\Big)\,x^3-\Big(\frac{r_{4,3}}{21}+\frac{g_2}{84}\Big)\,z\,x^4+O(x^5),\ r_{4,3}\in\mathbb{C}.$$

To obtain the results for the case $h_3=-2\sqrt{3g_2}$ simply replace $\sqrt{g_2}$ by $-\sqrt{g_2}$ in all expressions above.

4.3. **Example 3.** $g=4$. Differential expressions

$$L_3=\frac{d^3}{dx^3}-24\wp(x)\frac{d}{d\,x}-12\wp'(x),$$
$$P_5=\frac{d^5}{dx^5}-40\wp(x)\frac{d^3}{dx^3}-60\wp'(x)\frac{d^2}{dx^2}+(38g_2+40\wp(x)^2)\frac{d}{dx}+160\wp(x)\wp'(x).$$

Curve

$$\mathcal{F}_4(z,y)=y^3-z^5+208\,g_3z^3+y(3136g_2g_3-44\,g_2z^2)-3136({g_2}^3+4{g_3}^2)z=0. \tag{4.3}$$

Elliptic solution of the second kind

$$\psi_a(z,x)=e^{\lambda_a(z)x}\prod_{j=1}^{4}\frac{\sigma(x-a_j(z))}{\sigma(x)\sigma(a_j(z))},\quad a(z)=(a_1(z),\ldots,a_4(z)),$$
$$z=7(\wp'(a_1)+\wp'(a_2)+\wp'(a_3)+\wp'(a_4)),$$
$$\lambda_a=\zeta(a_1)+\zeta(a_2)+\zeta(a_3)+\zeta(a_4),$$
$$0=(\wp(a_1)+\wp(a_2)+\wp(a_3)+\wp(a_4)),$$
$$18\wp(a_j)=\frac{3}{4}\sum_{\ell=1,\ \ell\neq j}^{4}\Big(\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\Big)^2+\frac{3}{4}\Big(\sum_{\ell=1,\ \ell\neq j}^{4}\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\Big)^2,$$

$1\leq j\leq 4$. Series solutions

$$\psi_1=x^6+\frac{z}{312}x^9+O(x^{10}),$$

$$\psi_2 = x - \frac{z}{48}\,x^4 - \frac{g_2}{15}\,x^5 + r_{5,2}\,x^6 + \Big(\frac{3\,g_3}{77} - \frac{z^2}{3168}\Big)x^7 + O(x^8),\ \ r_{5,2}\in\mathbb{C},$$
$$\psi_3 = \tfrac{1}{x^4} + \frac{z}{42}\,\tfrac{1}{x} - \frac{3\,g_2}{20} + r_{5,3}\,x + O(x^2),\ \ r_{5,3}\in\mathbb{C}.$$

4.4. **Example 4.** $g=6$. Differential expressions

$$L_3 = \frac{d^3}{dx^3} + \big(\frac{30}{7}\sqrt{3g_2} - 48\,\wp(x)\big)\,\frac{d}{dx} - 24\wp'(x),$$
$$\begin{aligned}P_7 = {} & \frac{d^7}{dx^7} + \big(\frac{4}{3}\sqrt{3g_2} - 112\,\wp(x)\big)\,\frac{d^5}{dx^5} - 280\,\wp'(x)\,\frac{d^4}{dx^4}\\
& + \Big(\frac{316\,g_2}{3} + \frac{160}{3}\sqrt{3\,g_2}\,\wp(x) + 1120\,\wp(x)^2\Big)\frac{d^3}{dx^3} + \Big(80\sqrt{3\,g_2}\,\wp'(x)\\
& + 6720\,\wp(x)\,\wp'(x)\Big)\frac{d^2}{dx^2} - \frac{8}{49}\Big(3333\sqrt{3}\,{g_2}^{\frac{3}{2}} + 23030\,g_3 + 5614\,g_2\,\wp(x)\\
& + 30380\sqrt{3\,g_2}\,\wp(x)^2 - 150920\,\wp(x)^3\Big)\frac{d}{dx}\\
& + \frac{512}{7}\Big(19\,g_2 - 70\sqrt{3\,g_2}\,\wp(x)\Big)\ \wp'(x).\end{aligned}$$

Curve

$$\begin{aligned}\mathcal{F}_6(z,y) = {} & y^3 - z^7 + \Big(\frac{1172432}{441}\sqrt{3}\,{g_2}^{\frac{3}{2}} + 2992\,g_3\Big)\,z^5\\
& - \Big(\frac{389275254016}{453789}\,{g_2}^3 + \frac{8716731904}{3087}\sqrt{3}\,{g_2}^{\frac{3}{2}}\,g_3 + 2972416\,{g_3}^2\Big)\,z^3\\
& + \Big(26\sqrt{3\,g_2}\,z^4 - \frac{20521280}{1029}\,{g_2}^2\,z^2 + \frac{308472947200}{823543}\sqrt{3}\,{g_2}^{\frac{7}{2}}\\
& - \frac{225472}{7}\sqrt{3\,g_2}\,g_3\,z^2 + \frac{14301619200}{2401}\,{g_2}^2\,g_3 + \frac{41817600}{7}\sqrt{3\,g_2}\,{g_3}^2\Big)\,y\\
& + \Big(\frac{791904252620800}{17294403}\sqrt{3}\,{g_2}^{\frac{9}{2}} + \frac{77133027840000}{117649}\,{g_2}^3\,g_3\\
& + \frac{346472755200}{343}\sqrt{3}\,{g_2}^{\frac{3}{2}}\,{g_3}^2 + 1003622400\,{g_3}^3\Big)\,z = 0.\end{aligned}\tag{4.4}$$

Elliptic solution of the second kind

$$\psi_a(z,x) = e^{\lambda_a(z)x}\prod_{j=1}^{6}\frac{\sigma(x-a_j(z))}{\sigma(x)\sigma(a_j(z))},\quad a(z) = (a_1(z),\dots,a_6(z)),$$
$$z = 22(\wp'(a_1)+\wp'(a_2)+\wp'(a_3)+\wp'(a_4)+\wp'(a_5)+\wp'(a_6)),$$
$$\lambda_a = \zeta(a_1)+\zeta(a_2)+\zeta(a_3)+\zeta(a_4)+\zeta(a_5)+\zeta(a_6),$$

$$\frac{30}{7}\sqrt{3g_2} = -10(\wp(a_1)+\wp(a_2)+\wp(a_3)+\wp(a_4)+\wp(a_5)+\wp(a_6)),$$
$$0 = -13\sum_{j=1}^{6}\wp(a_j) - 42\wp(a_j) + \frac{3}{4}\sum_{\ell=1,\ \ell\neq j}^{6}\Big(\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\Big)^2$$
$$+\frac{3}{4}\Big(\sum_{\ell=1,\ \ell\neq j}^{6}\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\Big)^2, \quad 1\le j\le 6.$$

Series solutions

$$\psi_1 = x^8 + O(x^{10}),$$
$$\psi_2 = x + \frac{\sqrt{3\,g_2}}{21}x^3 - \frac{z}{120}x^4 - \frac{11\,g_2}{490}x^5 - \frac{z\sqrt{3\,g_2}}{630}x^6$$
$$+\left(\frac{z^2}{9360} - \frac{1609}{133770}\sqrt{3}\,{g_2}^{\frac{3}{2}} - \frac{6\,g_3}{91}\right)x^7 + r_{7,2}x^8 + O(x^9),\ r_{7,2}\in\mathbb{C},$$
$$\psi_3 = \tfrac{1}{x^6} + \frac{3}{14}\sqrt{3\,g_2}\,\tfrac{1}{x^4} + \frac{z}{132}\,\tfrac{1}{x^3} - \frac{2\,g_2}{245}\,\tfrac{1}{x^2} + \frac{4}{1155}\sqrt{3\,g_2}\,z\,\tfrac{1}{x} + \frac{z^2}{6336}$$
$$-\frac{461}{13720}\sqrt{3}\,{g_2}^{\frac{3}{2}} - \frac{g_3}{7} + r_{7,3}\,x + O(x^2),\ r_{7,3}\in\mathbb{C}.$$

To obtain the results for the case $h_6 = -\frac{30}{7}\sqrt{3g_2}$ simply replace $\sqrt{g_2}$ by $-\sqrt{g_2}$ in all expressions above.

Remark 9. If $g_2 = 0$, all curves above degenerate into cyclic coverings of the line (see, e.g., [30]); i.e.,

$$\mathcal{F}_g(z,y) = y^3 - T_{g+1}(z) = 0. \tag{4.5}$$

4.5. **Example 5.** $g = 7$.

$$L_3 = \frac{d^3}{d\,x^3} + \big(h_7 - 63\,\wp(x)\big)\frac{d}{dx} - \frac{63}{2}\wp'(x).$$

Series solutions $\psi_1 = x^9 + O(x^{11})$. Then $\psi_2 = \sum_{j=0}^{\infty} r_{j,2}x^{j+1}$ leads to the condition

$$0 = 54054000\,{g_2}^2 + \big(55296\,z^2 - 49420800\,g_3\big)\,h_7 - 1801800\,g_2\,{h_7}^2 + 3575\,{h_7}^4$$

for $r_{8,2}$ being finite. This equation does not have a solution h_7 which is independent of z if $g_2 \neq 0$.

Remark 10. This result does not imply that there exist no commuting pairs of differential expressions (L_3, P_7) with elliptic coefficients where $g_2 \neq 0$. For example, choose in (1.3) for the coefficients $(q_1(x), q_0(x))$ one of the pairs

$\{(-6\wp(x), \pm 3\wp'(x)), (-12\wp(x), \pm 6\wp'(x)), (-18\wp(x), \pm 15\wp'(x))\}$. Then there exist corresponding differential expressions P_7 such that the commutator $[L_3, P_7] = 0$.

All these calculations can be done easily by using *Mathematica* or another CAS.

Appendix A. The limit $\omega_1 \to \infty, \omega_3 \to \infty$

In the limiting case where the half-periods $\omega_1 \to \infty$, $\omega_3 \to \infty$, equation (3.2) degenerates into

$$\begin{gathered}\psi'''(z,x) - \frac{g(g+2)}{x^2}\psi'(z,x) + \big(\frac{g(g+2)}{x^3} - z\big)\psi(z,x) = 0, \\ z \in \mathbb{C}, \quad g \in \mathbb{N}, \quad g \not\equiv 2 \pmod 3.\end{gathered} \tag{A.1}$$

According to the theory of Fuchs, $x = 0$ is a regular singular point of (A.1). The method of Frobenius,

$$\psi(z,x) = x^\rho \sum_{\ell=0}^{\infty} r_\ell x^\ell, \quad r_0 \neq 0, \tag{A.2}$$

then yields from the indicial equation $\rho = -g, 1, (g+2)$. This directly leads to the following three linearly independent meromorphic solutions:

$$\begin{aligned}\psi_1(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell+g+2}, \quad & r_{3\ell+1} = r_{3\ell+2} = 0, \\ & r_{3\ell+3} = \frac{z r_{3\ell}}{(3\ell+3)(3\ell+g+4)(3\ell+2g+5)}, \quad \ell \in \mathbb{N}_0,\end{aligned} \tag{A.3}$$

$$\begin{aligned}\psi_2(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell+1}, \quad & r_{3\ell+1} = r_{3\ell+2} = 0, \\ & r_{3\ell+3} = \frac{z r_{3\ell}}{(3\ell+3)(3\ell+g+4)(3\ell-g+2)}, \quad \ell \in \mathbb{N}_0,\end{aligned} \tag{A.4}$$

$$\begin{aligned}\psi_3(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell-g}, \quad & r_{3\ell+1} = r_{3\ell+2} = 0, \\ & r_{3\ell+3} = \frac{z r_{3\ell}}{(3\ell+3)(3\ell-g+2)(3\ell-2g+1)}, \quad \ell \in \mathbb{N}_0.\end{aligned} \tag{A.5}$$

Note that the denominators in the coefficients $r_{3\ell+3}$ in (A.4) and (A.5) can not become zero since $g \not\equiv 2 \pmod 3$. Thus (A.1) possesses a meromorphic fundamental system. By another theorem of Halphen [20], [25, pp. 272–275] the general solution of (A.1) must therefore have the following form

$$\psi(z,x) = \sum_{j=1}^{3} c_j \frac{p_{g,j}(x)}{x^g} e^{\beta_j x} \tag{A.6}$$

where $c_j, \beta_j \in \mathbb{C}$, $j = 1,2,3$, and $p_{g,j}(x)$, $j = 1,2,3$, are polynomials of degree g. Equation (A.1) is invariant under the transformation $x \to \alpha_3^j kx$, $j = 1,2,3$, $k = z^{1/3}$, $\alpha_3 = e^{2\pi i/3}$, which finally yields the general solution of (A.1),

$$\psi(z,x) = \sum_{j=1}^{3} \frac{c_j p_g(\alpha_3^j kx)}{x^g} e^{\alpha_3^j kx}, \qquad p_g(\tilde{x}) = \sum_{\ell=0}^{g} \tilde{r}_\ell \tilde{x}^\ell, \tag{A.7}$$

$$\tilde{r}_{\ell+3} = \frac{(6g\ell + 11g - 3\ell^2 - 9\ell - 6 - 2g^2)\tilde{r}_{\ell+2} + 3(g-\ell-1)\tilde{r}_{\ell+1}}{(\ell+3)(\ell-g+2)(\ell-2g+1)},$$
$$\ell = 0,\dots,g-3, \quad \tilde{r}_1 = -\tilde{r}_0, \quad \tilde{r}_2 = \tilde{r}_0/2.$$

Remark 11. Halphen solved equation (A.1) by using a Darboux-type transformation expressing a solution ψ corresponding to $g{+}3$ in terms of a solution ψ for g, i.e.,

$$\psi_{g+3} = z\,\psi_g - \frac{2g+3}{x}\,\psi_g'' + \frac{(2g+3)(g+1)}{x^2}\,\psi_g' - \frac{(2g+3)(g+1)}{x^3}\,\psi_g. \tag{A.8}$$

Appendix B. Some theorems on elliptic functions

For convenience we recall some theorems representing an arbitrary elliptic function in terms of σ- and ζ-functions which are used in this text. For general references see, for instance, Akhiezer [2], Markushevich [26], and Whittaker and Watson [40].

Theorem 12. [[26], Theorem 5.12 (p. 181)]. *Given an elliptic function f of order n with fundamental periods $2\,\omega_1$ and $2\,\omega_3$, let $a_1,\dots,a_n$ and $b_1,\dots,b_n$ be the zeros and poles of f in the fundamental period parallelogram Δ repeated according to their multiplicities. Then*

$$f(x) = C\frac{\sigma(x-a_1)\cdots\sigma(x-a_n)}{\sigma(x-b_1)\cdots\sigma(x-b_{n-1})\sigma(x-b_n')},$$

where C is a suitable constant, σ is constructed from the fundamental periods $2\,\omega_1$ and $2\omega_3$ and where $b_n' - b_n = (a_1+\cdots+a_n)-(b_1+\cdots+b_n)$ is a period of f. Conversely, every such function is an elliptic function.

Theorem 13. [[26], Theorem 5.13 (p. 182)]. *Given an elliptic function f with fundamental periods $2\omega_1$ and $2\omega_3$, let $b_1, \dots, b_r$ be the distinct poles of f in Δ. Suppose the principal part of the Laurent expansion near b_k is given by*

$$\sum_{j=1}^{\beta_k} \frac{A_{j,k}}{(x-b_k)^j}, \quad k = 1, \dots, r. \tag{B.1}$$

Then

$$f(x) = C + \sum_{k=1}^{r} \sum_{j=1}^{\beta_k} (-1)^{j-1} \frac{A_{j,k}}{(j-1)!} \zeta^{(j-1)}(x-b_k), \tag{B.2}$$

where C is a suitable constant and $\zeta(x)$ is constructed from the fundamental periods $2\omega_1$ and $2\omega_3$. Conversely, every such function is an elliptic function if $\sum_{k=1}^{r} A_{1,k} = 0$.

Note that this theorem resembles the partial fraction expansions for rational functions.

Finally, we turn to elliptic functions of the second kind, the central object in our analysis. A meromorphic function $\psi : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ for which there exist two complex constants ω_1 and ω_3 with nonreal ratio and two complex constants ρ_1 and ρ_3 such that for $i = 1, 3$, $\psi(x + 2\omega_i) = \rho_i \psi(x)$ is called elliptic of the second kind. We call $2\,\omega_1$ and $2\,\omega_3$ the quasi-periods of ψ. Together with $2\,\omega_1$ and $2\,\omega_3$, $2\,m_1\omega_1 + 2\,m_3\omega_3$ are also quasi-periods of ψ if m_1 and m_3 are integers. If every quasi-period of ψ can be written as an integer linear combination of $2\,\omega_1$ and $2\,\omega_3$ then these are called fundamental quasi-periods.

Theorem 14. *A function ψ which is elliptic of the second kind and has fundamental quasi-periods $2\,\omega_1$ and $2\,\omega_3$ can always be put into the form*

$$\psi(x) = C \exp(\lambda x) \frac{\sigma(x-a_1) \cdots \sigma(x-a_n)}{\sigma(x-b_1) \cdots \sigma(x-b_n)}$$

for suitable constants C, λ, $a_1, \dots, a_n$ and $b_1, \dots, b_n$. Here σ is constructed from the fundamental periods $2\,\omega_1$ and $2\,\omega_3$. Conversely, every such function is elliptic of the second kind.

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