EXAMPLES OF COMMUTING PAIRS OF OPERATORS (L_3, P_r) WITH ELLIPTIC COEFFICIENTS

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We freely use the notation introduced in [1] and [2]. Furthermore we denote homogeneous quantities (obtained by setting integration constants zero) without hats and corresponding inhomogeneous quantities with hats "ˆ".

Consider the differential expressions

$$(L_3 f)(x) = f^{(3)}(x) + q_1(x)\, f'(x) + \big(\frac{1}{2}\, {q_1}'(x) + q_0(x)\big)\, f(x) \tag{1.1}$$

and

$$(P_1 f)(x) = f'(x), \tag{1.2}$$

$$(P_2 f)(x) = f''(x) + \frac{2}{3}\, q_1(x) f(x), \tag{1.3}$$

$$\begin{aligned}(P_4 f)(x) &= f^{(4)}(x) + \frac{4}{3}\, q_1(x)\, f''(x) + \frac{4}{3}\,(q_0(x) + q_1'(x))\, f'(x)\\ &\quad + \frac{1}{9}\,(2\, q_1^2(x) + 5q_1''(x) + 6\, q_0'(x))\, f(x),\end{aligned} \tag{1.4}$$

$$\begin{aligned}(P_5 f)(x) &= f^{(5)}(x) + \frac{5}{3}\, q_1(x)\, f^{(3)}(x) + \left(\frac{5}{3}\, q_0(x) + \frac{5}{2}\, {q_1}'(x)\right)\, f''(x)\\ &\quad + \frac{1}{18}\left(10\, q_1(x)^2 + 30\, {q_0}'(x) + 35\, {q_1}''(x)\right)\, f'(x)\\ &\quad + \frac{1}{9}\left(10\, q_0(x)\, q_1(x) + 5\, q_1(x)\, {q_1}'(x) + 10\, {q_0}''(x) + 5\, {q_1}^{(3)}(x)\right) f(x),\end{aligned} \tag{1.5}$$

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$$
\begin{aligned}
P_7 = {} & f^{(7)}(x) + \frac{7}{3}\, q_1(x)\, f^{(5)}(x) + \Big(\frac{7}{3}\, q_0(x) + \frac{35}{6}\, q_1'(x)\Big)\, f^{(4)}(x) \\
& \Big(\frac{14}{9}\, q_1(x)^2 + \frac{14}{3}\, q_0'(x) + \frac{77}{9}\, q_1''(x)\Big)\, f^{(3)}(x) \\
& + \Big(\frac{28}{9}\, q_0(x)\, q_1(x) + \frac{14}{3}\, q_1(x)\, q_1'(x) + \frac{56}{9}\, q_0''(x) + 7\, q_1^{(3)}(x)\Big)\, f''(x) \\
& + \Big(\frac{14}{9}\, q_0(x)^2 + \frac{14}{81}\, q_1(x)^3 + \frac{28}{9}\, q_1(x)\, q_0'(x) + \frac{28}{9}\, q_0(x)\, q_1'(x) \\
& + \frac{35}{18}\, q_1'(x)^2 + \frac{28}{9}\, q_1(x)\, q_1''(x) + \frac{35}{9}\, q_0^{(3)}(x) + \frac{161}{54}\, q_1^{(4)}(x)\Big)\, f'(x) \\
& + \Big(\frac{14}{27}\, q_0(x)\, q_1(x)^2 + \frac{14}{9}\, q_0(x)\, q_0'(x) + \frac{7}{27}\, q_1(x)^2\, q_1'(x) \\
& + \frac{49}{27}\, q_0'(x)\, q_1'(x) + \frac{14}{9}\, q_1(x)\, q_0''(x) + \frac{35}{27}\, q_0(x)\, q_1''(x) + \frac{7}{6}\, q_1'(x)\, q_1''(x) \\
& + \frac{7}{9}\, q_1(x)\, q_1^{(3)}(x) + \frac{28}{27}\, q_0^{(4)}(x) + \frac{14}{27}\, q_1^{(5)}(x)\Big)\, f(x).
\end{aligned} \tag{1.6}
$$

The commutator $[L_3, P_5]$ is a first order differential expression which yields the homogeneous (i.e., setting the constants $c_j = d_j = 0, j = 0, 1, 2$ in (1.8)) stationary s-Bsq$_5$ equation

$$
\widehat{\text{s-Bsq}_5}(q_0, q_1) = \begin{cases}
\frac{1}{9}\, q_{0,xxxxx} + \frac{5}{18}\, q_0 q_{1,xxx} + \frac{5}{9}\, q_1 q_{0,xxx} + \frac{5}{9}\, q_{1,xx} q_{0,x} \\
\qquad + \frac{5}{6}\, q_{1,x} q_{0,xx} + \frac{5}{9}\, q_1^2 q_{0,x} + \frac{10}{9}\, q_0 q_1 q_{1,x} + d_0 \big(\frac{1}{18}\, q_{1,xxxxx} \\
\qquad + \frac{1}{3}\, q_1 q_{1,xxx} + \frac{2}{3}\, q_{1,x} q_{1,xx} + \frac{4}{9}\, q_1^2 q_{1,x} - \frac{4}{3}\, q_0 q_{0,x}\big) \\
\qquad + c_1 \left(\frac{1}{6}\, q_{1,xxx} + \frac{2}{3}\, q_1 q_{1,x}\right) - d_1 q_{0,x} = 0, \\
\frac{1}{9}\, q_{1,xxxxx} + \frac{5}{9}\, q_1 q_{1,xxx} + \frac{25}{18}\, q_{1,x} q_{1,xx} + \frac{5}{9}\, q_1^2 q_{1,x} \\
\qquad - \frac{10}{3}\, q_0 q_{0,x} - d_0 \left(\frac{2}{3}\, q_{0,xxx} + \frac{4}{3}\, q_1 q_{0,x} + \frac{4}{3}\, q_{1,x} q_0\right) \\
\qquad - c_1\, 2 q_{0,x} - d_1\, q_{1,x} = 0.
\end{cases} \tag{1.7}
$$

The homogeneous s-Bsq$_7$ equation reads

(1.8)
$$\text{s-Bsq}_7(q_0,q_1) = \begin{cases} -\frac{28}{9}\,q_0(x)\,q_1(x)\,q_0{}'(x) - \frac{14}{9}\,q_0(x)^2\,q_1{}'(x) + \frac{28}{81}\,q_1(x)^3\,q_1{}'(x) \\ +\frac{35}{54}\,q_1{}'(x)^3 - \frac{28}{9}\,q_0{}'(x)\,q_0{}''(x) + \frac{7}{3}\,q_1(x)\,q_1{}'(x)\,q_1{}''(x) \\ -\frac{14}{9}\,q_0(x)\,q_0{}^{(3)}(x) + \frac{14}{27}\,q_1(x)^2\,q_1{}^{(3)}(x) + \frac{14}{9}\,q_1{}''(x)\,q_1{}^{(3)}(x) \\ +\frac{49}{54}\,q_1{}'(x)\,q_1{}^{(4)}(x) + \frac{7}{27}\,q_1(x)\,q_1{}^{(5)}(x) + \frac{1}{27}q_1{}^{(7)}(x) = 0, \\ -\frac{14}{9}\,q_0(x)^2\,q_0{}'(x) + \frac{28}{81}\,q_1(x)^3\,q_0{}'(x) + \frac{28}{27}\,q_0(x)\,q_1(x)^2\,q_1{}'(x) \\ +\frac{7}{6}\,q_0{}'(x)\,q_1{}'(x)^2 + \frac{49}{27}\,q_1(x)\,q_1{}'(x)\,q_0{}''(x) + \frac{14}{9}\,q_1(x)\,q_0{}'(x)\,q_1{}''(x) \\ +\frac{14}{9}\,q_0(x)\,q_1{}'(x)\,q_1{}''(x) + \frac{14}{27}\,q_1(x)^2\,q_0{}^{(3)}(x) + \frac{49}{54}\,q_1{}''(x)\,q_0{}^{(3)}(x) \\ +\frac{7}{9}\,q_0(x)\,q_1(x)\,q_1{}^{(3)}(x) + \frac{7}{9}\,q_0{}''(x)\,q_1{}^{(3)}(x) + \frac{35}{54}\,q_1{}'(x)\,q_0{}^{(4)}(x) \\ +\frac{7}{18}\,q_0{}'(x)\,q_1{}^{(4)}(x) + \frac{7}{27}\,q_1(x)\,q_0{}^{(5)}(x) + \frac{7}{54}\,q_0(x)\,q_1{}^{(5)}(x) \\ +\frac{1}{27}\,q_0{}^{(7)}(x) = 0. \end{cases}$$

Note that

$$[\hat{P}_r, L_3] = 3\,\hat{f}_{n+1,x}\,\frac{d}{d\,x} + \frac{3}{2}\,\hat{f}_{n+1,xx} + 3\,\hat{g}_{n+1,x}, \qquad r = 3\,n+1 \text{ or } r = 3\,n+2, \quad n \in \mathbb{N}_0 \tag{1.9}$$

and that the time-dependent Bsq hierarchy is defined as a collection of evolution equations (varying $r = 3\,n+1$ or $r = 3\,n+2,\ n \in \mathbb{N}_0$)

$$\frac{d}{d\,t_r} L_3(t_r) - [\hat{P}_r(t_r), L_3(t_r)] = 0, \quad r = 3\,n+1 \text{ or } r = 3\,n+2, \quad n \in \mathbb{N}_0, \tag{1.10}$$

or equivalently, by

$$\widehat{\text{Bsq}}_r(q_0,q_1) = \begin{cases} q_{0,t_r} - 3\,\hat{g}_{n+1,x} = 0 \\ q_{1,t_r} - 3\,\hat{f}_{n+1,x} = 0 \end{cases},$$

$$(x,t_r) \in \mathbb{R}^2, \quad r = 3\,n+1 \text{ or } r = 3\,n+2, \quad n \in \mathbb{N}_0. \tag{1.11}$$

2. Commuting pairs of operators (L_3, P_5) with elliptic coefficients

We give now some examples of commuting pairs of operators $(L_3, \hat{P}_5)$ with elliptic coefficients, which are not covered by [3].

In the following $\hat{P}_5$ denotes the inhomogeneous differential expression

$$\hat{P}_5 = P_5 + d_1 P_1. \tag{2.1}$$

Example 1

$$\begin{aligned}
&q_1 = -6\,\wp, \quad q_0 = 3\,\wp', \quad d_1 = -\frac{1}{3}\,g_2,\\
&(L_3 f)(x) = f^{(3)}(x) - 6\,\wp(x)\,f'(x),\\
&(\hat{P}_5 f)(x) = f^{(5)}(x) - 10\,\wp(x)\,f'''(x) - 10\,\wp'(x)\,f''(x) + (3\,g_2 - 20\wp(x)^2)f'(x).
\end{aligned}$$

Curve

$$y^3 + \left(g_2\,z^2 + 4\,g_2\,g_3\right)\,y - z^5 - 8\,g_3\,z^3 - 16\,{g_3}^2\,z = 0. \tag{2.2}$$

Example 2

$$\begin{aligned}
&q_1 = -6\,\wp, \quad q_0 = -3\,\wp', \quad d_1 = -\frac{1}{3}\,g_2,\\
&(L_3 f)(x) = f^{(3)}(x) - 6\,\wp(x)\,f'(x) - 6\,\wp'(x)\,f(x),\\
&(\hat{P}_5 f)(x) = f^{(5)}(x) - 10\,\wp(x)\,f'''(x) - 20\,\wp'(x)\,f''(x) + (8\,g_2 - 80\wp(x)^2)f'(x)\\
&\qquad - 40\,\wp(x)\wp'(x)f(x).
\end{aligned}$$

Curve

$$y^3 + \left(g_2\,z^2 + 4\,g_2\,g_3\right)\,y - z^5 - 8\,g_3\,z^3 - 16\,{g_3}^2\,z = 0. \tag{2.3}$$

Example 3

$$\begin{aligned}
&q_1 = -12\,\wp, \quad q_0 = 6\,\wp', \quad d_1 = \frac{4}{3}\,g_2,\\
&(L_3 f)(x) = f^{(3)}(x) - 12\,\wp(x)\,f'(x),\\
&(\hat{P}_5 f)(x) = f^{(5)}(x) - 20\,\wp(x)\,f'''(x) - 20\,\wp'(x)\,f''(x) + 8\,g_2 f'(x).
\end{aligned}$$

Curve

$$y^3 - 4\,g_2\,z^2\,y - z^5 - 16\,g_3\,z^3 = 0. \tag{2.4}$$

Example 4

$$\begin{aligned}
&q_1 = -12\,\wp, \quad q_0 = -6\,\wp', \quad d_1 = \frac{4}{3}\,g_2,\\
&(L_3 f)(x) = f^{(3)}(x) - 12\,\wp(x)\,f'(x) - 12\wp'(x)f(x),\\
&(\hat{P}_5 f)(x) = f^{(5)}(x) - 20\,\wp(x)\,f'''(x) - 40\,\wp'(x)\,f''(x) + (18\,g_2 - 120\wp(x)^2)f'(x).
\end{aligned}$$

Curve

$$y^3 - 4\,g_2\,z^2\,y - z^5 - 16\,g_3\,z^3 = 0. \tag{2.5}$$

3. Commuting pairs of operators (L_3, P_7) with elliptic coefficients

We give now some examples of commuting pairs of operators $(L_3, \hat{P}_7)$ with elliptic coefficients, which are mentioned in Remark 10 of [3].

$\hat{P}_7$ denotes the inhomogeneous differential expression

$$\hat{P}_7 = P_7 + d_1 P_1. \tag{3.1}$$

Example 5

$$
\begin{aligned}
&q_1 = -6\,\wp, \quad q_0 = 3\,\wp', \quad d_1 = \frac{4}{3}\,g_3,\\
&(L_3 f)(x) = f^{(3)}(x) - 6\,\wp(x)\,f'(x),\\
&(\hat{P}_7 f)(x) = f^{(7)}(x) - 14\,\wp(x)\,f^{(5)}(x) - 28\,\wp'(x)\,f^{(4)}(x) + \left(\frac{56}{3}\,g_2 - 168\,\wp(x)^2\right)\,f^{(3)}(x)\\
&\qquad - 168\,\wp(x)\,\wp'(x)\,f''(x) + \left(48\,g_3 + 56\,g_2\,\wp(x) - 336\,\wp(x)^3\right)\,f'(x).
\end{aligned}
$$

Curve

$$
y^3 - \frac{1}{3}\,{g_2}^2\,z^2\,y - z^7 - 4\,g_3\,z^5 - \frac{2}{27}\,{g_2}^3\,z^3 = 0. \tag{3.2}
$$

Example 6

$$
\begin{aligned}
&q_1 = -6\,\wp, \quad q_0 = -3\,\wp', \quad d_1 = \frac{4}{3}\,g_3,\\
&(L_3 f)(x) = f^{(3)}(x) - 6\,\wp(x)\,f'(x) - 6\,\wp'(x)\,f(x),\\
&(\hat{P}_7 f)(x) = f^{(7)}(x) - 14\,\wp(x)\,f^{(5)}(x) - 42\,\wp'(x)\,f^{(4)}(x) + \left(\frac{98}{3}\,g_2 - 336\,\wp(x)^2\right)\,f^{(3)}(x)\\
&\qquad - 504\,\wp(x)\,\wp'(x)\,f''(x) + \left(216\,g_3 + 308\,g_2\,\wp(x) - 2016\,\wp(x)^3\right)\,f'(x)\\
&\qquad + \left(56\,g_2\,\wp'(x) - 1008\,\wp(x)^2\,\wp'(x)\right)\,f(x).
\end{aligned}
$$

Curve

$$
y^3 - \frac{1}{3}\,{g_2}^2\,z^2\,y - z^7 - 4\,g_3\,z^5 - \frac{2}{27}\,{g_2}^3\,z^3 = 0. \tag{3.3}
$$

Example 7

$$
\begin{aligned}
&q_1 = -12\,\wp, \quad q_0 = 6\,\wp', \quad d_1 = \frac{32}{3}\,g_3,\\
&(L_3 f)(x) = f^{(3)}(x) - 12\,\wp(x)\,f'(x),\\
&(\hat{P}_7 f)(x) = f^{(7)}(x) - 28\,\wp(x)\,f^{(5)}(x) - 56\,\wp'(x)\,f^{(4)}(x) + \left(\frac{112}{3}\,g_2 - 224\,\wp(x)^2\right)\,f^{(3)}(x)\\
&\qquad - 112\,\wp(x)\,\wp'(x)\,f''(x) + 48\,g_3\,f'(x).
\end{aligned}
$$

Curve

$$
y^3 - \frac{16}{3}\,{g_2}^2\,z^2\,y - z^7 - 32\,g_3\,z^5 + \frac{128}{27}\,\left({g_2}^3 - 54\,{g_3}^2\right)\,z^3 = 0. \tag{3.4}
$$

Example 8

$$
\begin{aligned}
&q_1 = -12\,\wp, \quad q_0 = -6\,\wp', \quad d_1 = \frac{32}{3}\,g_3,\\
&(L_3 f)(x) = f^{(3)}(x) - 12\,\wp(x)\,f'(x) - 12\,\wp'(x)\,f(x),\\
&(\hat{P}_7 f)(x) = f^{(7)}(x) - 28\,\wp(x)\,f^{(5)}(x) - 84\,\wp'(x)\,f^{(4)}(x) + \left(\frac{196}{3}\,g_2 - 560\,\wp(x)^2\right)\,f^{(3)}(x)\\
&\qquad - 560\,\wp(x)\,\wp'(x)\,f''(x) + \left(160\,g_3 + 168\,g_2\,\wp(x) - 1120\,\wp(x)^3\right)\,f'(x).
\end{aligned}
$$

Curve

$$y^3 - \frac{16}{3}\, {g_2}^2\, z^2\, y - z^7 - 32\, g_3\, z^5 + \frac{128}{27}\, \left({g_2}^3 - 54\, {g_3}^2\right)\, z^3 = 0. \tag{3.5}$$

Example 9

$$\begin{aligned} q_1 &= -18\,\wp, \quad q_0 = 15\,\wp', \quad d_1 = 76\, g_3, \\ (L_3 f)(x) &= f^{(3)}(x) - 18\,\wp(x)\, f'(x) + 6\,\wp'(x)\, f(x), \\ (\hat{P}_7 f)(x) &= f^{(7)}(x) - 42\,\wp(x)\, f^{(5)}(x) - 70\,\wp'(x)\, f^{(4)}(x) + 42\, g_2\, f^{(3)}(x) \\ &\quad + 280\,\wp(x)\,\wp'(x)\, f''(x) - \Big(120\, g_3 + 308\, g_2\,\wp(x) - 1120\,\wp(x)^3\Big)\, f'(x) \\ &\quad - \Big(56\, g_2\,\wp'(x) - 560\,\wp(x)^2\,\wp'(x)\Big)\, f(x). \end{aligned}$$

Curve

$$\begin{aligned} &y^3 + \left(13\, {g_2}^2\, z^2 + 1600\, {g_2}^2\, g_3\right)\, y - z^7 - 228\, g_3\, z^5 + 2\, \left(17\, {g_2}^3 - 8448\, {g_3}^2\right)\, z^3 \\ &\quad + \left(3200\, {g_2}^3\, g_3 - 409600\, {g_3}^3\right)\, z = 0. \end{aligned} \tag{3.6}$$

Example 10

$$\begin{aligned} q_1 &= -18\,\wp, \quad q_0 = -15\,\wp', \quad d_1 = 76\, g_3, \\ (L_3 f)(x) &= f^{(3)}(x) - 18\,\wp(x)\, f'(x) - 24\,\wp'(x)\, f(x), \\ (\hat{P}_7 f)(x) &= f^{(7)}(x) - 42\,\wp(x)\, f^{(5)}(x) - 140\,\wp'(x)\, f^{(4)}(x) + \Big(112\, g_2 - 840\,\wp(x)^2\Big)\, f^{(3)}(x) \\ &\quad - 280\,\wp(x)\,\wp'(x)\, f''(x) - \Big(400\, g_3 + 728\, g_2\,\wp(x) - 3920\,\wp(x)^3\Big)\, f'(x) \\ &\quad - \Big(336\, g_2\,\wp'(x) - 4480\,\wp(x)^2\,\wp'(x)\Big)\, f(x). \end{aligned}$$

Curve

$$\begin{aligned} &y^3 + \left(13\, {g_2}^2\, z^2 + 1600\, {g_2}^2\, g_3\right)\, y - z^7 - 228\, g_3\, z^5 + 2\, \left(17\, {g_2}^3 - 8448\, {g_3}^2\right)\, z^3 \\ &\quad + \left(3200\, {g_2}^3\, g_3 - 409600\, {g_3}^3\right)\, z = 0. \end{aligned} \tag{3.7}$$

In the following examples the invariant $g_2 = 0$.

Example 11

$$\begin{aligned} q_1 &= -30\,\wp, \quad q_0 = 15\,\wp', \quad g_2 = 0, \quad d_1 = -\frac{220}{3}\, g_3, \\ (L_3 f)(x) &= f^{(3)}(x) - 30\,\wp(x)\, f'(x), \\ (\hat{P}_7 f)(x) &= f^{(7)}(x) - 70\,\wp(x)\, f^{(5)}(x) - 140\,\wp'(x)\, f^{(4)}(x) + 280\,\wp(x)^2\, f^{(3)}(x) \\ &\quad + 1400\,\wp(x)\,\wp'(x)\, f''(x) - \Big(400\, g_3 - 2800\,\wp(x)^3\Big)\, f'(x). \end{aligned}$$

Curve

$$y^3 - z^7 + 220\, g_3\, z^5 - 25600\, {g_3}^2\, z^3 = 0. \tag{3.8}$$

Example 12

$$
\begin{aligned}
&q_1 = -30\,\wp, \quad q_0 = -15\,\wp', \quad g_2 = 0, \quad d_1 = -\frac{220}{3}\, g_3,\\
&(L_3 f)(x) = f^{(3)}(x) - 30\,\wp(x)\, f'(x) - 30\,\wp'(x)\, f(x),\\
&(\hat{P}_7 f)(x) = f^{(7)}(x) - 70\,\wp(x)\, f^{(5)}(x) - 210\,\wp'(x)\, f^{(4)}(x) - 560\,\wp(x)^2\, f^{(3)}(x)\\
&\qquad + 1960\,\wp(x)\,\wp'(x)\, f''(x) - \Big(1800\, g_3 - 16800\,\wp(x)^3\Big)\, f'(x) + 8400\,\wp(x)^2\,\wp'(x)\, f(x).
\end{aligned}
$$

Curve

$$
y^3 - z^7 + 220\, g_3\, z^5 - 25600\, {g_3}^2\, z^3 = 0. \tag{3.9}
$$

References

[1] R. Dickson, F. Gesztesy, and K. Unterkofler, *A new approach to the Boussinesq hierarchy*, Math. Nachr. **198** (1999), 51–108.

[2] R. Dickson, F. Gesztesy, and K. Unterkofler, *Algebro-geometric solutions of the Boussinesq hierarchy*, Rev. Math. Phys. **11** (1999), 823–879.

[3] K. Unterkofler, *On the solutions of Halphen's equation*, preprint 1999, Diff. Integral Eqs. in print.

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