EXAMPLES OF COMMUTING PAIRS OF OPERATORS (L_4, P_5) WITH ELLIPTIC COEFFICIENTS

K. UNTERKOFLER

First a remark on Halphen' s fourth order equation.

Remark 1. *Let*

$$L_2 = \frac{d^2}{dx^2} - n(n+1)\wp(x). \tag{1.1}$$

Then

$$((L_2{}^2 - \frac{1}{12}\, g_2\, n^2(n+1)^2 - z)f)(x) = 0 \tag{1.2}$$

is identical with Halphen' s fourth order equation [2]*, page 272 and* [1] *page 464, Ex. 13. (Note that* $g_2 = 12\wp^2 - 2\wp''$*.)*

$$\begin{aligned}(L_2{}^2 f)(x) &= f^{(4)}(x) - 2\,n(n+1)\,\wp(x)\,f''(x) - 2\,n(n+1)\,\wp'(x)\,f'(x) \\ &\quad + \Big((n-2)n(n+1)(n+3)\wp^2(x) + \frac{g_2}{2}\,(n+1)\Big)\ f(x),\end{aligned} \tag{1.3}$$

i.e., for $n = 1, \dots, 5$,

$$\begin{aligned}(L_2{}^2 f)(x) &= f^{(4)}(x) - 4\,\wp(x)\,f''(x) - 4\,\wp'(x)\,f'(x) - 8\wp^2(x)\,f(x),\\ (L_2{}^2 f)(x) &= f^{(4)}(x) - 12\,\wp(x)\,f''(x) - 12\,\wp'(x)\,f'(x),\\ (L_2{}^2 f)(x) &= f^{(4)}(x) - 24\,\wp(x)\,f''(x) - 24\,\wp'(x)\,f'(x) + 72\wp^2(x)\,f(x),\\ (L_2{}^2 f)(x) &= f^{(4)}(x) - 40\,\wp(x)\,f''(x) - 40\,\wp'(x)\,f'(x) + 280\wp^2(x)\,f(x),\\ (L_2{}^2 f)(x) &= f^{(4)}(x) - 60\,\wp(x)\,f''(x) - 60\,\wp'(x)\,f'(x) + 720\wp^2(x)\,f(x),\end{aligned}$$

where we dropped the constant term.

Hermite ([3] *page ??) enummerates four examples of order four. Note that Forsyth's other example* [1] *Ex. 11 is not included in our list.*

Consider

$$\begin{aligned}(L_4 f)(x) &= f^{(4)}(x) + q_2(x)\,f''(x) + (q_1(x) + q_2{}'(x))\ f'(x) \\ &\quad + \left(q_0(x) + \frac{1}{2}\,q_1{}'(x) + \frac{1}{2}\,q_2{}''(x)\right)\ f(x)\end{aligned} \tag{1.4}$$

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and

$$
\begin{aligned}
(P_5 f)(x) = {}& f^{(5)}(x) + \frac{5}{4}\, q_2(x)\, f^{(3)}(x) + \left(\frac{5}{4}\, q_1(x) + \frac{15}{8}\, {q_2}'(x)\right) f''(x) \\
&+ \frac{1}{4}\left(5\, q_0(x) + \frac{5}{8}\, q_2(x)^2 + 5\, {q_1}'(x) + \frac{25}{4}\, {q_2}''(x)\right) f'(x) \\
&+ \frac{1}{4}\left(\frac{5}{4}\, q_1(x)\, q_2(x) + \frac{5}{2}\, {q_0}'(x) + \frac{5}{8}\, q_2(x)\, {q_2}'(x) + \frac{5}{2}\, {q_1}''(x)\right. \\
&\left. + \frac{15}{8}\, {q_2}^{(3)}(x)\right) f(x).
\end{aligned}
\tag{1.5}
$$

Then the commutator $[L_4, P_5]$ is a second order differential expression. But the resulting stationary equations are a little bit lengthy.

(**Note** that this are only the homogeneous equations!)

$$
\begin{aligned}
0 = {}& \frac{-21\, q_0\, q_0'}{4} + \frac{21\, {q_2}^2\, q_0'}{32} + \frac{21\, q_1\, q_2\, q_1'}{16} + \frac{21\, {q_1}^2\, q_2'}{32} + \frac{21\, q_0\, q_2\, q_2'}{16} - \frac{35\, {q_2}^3\, q_2'}{128} \\
& - \frac{91\, {q_2'}^3}{128} + \frac{7\, q_2'\, q_0''}{8} + \frac{49\, q_1'\, q_1''}{16} - \frac{35\, q_2\, q_2'\, q_2''}{16} + \frac{7\, q_2\, q_0^{(3)}}{16} + \frac{21\, q_1\, q_1^{(3)}}{16} - \frac{7\, q_0\, q_2^{(3)}}{16} \\
& - \frac{49\, {q_2}^2\, q_2^{(3)}}{128} - \frac{175\, q_2''\, q_2^{(3)}}{64} - \frac{91\, q_2'\, q_2^{(4)}}{64} + \frac{7\, q_0^{(5)}}{16} - \frac{21\, q_2\, q_2^{(5)}}{64} - \frac{q_2^{(7)}}{64},
\end{aligned}
\tag{1.6}
$$

$$
\begin{aligned}
0 = {}& \frac{21\, q_1\, q_2\, q_0'}{16} - \frac{21\, {q_0'}^2}{4} + \frac{21\, {q_1}^2\, q_1'}{32} + \frac{21\, q_0\, q_2\, q_1'}{16} - \frac{35\, {q_2}^3\, q_1'}{128} + \frac{21\, q_2\, {q_1'}^2}{16} + \frac{21\, q_0\, q_1\, q_2'}{16} \\
& - \frac{105\, q_1\, {q_2}^2\, q_2'}{128} + \frac{21\, q_2\, q_0'\, q_2'}{8} + \frac{21\, q_1\, q_1'\, q_2'}{8} + \frac{21\, q_0\, {q_2'}^2}{16} - \frac{105\, {q_2}^2\, {q_2'}^2}{128} - \frac{217\, q_1'\, {q_2'}^2}{128} \\
& - \frac{21\, q_0\, q_0''}{4} + \frac{21\, {q_2}^2\, q_0''}{32} + \frac{35\, q_1'\, q_0''}{16} + \frac{21\, q_1\, q_2\, q_1''}{16} + \frac{7\, q_0'\, q_1''}{4} - \frac{35\, q_2\, q_2'\, q_1''}{16} + \frac{49\, {q_1''}^2}{16} \\
& + \frac{21\, {q_1}^2\, q_2''}{32} + \frac{21\, q_0\, q_2\, q_2''}{16} - \frac{35\, {q_2}^3\, q_2''}{128} - \frac{119\, q_2\, q_1'\, q_2''}{64} - \frac{63\, q_1\, q_2'\, q_2''}{32} - \frac{553\, {q_2'}^2\, q_2''}{128} \\
& + \frac{7\, q_0''\, q_2''}{8} - \frac{35\, q_2\, {q_2''}^2}{16} + \frac{21\, q_1\, q_0^{(3)}}{16} + \frac{21\, q_2'\, q_0^{(3)}}{16} + \frac{7\, q_0\, q_1^{(3)}}{8} - \frac{35\, {q_2}^2\, q_1^{(3)}}{64} + \frac{35\, q_1'\, q_1^{(3)}}{8} \\
& - \frac{63\, q_2''\, q_1^{(3)}}{32} - \frac{49\, q_1\, q_2\, q_2^{(3)}}{64} - \frac{7\, q_0'\, q_2^{(3)}}{16} - \frac{189\, q_2\, q_2'\, q_2^{(3)}}{64} - \frac{7\, q_1''\, q_2^{(3)}}{4} - \frac{175\, {q_2^{(3)}}^2}{64} \\
& + \frac{7\, q_2\, q_0^{(4)}}{16} + \frac{21\, q_1\, q_1^{(4)}}{16} - \frac{21\, q_2'\, q_1^{(4)}}{16} - \frac{7\, q_0\, q_2^{(4)}}{16} - \frac{49\, {q_2}^2\, q_2^{(4)}}{128} - \frac{49\, q_1'\, q_2^{(4)}}{64} \\
& - \frac{133\, q_2''\, q_2^{(4)}}{32} - \frac{7\, q_2\, q_1^{(5)}}{16} - \frac{7\, q_1\, q_2^{(5)}}{64} - \frac{7\, q_2'\, q_2^{(5)}}{4} + \frac{7\, q_0^{(6)}}{16} - \frac{21\, q_2\, q_2^{(6)}}{64} - \frac{q_1^{(7)}}{8} - \frac{q_2^{(8)}}{64},
\end{aligned}
\tag{1.7}
$$

$$
\begin{aligned}
0 = {} & \frac{21\,{q_1}^2\,q_0'}{32} - \frac{21\,q_0\,q_2\,q_0'}{16} + \frac{7\,{q_2}^3\,q_0'}{128} + \frac{21\,q_0\,q_1\,q_1'}{16} - \frac{21\,q_1\,{q_2}^2\,q_1'}{128} + \frac{21\,q_2\,q_0'\,q_1'}{16} \\
& + \frac{21\,q_1\,{q_1'}^2}{32} - \frac{21\,{q_1}^2\,q_2\,q_2'}{64} + \frac{21\,q_1\,q_0'\,q_2'}{16} + \frac{21\,q_0\,q_1'\,q_2'}{16} - \frac{105\,{q_2}^2\,q_1'\,q_2'}{128} + \frac{21\,{q_1'}^2\,q_2'}{32} \\
& - \frac{105\,q_1\,q_2\,{q_2'}^2}{128} + \frac{119\,q_0'\,{q_2'}^2}{128} - \frac{21\,q_2\,{q_2'}^3}{64} + \frac{21\,q_1\,q_2\,q_0''}{32} - \frac{91\,q_0'\,q_0''}{16} + \frac{21\,q_2\,q_2'\,q_0''}{32} \\
& + \frac{21\,{q_1}^2\,q_1''}{64} + \frac{21\,q_0\,q_2\,q_1''}{32} - \frac{35\,{q_2}^3\,q_1''}{256} + \frac{21\,q_2\,q_1'\,q_1''}{16} + \frac{7\,q_1\,q_2'\,q_1''}{32} - \frac{497\,{q_2'}^2\,q_1''}{256} \\
& + \frac{63\,q_0''\,q_1''}{32} + \frac{21\,q_0\,q_1\,q_2''}{32} - \frac{105\,q_1\,{q_2}^2\,q_2''}{256} + \frac{49\,q_2\,q_0'\,q_2''}{64} + \frac{21\,q_1\,q_1'\,q_2''}{64} + \frac{21\,q_0\,q_2'\,q_2''}{32} \\
& - \frac{147\,{q_2}^2\,q_2'\,q_2''}{256} - \frac{231\,q_1'\,q_2'\,q_2''}{64} - \frac{259\,q_2\,q_1''\,q_2''}{128} - \frac{63\,q_1\,{q_2''}^2}{64} - \frac{315\,q_2'\,{q_2''}^2}{128} - \frac{7\,q_0\,q_0^{(3)}}{4} \\
& + \frac{7\,q_1'\,q_0^{(3)}}{4} + \frac{7\,q_1\,q_2\,q_1^{(3)}}{32} + \frac{21\,q_0'\,q_1^{(3)}}{16} - \frac{105\,q_2\,q_2'\,q_1^{(3)}}{64} + \frac{49\,q_1''\,q_1^{(3)}}{16} - \frac{7\,{q_2}^3\,q_2^{(3)}}{128} \\
& - \frac{21\,q_2\,q_1'\,q_2^{(3)}}{16} - \frac{175\,q_1\,q_2'\,q_2^{(3)}}{128} - \frac{399\,{q_2'}^2\,q_2^{(3)}}{256} - \frac{7\,q_0''\,q_2^{(3)}}{16} - \frac{63\,q_2\,q_2''\,q_2^{(3)}}{32} \\
& - \frac{119\,q_1^{(3)}\,q_2^{(3)}}{64} + \frac{21\,q_1\,q_0^{(4)}}{32} + \frac{7\,q_0\,q_1^{(4)}}{16} - \frac{35\,{q_2}^2\,q_1^{(4)}}{128} + \frac{49\,q_1'\,q_1^{(4)}}{32} - \frac{105\,q_2''\,q_1^{(4)}}{64} \\
& - \frac{49\,q_1\,q_2\,q_2^{(4)}}{128} - \frac{35\,q_0'\,q_2^{(4)}}{64} - \frac{63\,q_2\,q_2'\,q_2^{(4)}}{64} - \frac{161\,q_1''\,q_2^{(4)}}{128} - \frac{21\,q_2^{(3)}\,q_2^{(4)}}{8} + \frac{7\,q_1\,q_1^{(5)}}{32} \\
& - \frac{7\,q_2'\,q_1^{(5)}}{8} - \frac{7\,q_0\,q_2^{(5)}}{32} - \frac{35\,{q_2}^2\,q_2^{(5)}}{256} - \frac{7\,q_1'\,q_2^{(5)}}{16} - \frac{105\,q_2''\,q_2^{(5)}}{64} - \frac{7\,q_2\,q_1^{(6)}}{32} - \frac{7\,q_1\,q_2^{(6)}}{128} \\
& - \frac{77\,q_2'\,q_2^{(6)}}{128} + \frac{3\,q_0^{(7)}}{32} - \frac{7\,q_2\,q_2^{(7)}}{64} - \frac{q_1^{(8)}}{16} - \frac{q_2^{(9)}}{128}.
\end{aligned} \tag{1.8}
$$

2. Commuting pairs of operators (L_4, P_5) with rational coefficients

First we look for rational solutions of the stationary equations by the Ansatz

$$
q_2(x) = \frac{a}{x^2}, \quad q_1(x) = \frac{b}{x^3}, \quad q_0(x) = \frac{c}{x^4}, \qquad a, b, c \in \mathbb{C}. \tag{2.1}
$$

This yields

$$
\begin{aligned}
(L_4 f)(x) = {} & \left(\frac{3\,a}{x^4} - \frac{3\,b}{2\,x^4} + \frac{c}{x^4} \right) f(x) + \left(-\frac{2\,a}{x^3} + \frac{b}{x^3} \right) f'(x) \\
& + \frac{a}{x^2}\, f''(x) + f^{(4)}(x)
\end{aligned} \tag{2.2}
$$

and

$$
\begin{aligned}
(P_5 f)(x) = {} & -\left(\frac{45\,a}{x^5} + \frac{5\,a^2}{4\,x^5} - \frac{30\,b}{x^5} - \frac{5\,a\,b}{4\,x^5} + \frac{10\,c}{x^5} \right) \frac{1}{4} f(x) \\
& + \left(\frac{75\,a}{2\,x^4} + \frac{5\,a^2}{8\,x^4} - \frac{15\,b}{x^4} + \frac{5\,c}{x^4} \right) \frac{1}{4}\, f'(x) \\
& + \left(-\frac{15\,a}{4\,x^3} + \frac{5\,b}{4\,x^3} \right) f''(x) + \frac{5\,a}{4\,x^2}\, f^{(3)(x)} + f^{(5)}(x).
\end{aligned} \tag{2.3}
$$

Setting $[L_4, P_5] = 0$ yields

$$0 = 1260\,a - \frac{375\,a^2}{2} - \frac{85\,a^3}{8} + 2520\,b + \frac{735\,a\,b}{2} + \frac{105\,a^2\,b}{8} + \frac{45\,b^2}{2} - \frac{25\,a\,b^2}{16}$$
$$+ 2520\,c + \frac{355\,a\,c}{2} + \frac{5\,a^2\,c}{8} - 35\,b\,c + 5\,c^2,$$
$$0 = -315\,a + \frac{315\,a^2}{2} + \frac{105\,a^3}{16} - 630\,b - \frac{735\,a\,b}{8} - \frac{105\,a^2\,b}{32} - \frac{105\,b^2}{4}$$
$$- 1050\,c - \frac{105\,a\,c}{2} + \frac{35\,b\,c}{4},$$
$$0 = 45\,a - \frac{45\,a^2}{2} - \frac{15\,a^3}{16} + \frac{15\,b^2}{4} + 150\,c + \frac{15\,a\,c}{2}. \tag{2.4}$$

Mathematica yields for $\{a, b, c\}$ the following set of solutions

$$\begin{aligned}
&\{\{c \to 0, b \to -24, a \to -12\},\\
&\{c \to 0, b \to 0, a \to 0\},\\
&\{c \to 0, b \to 24, a \to -12\},\\
&\{c \to 4, b \to 0, a \to -4\},\\
&\{c \to 12, b \to -8, a \to -8\},\\
&\{c \to 12, b \to 8, a \to -8\},\\
&\{c \to 28, b \to 0, a \to -28\},\\
&\{c \to 36, b \to -24, a \to -24\},\\
&\{c \to 36, b \to 0, a \to -12\},\\
&\{c \to 36, b \to 24, a \to -24\},\\
&\{c \to 88, b \to 0, a \to -16\},\\
&\{c \to 180, b \to -48, a \to -36\},\\
&\{c \to 180, b \to 48, a \to -36\},\\
&\{c \to 396, b \to 0, a \to -60\}\}.
\end{aligned} \tag{2.5}$$

This list seems to be not complete, e.g., it does not contain all examples of Remark 1). (I will check this with Maple. checked: Maple yields the same set of solutions). Hence we conclude that this Ansatz does not yield all solutions.

3. Commuting pairs of operators (L_4, P_5) with elliptic coefficients

These rational examples give rise to the following commuting pairs (L_4, P_5) of differential expressions with elliptic coefficients, i.e., $[L_4, P_5] = 0$.

(**Note** that some of these differential expressions are not homogeneous!)

Example 1

Let

$$q_2 = -12\,\wp, \quad q_1 = 12\,\wp, \quad q_0 = 0,$$
$$(L_4 f)(x) = f^{(4)}(x) - 12\,\wp(x)\,f''(x),$$
$$(P_5 f)(x) = f^{(5)}(x) - 15\,\wp(x)\,f'''(x) - \frac{15}{2}\,\wp'(x)\,f''(x) + 3\,g_2\,f'(x). \tag{3.1}$$

Curve

$$y^4 - z^5 - \frac{9\,g_2}{2}\,z^4 - \frac{81\,{g_2}^2}{16}\,z^3 - \frac{27\,g_3}{2} z\; y^2 - \left(\frac{27\,{g_2}^3}{16} - \frac{729\,{g_3}^2}{16}\right)\,z^2 = 0. \tag{3.2}$$

Example 2

$$\begin{aligned} &q_2 = -12\,\wp, \quad q_1 = -12\,\wp, \quad q_0 = 0,\\ &(L_4 f)(x) = f^{(4)}(x) - 12\,\wp(x)\,f''(x) - 24\,\wp'(x)\,f'(x) - 72\wp(x)^2\,f(x),\\ &(P_5 f)(x) = f^{(5)}(x) - 15\,\wp(x)\,f^{(3)}(x) - \frac{75}{2}\,\wp'(x)\,f''(x)\\ &\qquad + \left(18\,g_2 - 180\,\wp(x)^2\right)\,f'(x) - 90\,\wp(x)\,\wp'(x)\,f(x). \end{aligned} \tag{3.3}$$

Curve

$$\begin{aligned} &y^4 - z^5 - \frac{69}{2}\,g_2\,z^4 - \frac{7569}{16}\,{g_2}^2\,z^3 - \frac{59049}{4}\,{g_2}^5 - \left(\frac{27}{2}\,g_3\,z + 81\,g_2\,g_3\right)\,y^2\\ &\quad + \frac{6561}{4}\,{g_2}^2\,{g_3}^2 - \frac{81}{16}\,\left(637\,{g_2}^3 - 9\,{g_3}^2\right)\,z^2 - \left(10935\,{g_2}^4 - \frac{2187}{4}\,g_2\,{g_3}^2\right)\,z = 0. \end{aligned} \tag{3.4}$$

Example 3

$$\begin{aligned} &q_2 = -4\,\wp, \quad q_1 = 0, \quad q_0 = 4\wp^2,\\ &(L_4 f)(x) = f^{(4)}(x) - 4\,\wp(x)\,f''(x) - 4\,f'(x)\,\wp'(x) - 8\,\wp(x)^2 f(x),\\ &(P_5 f)(x) = f^{(5)}(x) - 5\,\wp(x)\,f^{(3)}(x) - \frac{15}{2}\,\wp'(x)\,f''(x) + \left(3\,g_2 - 30\,\wp(x)^2\right)\,f'(x)\\ &\qquad - 15\,\wp(x)\,\wp'(x)\,f(x). \end{aligned} \tag{3.5}$$

Curve

$$\begin{aligned} &y^4 - z^5 - \frac{9}{2}\,g_2\,z^4 - \frac{129}{16}\,{g_2}^2\,z^3 - \frac{9}{16}\,{g_2}^5 + \left(\frac{g_3}{2}\,z + \frac{g_2\,g_3}{2}\right)\,y^2 + \frac{{g_2}^2\,{g_3}^2}{16}\\ &\quad - \frac{115\,{g_2}^3 - {g_3}^2}{16}\,z^2 - \frac{51\,{g_2}^4 - 2\,g_2\,{g_3}^2}{16}\,z = 0. \end{aligned} \tag{3.6}$$

Example 4

$$\begin{aligned} &q_2 = -8\,\wp, \quad q_1 = -4\,\wp', \quad q_0 = 12\wp^2,\\ &(L_4 f)(x) = f^{(4)}(x) - 8\,\wp(x)\,f''(x) - 12\,\wp'(x)\,f'(x) - 24\,\wp(x)^2 f(x),\\ &(P_5 f)(x) = f^{(5)}(x) - 10\,\wp(x)\,f^{(3)}(x) - 20\,\wp'(x)\,f''(x) + \left(8\,g_2 - 80\,\wp(x)^2\right)\,f'(x)\\ &\qquad - 40\,\wp(x)\,\wp(x)'\,f(x). \end{aligned} \tag{3.7}$$

Curve

$$
\begin{aligned}
& y^4 - z^5 - 12\, g_2\, z^4 - 57\, {g_2}^2\, z^3 - 134\, {g_2}^3\, z^2 - 156\, {g_2}^4\, z - 72\, {g_2}^5 \\
& \quad - (4\, g_3\, z + 8\, g_2\, g_3)\, y^2 = 0.
\end{aligned} \tag{3.8}
$$

Example 5

$$
\begin{aligned}
& q_2 = -8\,\wp, \quad q_1 = 4\,\wp', \quad q_0 = 12\wp^2, \\
& (L_4 f)(x) = f^{(4)}(x) - 8\,\wp(x)\, f''(x) - 4\,\wp'(x)\, f'(x), \\
& (P_5 f)(x) = f^{(5)}(x) - 10\,\wp(x)\, f^{(3)}(x) - 10\,\wp'(x)\, f''(x) + \Big(3\, g_2 - 20\,\wp(x)^2\Big)\, f'(x).
\end{aligned} \tag{3.9}
$$

Curve

$$
y^4 - z^5 - 2\, g_2\, z^4 - {g_2}^2\, z^3 - 4\, g_3\, z\, y^2 = 0. \tag{3.10}
$$

Example 6a

$$
\begin{aligned}
q_2 &= -28\,\wp, \quad q_1 = -12\,\sqrt{2\, g_3}, \quad q_0 = 28\wp^2, \\
(L_4 f)(x) &= f^{(4)}(x) - 28\,\wp(x)\, f''(x) - \Big(12\,\sqrt{2\, g_3} + 28\,\wp'(x)\Big)\, f'(x) \\
&\quad - 56\,\wp(x)^2\, f(x), \\
(P_5 f)(x) &= f^{(5)}(x) - 35\,\wp(x)\, f^{(3)}(x) - \left(5\,\sqrt{2\, g_3} + \frac{105}{2}\,\wp'(x)\right)\, f''(x) \\
&\quad + \left(\frac{147}{4}\, g_2 - 105\,\wp(x)^2\right)\, f'(x) - 35\,\sqrt{2\, g_3}\,\wp(x)\, f(x).
\end{aligned} \tag{3.11}
$$

Curve

$$
\begin{aligned}
& y^4 - z^5 - \frac{189}{2}\, g_2\, z^4 - \frac{54705}{16}\, {g_2}^2\, z^3 - \frac{3651921}{4}\, {g_2}^5 \\
& \quad - \left(40\, z^3 + 2156\, g_2\, z^2 + \frac{71981}{2}\, {g_2}^2\, z + 173901\, {g_2}^3\right)\, \sqrt{2\, g_3}\, y \\
& \quad + 5250987\, {g_2}^2\, {g_3}^2 - \left(\frac{763\, z\, g_3}{2} + 4214\, g_2\, g_3\right)\, y^2 \\
& \quad - \left(\frac{923209\, {g_2}^3}{16} - \frac{607257\, {g_3}^2}{16}\right)\, z^2 - \left(428064\, {g_2}^4 - \frac{1750329\, g_2\, {g_3}^2}{2}\right)\, z = 0.
\end{aligned} \tag{3.12}
$$

Example 6b

$$\begin{aligned}
&q_2 = -28\,\wp, \quad q_1 = 12\sqrt{2\,g_3}, \quad q_0 = 28\wp^2,\\
&(L_4 f)(x) = f^{(4)}(x) - 28\,\wp(x)\,f''(x) + \left(12\sqrt{2\,g_3} - 28\,\wp'(x)\right) f'(x)\\
&\qquad - 56\,\wp(x)^2\,f(x),\\
&(P_5 f)(x) = f^{(5)}(x) - 35\,\wp(x)\,f^{(3)}(x) + \left(5\sqrt{2\,g_3} - \frac{105}{2}\,\wp'(x)\right) f''(x)\\
&\qquad + \left(\frac{147}{4}\,g_2 - 105\,\wp(x)^2\right) f'(x) + 35\sqrt{2\,g_3}\,\wp(x)\,f(x).
\end{aligned} \tag{3.13}$$

Curve

$$\begin{aligned}
&y^4 - z^5 - \frac{189}{2}\,g_2\,z^4 - \frac{54705}{16}\,{g_2}^2\,z^3 - \frac{3651921}{4}\,{g_2}^5\\
&+ \left(40\,z^3 + 2156\,g_2\,z^2 + \frac{71981}{2}\,{g_2}^2\,z + 173901\,{g_2}^3\right)\sqrt{2\,g_3}\,y\\
&+ 5250987\,{g_2}^2\,{g_3}^2 - \left(\frac{763\,z\,g_3}{2} + 4214\,g_2\,g_3\right) y^2\\
&- \left(\frac{923209\,{g_2}^3}{16} - \frac{607257\,{g_3}^2}{16}\right) z^2 - \left(428064\,{g_2}^4 - \frac{1750329\,g_2\,{g_3}^2}{2}\right) z = 0.
\end{aligned} \tag{3.14}$$

Example 7

$$\begin{aligned}
&q_2 = -24\,\wp, \quad q_1 = 12\wp', \quad q_0 = 36\wp^2,\\
&(L_4 f)(x) = f^{(4)}(x) - 24\,\wp(x)\,f''(x) - 12\,\wp'(x)\,f'(x),\\
&(P_5 f)(x) = f^{(5)}(x) - 30\,\wp(x)\,f^{(3)}(x) - 30\,\wp'(x)\,f''(x) + 18\,g_2\,f'(x).
\end{aligned} \tag{3.15}$$

Curve

$$\begin{aligned}
&y^4 - z^5 - 42\,g_2\,z^4 - 657\,{g_2}^2\,z^3 - 4536\,{g_2}^3\,z^2 - 11664\,{g_2}^4\,z\\
&+ (108\,g_3\,z + 1296\,g_2\,g_3)\;y^2 = 0.
\end{aligned} \tag{3.16}$$

Example 8

$$\begin{aligned}
&q_2 = -12\,\wp, \quad q_1 = 0, \quad q_0 = 36\wp^2,\\
&(L_4 f)(x) = f^{(4)}(x) - 12\,\wp(x)\,f''(x) - 12\,\wp'(x)\,f'(x),\\
&(P_5 f)(x) = f^{(5)}(x) - 15\,\wp(x)\,f^{(3)}(x) - \frac{45}{2}\,\wp'(x)\,f''(x) \left(\frac{27}{4}\,g_2 - 45\,\wp(x)^2\right) f'(x).
\end{aligned} \tag{3.17}$$

Curve

$$y^4 - z^5 - \frac{9}{2}\,g_2\,z^4 - \frac{81}{16}\,{g_2}^2\,z^3 - \frac{27}{2}\,g_3\,z\,y^2 - \frac{27}{16}\left({g_2}^3 - 27\,{g_3}^2\right) z^2 = 0. \tag{3.18}$$

Example 9

$$
\begin{aligned}
&q_2 = -24\,\wp, \quad q_1 = -12\wp', \quad q_0 = 36\wp^2,\\
&(L_4 f)(x) = f^{(4)}(x) - 24\,\wp(x)\,f''(x) - 36\,\wp'(x)\,f'(x) - 72\,\wp(x)^2 f(x),,\\
&(P_5 f)(x) = f^{(5)}(x) - 30\,\wp(x)\,f^{(3)}(x) - 60\,\wp'(x)\,f''(x) + \Big(33\,g_2 - 180\,\wp(x)^2\Big)\; f'(x).
\end{aligned} \tag{3.19}
$$

Curve

$$
\begin{aligned}
&y^4 - z^5 - 72\,g_2\,z^4 - 2025\,{g_2}^2\,z^3 - 27594\,{g_2}^3\,z^2 - 179820\,{g_2}^4\,z - 437400\,{g_2}^5\\
&+ (108\,g_3\,z + 1944\,g_2\,g_3)\;y^2 = 0.
\end{aligned} \tag{3.20}
$$

Example 10

$$
\begin{aligned}
q_2 &= -16\,\wp, \quad q_1 = 12\,i\,\sqrt{g_3}, \quad q_0 = 88\wp^2,\\
(L_4 f)(x) &= f^{(4)}(x) - 16\,\wp(x)\,f''(x) + (12\,i\,\sqrt{g_3} - 16\,\wp'(x))\,f'(x) + 40\,\wp(x)^2\,f(x),\\
(P_5 f)(x) &= f^{(5)}(x) - 20\,\wp(x)\,f^{(3)}(x) + (10\,i\,\sqrt{g_3} - 30\,\wp'(x))\;f''(x) + 3\,g_2\,f'(x)\\
&\quad - (20\,i\,\sqrt{g_3}\,\wp(x) - 60\,\wp(x)\,\wp'(x))\;f(x).
\end{aligned} \tag{3.21}
$$

P_5 is inhomogeneous, i.e.,

$$
P_{5,inh} = P_{5,hom} - 5\,i\,\sqrt{g_3}\,P_{2,hom} - \frac{19}{2}\,P_{2,hom}. \tag{3.22}
$$

Curve

$$
\begin{aligned}
&y^4 - z^5 + 18\,g_2\,z^4 - 102\,{g_2}^2\,z^3 + 80\,{g_2}^3\,z^2 + 975\,{g_2}^4\,z - 2250\,{g_2}^5\\
&+ \big(20\;z^3 - 284\,z^2\,g_2\; + 1340\,z\,{g_2}^2 - 2100\,{g_2}^3\big)\;i\,\sqrt{g_3}\,y\\
&- (8\,g_3\,z - 40\,g_2\,g_3)\;y^2 = 0.
\end{aligned} \tag{3.23}
$$

In the following examples g_2 must be zero!!!

Example 11

$$
\begin{aligned}
q_2 &= -36\,\wp, \quad q_1 = -24\,\wp, \quad q_0 = 180\wp^2,\\
(L_4 f)(x) &= f^{(4)}(x) - 36\,\wp(x)\,f''(x) - 60\,f'(x)\,\wp'(x),\\
(P_5 f)(x) &= f^{(5)}(x) - 45\,\wp(x)\,f^{(3)}(x) - \frac{195}{2}\,\wp'(x)\,f''(x) - 90\,\wp(x)^2\,f'(x)\\
&\quad + 315\,\wp(x)\,\wp'(x)\,f(x).
\end{aligned} \tag{3.24}
$$

Curve

$$
y^4 - z^5 - \frac{351}{2}\,g_3\,z\,y^2 - \frac{250047}{16}\,{g_3}^2\,z^2 = 0. \tag{3.25}
$$

Example 12

$$
\begin{aligned}
q_2 &= -36\,\wp, \quad q_1 = 24\,\wp, \quad q_0 = 180\wp^2, \\
(L_4 f)(x) &= f^{(4)}(x) - 36\,\wp(x)\,f''(x) - 12\,\wp'(x)\,f'(x) + 144\,\wp(x)^2\,f(x), \\
(P_5 f)(x) &= f^{(5)}(x) - 45\,\wp(x)\,f^{(3)}(x) - \frac{75}{2}\,\wp'(x)\,f''(x) + 270\,\wp(x)^2\,f'(x) \\
&\quad + 135\,\wp(x)\,\wp'(x)\,f(x).
\end{aligned}
\tag{3.26}
$$

Curve

$$
y^4 - z^5 - \frac{351}{2}\,g_3\,z\,y^2 - \frac{250047}{16}\,{g_3}^2\,z^2 = 0.
\tag{3.27}
$$

Note that in Example 11 and Example 12 different differential expressions yield the same curve!

Example 13a

$$
\begin{aligned}
q_2 &= -60\,\wp, \quad q_1 = \frac{84}{5}\sqrt{6\,g_3}, \quad q_0 = 396\wp^2, \\
(L_4 f)(x) &= f^{(4)}(x) - 60\,\wp(x)\,f''(x) + \left(\frac{84}{5}\,\sqrt{6\,g_3} - 60\,\wp'(x)\right)\,f'(x) \\
&\quad + 216\,\wp(x)^2\,f(x), \\
(P_5 f)(x) &= f^{(5)}(x) - 75\,\wp(x)\,f^{(3)}(x) + \left(3\,\sqrt{6\,g_3} - \frac{225}{2}\,\wp'(x)\right)\,f''(x) \\
&\quad + 495\,\wp(x)^2\,f'(x) + \left(225\,\sqrt{6\,g_3}\,\wp(x) + 720\,\wp(x)\,\wp'(x)\right)\,f(x).
\end{aligned}
\tag{3.28}
$$

Curve

$$
\begin{aligned}
&y^4 - z^5 - \frac{57159}{10}\,g_3\,z\,y^2 + \left(72\,\sqrt{6\,g_3}\,z^3 + \frac{431532541 44}{3125}\,\sqrt{6}\,{g_3}^{\frac{5}{2}}\right)\,y \\
&\quad + \frac{1486457973}{2000}\,{g_3}^2\,z^2 = 0.
\end{aligned}
\tag{3.29}
$$

Example 13b

$$
\begin{aligned}
q_2 &= -60\,\wp, \quad q_1 = -\frac{84}{5}\sqrt{6\,g_3}, \quad q_0 = 396\wp^2, \\
(L_4 f)(x) &= f^{(4)}(x) - 60\,\wp(x)\,f''(x) - \left(\frac{84}{5}\,\sqrt{6\,g_3} + 60\,\wp'(x)\right)\,f'(x) \\
&\quad + 216\,\wp(x)^2\,f(x), \\
(P_5 f)(x) &= f^{(5)}(x) - 75\,\wp(x)\,f^{(3)}(x) - \left(3\,\sqrt{6\,g_3} + \frac{225}{2}\,\wp'(x)\right)\,f''(x) \\
&\quad + 495\,\wp(x)^2\,f'(x) - \left(225\,\sqrt{6}\,\sqrt{g_3}\,\wp(x) - 720\,\wp(x)\,\wp'(x)\right)\,f(x).
\end{aligned}
\tag{3.30}
$$

Curve

$$y^4 - z^5 - \frac{57159}{10}\, g_3\, z\, y^2 - \left(72\,\sqrt{6\, g_3}\, z^3 + \frac{43153254144}{3125}\,\sqrt{6}\, g_3{}^{\frac{5}{2}}\right) y$$

$$+ \frac{1486457973}{2000}\, g_3{}^2\, z^2 = 0. \tag{3.31}$$

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Institute for Theoretical Physics, Technical University of Graz, A–8010 Graz, Austria

E-mail address: `karl@itp.tu-graz.ac.at`