ON A THEOREM OF HALPHEN AND ITS APPLICATION TO INTEGRABLE SYSTEMS

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ABSTRACT. We extend Halphen's theorem which characterizes the solutions of certain nth-order differential equations with rational coefficients and meromorphic fundamental systems to a first-order $n \times n$ system of differential equations. As an application of this circle of ideas we consider stationary rational algebro-geometric solutions of the KdV hierarchy and illustrate some of the connections with completely integrable models of the Calogero-Moser-type. In particular, our treatment recovers the complete characterization of the isospectral class of such rational KdV solutions in terms of a precise description of the Airault-McKean-Moser locus of their poles.

1. INTRODUCTION

The purpose of this paper is twofold. First we prove an extension of Halphen's theorem, which characterizes the fundamental system of solutions of certain nth-order ordinary differential equations with rational coefficients to first-order $n \times n$ systems. In the second part of this paper we show how to apply Halphen's theorem to completely integrable systems of the Calogero-Moser-type, recovering a complete characterization of the isospectral class of all algebro-geometric rational solutions of the KdV hierarchy.

We start by describing Halphen's original result. Consider the following nth-order differential equation

$$q_n(z)y^{(n)}(z) + q_{n-1}(z)y^{(n-1)}(z) + \cdots + q_0(z)y(z) = 0, \tag{1.1}$$

where $q_j(z)$ are polynomials, and the order of $q_n(z)$ is at least the order of $q_j(z)$ for all $0 \leq j \leq (n-1)$, that is,

$$q_m(z) \text{ are polynomials, } 0 \leq m \leq n, \tag{1.2a}$$

$$q_m(z)/q_n(z) \text{ are bounded near } \infty \text{ for all } 0 \leq m \leq n-1. \tag{1.2b}$$

Then the zeros of $q_n(z)$ are the possible singularities of solutions of (1.1).

Assuming the fundamental system of solutions of (1.1) to be meromorphic, the following theorem due to Halphen holds.

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Theorem 1.1. (*Halphen* [21], *Ince* [22, p. 372–375]) *Assume* (1.2) *and suppose* (1.1) *has a meromorphic fundamental system of solutions. Then the general solution of* (1.1) *is of the form*

$$y(z) = \sum_{m=1}^{n} c_m r_m(z) e^{\lambda_m z}, \tag{1.3}$$

where $r_m(z)$ are rational functions of z, $\lambda_m \in \mathbb{C}$, $1 \leq m \leq n$, and c_m, $1 \leq m \leq n$ are arbitrary complex constants.

Moreover, the converse of Halphen's theorem holds as well.

Theorem 1.2. (*Ince* [22, p. 374–375]) *Suppose $r_m(z)$ are rational functions of z and $\lambda_m, c_m \in \mathbb{C}$, $1 \leq m \leq n$. If $r_1(z)e^{\lambda_1 z}, \dots, r_n(z)e^{\lambda_n z}$ are linearly independent, then*

$$y(z) = \sum_{m=1}^{n} c_m r_m(z) e^{\lambda_m z} \tag{1.4}$$

is the general solution of an nth-order equation of the type (1.1)*, whose coefficients satisfy* (1.2).

Remark 1.3. We note that Halphen's main idea of proof in [21] consists of replacing the rational coefficients in (1.1) by appropriate elliptic coefficients (as discussed in [20]) followed by an application of Picard's theorem (cf., e.g., [22, p. 375–378]). A closer examination of his argument seems to reveal a lack of proof of the crucial fact that the associated differential equation with elliptic coefficients necessarily has a meromorphic fundamental system of solutions. A proof of Theorem 1.1 (and Theorem 1.2), using a completely different strategy, is provided in Ince's monograph [22, p. 372–375].

One of the principal aims of this note is to prove a first-order $n \times n$ system generalization of Halphen's Theorem 1.1 and its converse, Theorem 1.2, in Section 2.

Analogous results hold for nth-order equations and first-order systems with periodic and elliptic coefficients. For a glimpse at the vast literature in these cases and their applications to completely integrable systems we refer the interested reader to [15]–[18], [38], [39] and the literature therein.

In Section 3 we then apply Halphen's theorem to the problem of characterizing the isospectral class of all stationary rational KdV solutions. All such (nonconstant) solutions q are well-known to be necessarily of the form

$$q(z) = q_\infty - \sum_{\ell=1}^{M} s_\ell(s_\ell + 1)(z - \zeta_\ell)^{-2} \tag{1.5}$$

for some $q_\infty \in \mathbb{C}$, $\{\zeta_\ell\}_{1 \leq \ell \leq M} \subset \mathbb{C}$, $\zeta'_\ell \neq \zeta_\ell$ for $\ell' \neq \ell$, and

$$s_\ell \in \mathbb{N}, \ 1 \leq \ell \leq M \text{ with } \sum_{\ell=1}^{M} s_\ell(s_\ell + 1) = g(g+1) \tag{1.6}$$

for some $g \in \mathbb{N}$, and the underlying spectral curve is then of the especially simple rational type

$$y^2 = (E - q_\infty)^{2g+1}. \tag{1.7}$$

On the other hand, not every q of the type (1.5), (1.6) is an algebro-geometric solution of the KdV hierarchy. In general, the points ζ_ℓ must satisfy a set of intricate constraints. In fact, necessary and sufficient conditions on ζ_ℓ for q in (1.5) to be a rational KdV solution are given by

$$\sum_{\substack{\ell'=1\\ \ell'\neq\ell}}^{M} \frac{s_{\ell'}(s_{\ell'}+1)}{(\zeta_\ell-\zeta_{\ell'})^{2k+1}} = 0 \quad \text{for } k=1,...,s_{\ell'} \text{ and } \ell=1,\dots,M. \tag{1.8}$$

This result was first derived by Duistermaat and Grünbaum [10] (cf. p. 199) in 1986, as a by-product of their investigations of bispectral pairs of differential operators. We will provide an elementary derivation of this result on the basis of Halphen's theorem and an explicit Frobenius-type analysis in Section 3.

For a fixed $g \in \mathbb{N}$, (1.6) and (1.8) yield a complete parametrization of all rational KdV solutions belonging to the spectral curve (1.7). In other words, they provide a complete characterization of the isospectral class of KdV solutions corresponding to (1.7). The constraints (1.8) represent the proper generalization of the locus of poles introduced by Airault, McKean, and Moser [5] in the sense that they explicitly describe the situation where poles are permitted to collide (i.e., where some of the $s_\ell > 1$).

2. Halphen's theorem for first-order systems

This section is devoted to a generalization of Halphen's theorem (and its converse) to first-order systems.

We briefly describe some of the notation used in this section. I_n denotes the identity in $\mathbb{C}^n$. An $m \times m$ diagonal matrix $D = (d_j\delta_{j,k})_{1\le j,k\le m}$ will occasionally be denoted by $\operatorname{diag}(d_1,\dots,d_m)$. The operation of transposition is denoted by the superscript t. Moreover, it will be convenient to denote the set of all $m \times n$ matrices whose entries are rational functions with respect to $z \in \mathbb{C}$ by $\mathcal{R}^{m\times n}$, the subset of $\mathcal{R}^{m\times n}$ with rational entries bounded at infinity by $\mathcal{R}^{m\times n}_\infty$.

We recall that for $T \in \mathcal{R}^{n\times n}$ invertible and differentiable with respect to z, the transformation $y(z) = T(z)u(z)$ turns the first-order system of differential equations $y'(z) = A(z)y(z)$ into the system $u'(z) = B(z)u(z)$, where $B(z) = T(z)^{-1}(A(z)T(z)-T'(z))$.

Definition 2.1. *(i) Two matrices $A, B \in \mathcal{R}^{n\times n}$ are called of the same kind if there exists an invertible matrix $T \in \mathcal{R}^{n\times n}$ such that*

$$B(z) = T(z)^{-1}(A(z)T(z) - T'(z)). \tag{2.1}$$

(ii) $B \in \mathcal{R}^{n\times n}$ is called reduced of order k if $B_{j,\ell} = \delta_{j+1,\ell}$ for all $1 \le j \le k$ and $1 \le \ell \le n$.

Our approach, including the notion of matrices being "of the same kind", was inspired by Loewy [29]. The relation of being of the same kind is obviously an equivalence relation on $\mathcal{R}^{n\times n}$. The relation of being of the same kind is obviously an equivalence relation on $\mathcal{R}^{n\times n}$.

Lemma 2.2. *Suppose that* $A \in \mathcal{R}_\infty^{n\times n}$ *is reduced of order* $k-1$. *Then either* $A_{k,k+1} = \dots = A_{k,n} = 0$, *or else there exists a matrix* $B \in \mathcal{R}_\infty^{n\times n}$ *of the same kind as* A *and also reduced of order* $k-1$ *but with the additional property that* $B_{k,k+1}(\infty) \neq 0$. *Moreover,* $A(\infty)$ *and* $B(\infty)$ *have the same eigenvalues counting algebraic multiplicities.*

Proof. We assume not all of the entries $A_{k,k+1}, \dots , A_{k,n}$ are equal to zero. Consider the $(n-k+1)\times(n-k)$ matrix in the lower right corner of A and denote it by R. Suppose that r is the largest nonnegative integer such that $z^r R_{1,j}(z)$ remains bounded near infinity for every $j \in \{1,\dots,n-k\}$. Then there exists an $\ell \in \{1,\dots,n-k\}$ such that $z^r R_{1,\ell}(z)$ does not vanish at infinity. Denote the constant $(n-k)\times(n-k)$ matrix, which achieves the exchange of columns 1 and ℓ of $R(z)$, by C. Then the first row of $z^r R(z)C$ is bounded at infinity and the first entry in that row does not vanish at infinity. Next, define

$$T(z) = \begin{pmatrix} I_k & 0 \\ 0 & z^r C \end{pmatrix}, \tag{2.2}$$

where I_k is the $k\times k$ identity matrix. Let

$$A(z) = \begin{pmatrix} \tilde A_{1,1}(z) & \tilde A_{1,2}(z) \\ \tilde A_{2,1}(z) & \tilde A_{2,2}(z) \end{pmatrix}, \tag{2.3}$$

where $\tilde A_{1,1}(z)$ and $\tilde A_{2,2}(z)$ are square matrices with k and $n-k$ rows, respectively. Then

$$T(z)^{-1}A(z)T(z) = \begin{pmatrix} \tilde A_{1,1}(z) & z^r \tilde A_{1,2}(z)C \\ z^{-r}C^{-1}\tilde A_{2,1}(z) & C^{-1}\tilde A_{2,2}(z)C \end{pmatrix}. \tag{2.4}$$

Since only the last row of $\tilde A_{1,2}(z)$ is different from zero, and since that row equals the first row of $R(z)$, the matrix $T(z)^{-1}A(z)T(z)$ remains bounded at infinity and its first $k-1$ rows are the same as those of $A(z)$. The matrix C was chosen so that the first entry in the last row of $z^r\tilde A_{1,2}(z)C$ does not vanish at infinity. Since

$$\lim_{z\to\infty} T(z)^{-1}T'(z) = 0, \tag{2.5}$$

we conclude that $B = T^{-1}(AT - T') \in \mathcal{R}_\infty^{n\times n}$ is reduced of order $k-1$ and that $B_{k,k+1}(\infty) \neq 0$.

Finally we prove that $A(\infty)$ and $B(\infty)$ have the same eigenvalues counting algebraic multiplicities. Since $T(\infty)$ might not exist, we first compute

$$\begin{aligned} &\det(\lim_{z\to\infty}((T^{-1}AT)(z) - \lambda I_n)) = \lim_{z\to\infty}\det((T^{-1}AT)(z) - \lambda I_n) \\ &= \lim_{z\to\infty}\det(A(z) - \lambda I_n) = \det(\lim_{z\to\infty}(A(z) - \lambda I_n)). \end{aligned} \tag{2.6}$$

By (2.5), the left-hand side of (2.6) is the characteristic polynomial of $B(\infty)$, while the right-hand side is the characteristic polynomial of $A(\infty)$. This completes the proof. $\square$

Lemma 2.3. *Assume that* $A \in \mathcal{R}_\infty^{n\times n}$ *is reduced of order* $k-1$ *and suppose that* $A_{k,k+1}(\infty) \neq 0$. *Then there exists a matrix* $B \in \mathcal{R}_\infty^{n\times n}$ *of the same kind as* A*, which is reduced of order* k. *Moreover,* $A(\infty)$ *and* $B(\infty)$ *are similar and hence isospectral* (*i.e., their eigenvalues, including algebraic and geometric multiplicities, coincide*).

Proof. Let $T \in \mathcal{R}^{n\times n}$ denote the $n \times n$ matrix obtained from the identity matrix I_n by replacing its $(k+1)$st row by

$$(-A_{k,1}, \dots, -A_{k,k}, 1, -A_{k,k+2}, \dots, -A_{k,n})/A_{k,k+1}. \tag{2.7}$$

T^{-1} is then the matrix obtained from the identity matrix I_n by replacing the $(k+1)$st row by $(A_{k,1}, \dots, A_{k,n})$. Note that the entries of T and T^{-1} are rational and bounded at infinity. Hence the matrix $B = T^{-1}(AT - T')$ has rational entries bounded at infinity. A straightforward calculation then shows that the first k rows of B have the desired form. Since T and T' are bounded at infinity, $\lim_{z\to\infty} T(z)^{-1}T'(z) = 0$ and hence $B(\infty) = T(\infty)^{-1}A(\infty)T(\infty)$. □

Theorem 2.4. *Let $Q \in \mathcal{R}_\infty^{n\times n}$ and suppose that the first-order system $y'(z) = Q(z)y(z)$ has a meromorphic fundamental system of solutions. Then $y'(z) = Q(z)y(z)$ has a fundamental matrix of the type*

$$Y(z) = R(z)\exp(\operatorname{diag}(\lambda_1 z, \dots, \lambda_n z)), \tag{2.8}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $Q(\infty)$ and $R \in \mathcal{R}^{n\times n}$.

Proof. The theorem will be proved by induction on n. Let $n = 1$. Any pole of $Q(z)$ must be of first-order with an integer residue, that is,

$$Q(z) = \lambda_1 + \sum_{\ell=1}^{N} \frac{m_\ell}{z - a_\ell}, \tag{2.9}$$

with $m_1, \dots, m_N \in \mathbb{Z}$. Then $Y(z) = \prod_{\ell=1}^{N}(z - a_\ell)^{m_\ell} \exp(\lambda_1 z)$ proves the claim for $n = 1$.

Next, let n be any natural number and assume that Theorem 2.4 has been proven for any natural number strictly less than n.
By hypothesis, $Q \in \mathcal{R}_\infty^{n\times n}$ and $Q(z)$ can be regarded to be reduced at least of order zero. We denote the eigenvalues of $Q(\infty)$ by $\lambda_1, \dots, \lambda_n$. Repeated, perhaps alternating, applications of Lemmas 2.2 and 2.3 then yield the existence of an integer $k \in \{1, \dots, n\}$, a $k \times k$ matrix $B_1(z)$, an $(n-k) \times k$ matrix $B_3(z)$, and an $(n-k) \times (n-k)$ matrix $B_4(z)$, such that

$$B(z) = \begin{pmatrix} B_1(z) & 0 \\ B_3(z) & B_4(z) \end{pmatrix} \tag{2.10}$$

has the following properties:

1. $B \in \mathcal{R}_\infty^{n\times n}$.
2. B is of the same kind as Q, that is, there exists an invertible matrix $T \in \mathcal{R}^{n\times n}$ such that $B(z) = T(z)^{-1}(Q(z)T(z) - T'(z))$.
3. $B_1(z)$ is reduced of order $k - 1$.
4. After a suitable relabeling of the eigenvalues of $Q(\infty)$ the eigenvalues of $B_1(\infty)$ are $\lambda_1, \dots, \lambda_k$ and the eigenvalues of $B_4(\infty)$ are $\lambda_{k+1}, \dots, \lambda_n$.
5. The first-order system $u'(z) = B(z)u(z)$ has a meromorphic fundamental system of solutions with respect to $z \in \mathbb{C}$.

We now have to distinguish whether $k = n$ or $k < n$. In the case $k = n$, $B(z) = B_1(z)$, and the system $u'(z) = B(z)u(z)$ is equivalent to the scalar equation

$$u_1^{(n)}(z) = B_{n,1}(z)u_1 + \dots + B_{n,n}(z)u_1^{(n-1)}(z). \tag{2.11}$$

In this case Halphen's theorem, Theorem 1.1, and the relations $u_k(z) = u_1^{(k-1)}(z)$ and $y(z) = T(z)u(z)$ prove our claim.
Next, assume that $k < n$. If w is any solution of $w'(z) = B_1(z)w(z)$, choose a solution v of the nonhomogeneous system

$$v'(z) - B_4(z)v(z) = B_3(z)w(z). \tag{2.12}$$

Then $u = (w, v)^t$ is a solution of $u'(z) = B(z)u(z)$ and hence meromorphic. Thus every solution of $w'(z) = B_1(z)w(z)$ and, choosing $w(z) = 0$ in (2.12), also every solution of $v'(z) = B_4(z)v(z)$ is meromorphic. By the induction hypothesis, there exist matrices $R_1 \in \mathcal{R}^{k\times k}$ and $R_4 \in \mathcal{R}^{(n-k)\times(n-k)}$ such that

$$U_1(z) = R_1(z)\,\mathrm{diag}(\exp(\lambda_1 z), \dots, \exp(\lambda_k z)) \tag{2.13}$$

is a fundamental matrix of $w'(z) = B_1(z)w(z)$ and

$$U_4(z) = R_4(z)\,\mathrm{diag}(\exp(\lambda_{k+1} z), \dots, \exp(\lambda_n z)) \tag{2.14}$$

is a fundamental matrix of $v'(z) = B_4(z)v(z)$.
Next define

$$U_3(z) = U_4(z)\int^z d\zeta\, U_4^{-1}(\zeta)B_3(\zeta)U_1(\zeta). \tag{2.15}$$

Then each column of

$$U(z) = \begin{pmatrix} U_1(z) & 0 \\ U_3(z) & U_4(z) \end{pmatrix} \tag{2.16}$$

is a solution of $u'(z) = B(z)u(z)$ and U is indeed a fundamental matrix of $u'(z) = B(z)u(z)$ since $\det(U(z)) = \det(U_1(z))\det(U_4(z)) \neq 0$. It remains to show that

$$U_3(z) = R_3(z)\,\mathrm{diag}(\exp(\lambda_1 z), \dots, \exp(\lambda_k z)) \tag{2.17}$$

for some matrix $R_3 \in \mathcal{R}^{(n-k)\times k}$. The entry in row j and column ℓ of the matrix $U_4^{-1}(\zeta)B_3(\zeta)U_1(\zeta)$ equals $\rho_{j,\ell}(\zeta)\exp[(\lambda_\ell - \lambda_{k+j})\zeta]$, where $\rho_{j,\ell}$ is a rational function, that is,

$$\rho_{j,\ell}(\zeta) = \sum_{r=0}^{N} a_{j,\ell,r}\zeta^r + \sum_{r=1}^{M}\sum_{s=1}^{M_r} \frac{b_{j,\ell,r,s}}{(\zeta - z_r)^s} \tag{2.18}$$

for appropriate choices of the parameters $a_{j,\ell,r}$, $b_{j,\ell,r,s}$, and pairwise distinct z_r.
Next we recall that

$$\int^z d\zeta\zeta^s e^{\lambda\zeta} = f(s,\lambda,z) \ (\mathrm{mod}\,(e^{\lambda z}\mathbb{C}(z))), \tag{2.19}$$

where

$$f(s,\lambda,z) = \begin{cases} 0 & \text{if } s \geq 0 \\ \frac{\lambda^{-s-1}}{(-s-1)!}\,\mathrm{Ei}(\lambda z) & \text{if } \lambda \neq 0 \text{ and } s \leq -1 \\ \ln(z) & \text{if } \lambda = 0 \text{ and } s = -1 \\ 0 & \text{if } \lambda = 0 \text{ and } s \leq -2 \end{cases} \tag{2.20}$$

and that the exponential integral $\mathrm{Ei}(\cdot)$ has a logarithmic branch point at zero. Therefore, if $\lambda_\ell \neq \lambda_{k+j}$,

$$(U_4^{-1}U_3)_{j,\ell}(z) = \int^z d\zeta\,\rho_{j,\ell}(\zeta)e^{(\lambda_\ell - \lambda_{k+j})\zeta} \tag{2.21}$$

$$= \mathrm{e}^{(\lambda_\ell - \lambda_{k+j})z} S_{j,\ell}(z) + \sum_{r=1}^{M} c_{j,\ell,r} \mathrm{e}^{(\lambda_\ell - \lambda_{k+j})z_r} \operatorname{Ei}((\lambda_\ell - \lambda_{k+j})(z - z_r)),$$

for appropriate rational functions $S_{j,\ell}$ and appropriate complex numbers $c_{j,\ell,r}$. However, since the entries of $U_3(z)$ and $U_4(z)$ must be meromorphic, all of the numbers $c_{j,\ell,r}$ must necessarily vanish. If $\lambda_\ell = \lambda_{k+j}$ a similar conclusion shows that no logarithmic terms appear so that in either case $(U_4^{-1}U_3)_{j,\ell}(z) \in \mathrm{e}^{(\lambda_\ell - \lambda_{k+j})z}\mathbb{C}(z)$. Hence we obtain

$$U_4^{-1}(z)U_3(z) = \operatorname{diag}(\mathrm{e}^{-\lambda_{k+1}z}, \dots, \mathrm{e}^{-\lambda_n z}) S(z) \operatorname{diag}(\mathrm{e}^{\lambda_1 z}, \dots, \mathrm{e}^{\lambda_k z}), \tag{2.22}$$

where $S \in \mathcal{R}^{(n-k)\times k}$ is the matrix with entries $S_{j,\ell}$. Thus, $R_3 = R_4 S \in \mathcal{R}^{(n-k)\times k}$. $\square$

Remark 2.5. If $Q(\infty) = 0$, the transformation $z = \frac{1}{\zeta}$ immediately shows that $\zeta = 0$, is a regular singular point of our differential equation. This implies that the fundamental system at $\zeta = 0$ is of the form (cf., e.g., [36, Sect. 23]) $Y(\zeta) = U(\zeta)\zeta^m$ for $|\zeta| < \zeta_0$, where $U(\zeta)$ is a holomorphic matrix for $|\zeta| < \zeta_0$. Hence our fundamental system is meromorphic on the whole Riemann sphere and must therefore be a purely rational matrix.

Next we consider two examples.

Example 2.6. *The first-order* 2×2 *system*

$$y'(z) = \begin{pmatrix} 1 & 0 \\ z & 1 + \frac{1}{z} \end{pmatrix} y(z)$$

is solved by

$$Y(z) = \begin{pmatrix} 1 & 1 \\ z^2 & z^2 + z \end{pmatrix} e^z.$$

This seems to suggest consideration of even more general systems of the type $z^{-q}Y'(z) = A(z)Y(z)$, with $q > 0$, rather than the case $q = 0$ only. But Theorem 2.4 can not hold in general for $q > 0$ as shown by the following elementary counterexample.

Example 2.7. *The first-order* 2×2 *system*

$$y'(z) = \begin{pmatrix} 0 & 1 \\ z^m & 0 \end{pmatrix} y(z), \quad m \in \mathbb{N}$$

has no solution in terms of elementary functions, although it clearly has a meromorphic fundamental system. The particular case $m = 1$ *represents the well-known Airy equation.*

Remark 2.8. In the case where all eigenvalues $1 \le \lambda_j \le n$ of $Q(\infty)$ are distinct, we now sketch an alternative proof of Theorem 2.4, based on Theorem 12.3 in Wasow's monograph [37]. Since Theorem 12.3 in [37] only applies to appropriate sectors of the complex plane with vertex at the origin, we argue as follows. First one can find a sufficiently small sector S_3, which does not contain any separation rays. (We recall that a ray (i.e., a half line), where $\operatorname{Re}(\lambda_j z - \lambda_k z) = 0$ for some pair of distinct integers j, k, is called a separation ray.) Then one chooses two other sectors S_1, S_2 with opening angles $\phi_j < \pi$, $j = 1, 2$, such that $S_1 \cup S_2 \cup S_3 = \mathbb{C}\backslash\{0\}$. It is then possible to show that the transition matrix from sector S_1 to sector S_2 equals the identity matrix. Hence, the solution of the form $Y(z) = R(z)\exp(\operatorname{diag}(\lambda_1, \dots, \lambda_n)z)$ in sector S_1 is valid in sector S_2 too and thus can be continued into S_3 since by hypothesis, the sector S_3 contains no separation rays.

Finally, we turn to a converse of Theorem 2.4.

Theorem 2.9. *Suppose $R \in \mathcal{R}^{n\times n}$, $\det(R) \neq 0$, and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then*

$$Y(z) = R(z)\exp(\operatorname{diag}(\lambda_1 z, \dots, \lambda_n z)) \tag{2.23}$$

is a fundamental matrix of a first-order linear system of differential equations $y'(z) = Q(z)y(z)$, where $Q \in \mathcal{R}^{n\times n}$ and $Q(z)$ is of the same kind as a matrix in $\mathcal{R}_\infty^{n\times n}$. In fact, $Q(z)$ is of the same kind as the constant diagonal matrix $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

Proof. Since

$$Q(z) = R(z)\operatorname{diag}(\lambda_1, \dots, \lambda_n)R(z)^{-1} + R'(z)R(z)^{-1}, \tag{2.24}$$

we choose $T(z) = R(z)^{-1}$ and hence obtain $T' = -R^{-1}R'R^{-1}$ and thus,

$$Q = T^{-1}(\operatorname{diag}(\lambda_1, ..., \lambda_n)T - T'). \tag{2.25}$$

Hence, $Q(z)$ is of the same kind as the constant matrix $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$. □

3. Some applications to rational solutions of the stationary KdV hierarchy

In this section we describe the connections between the preceding results and infinite-dimensional completely integrable Hamiltonian systems. For reasons of brevity we will only consider the simplest case of the KdV hierarchy, and in accordance with Sections 1, 2, only study its stationary rational solutions bounded at infinity (cf. [1], [3]–[7], [19], [23], [24]–[28], [30], [31], [32], [33], [35], [40] and the literature cited therein). The principal results on the stationary KdV hierarchy as needed in this section are summarized in the appendix, and we freely use these results and the notation established there in what follows.

The rational KdV solutions bounded at infinity are usually discussed in a time-dependent setting and the dynamics of their poles is in an intimate relationship with completely integrable systems of the Calogero-Moser-type. In our discussion below, the time-dependence will generally be suppressed and only occasionally be mentioned in connection with particular isospectral deformations of rational solutions of the KdV hierarchy. Our principal focus will be on stationary (isospectral) aspects of these rational KdV solutions and the implications of Halphen's theorem in this context.

We start by quoting a number of known results on stationary rational KdV solutions bounded at infinity.

Theorem 3.1. *Let $N \in \mathbb{N}$ and $\{z_j\}_{1\le j\le N} \subset \mathbb{C}$.*
(*i*) (*Airault, McKean, and Moser* [5]) *Any rational solution q of* (*some, and hence infinitely many equations of*) *the* KdV *hierarchy, or equivalently, any rational algebro-geometric potential q, is necessarily of the form*

$$q(z) = q_\infty - 2\sum_{j=1}^{N}(z - z_j)^{-2}, \tag{3.1}$$

for some $q_\infty \in \mathbb{C}$ *and with* $N \in \mathbb{N}$ *of the special type* $N = g(g+1)/2$ *for some* $g \in \mathbb{N}$.

(*ii*) (*Airault, McKean, and Moser* [5] (*see also* [38])) *If one allows for "collisions" between the* z_j, *that is, if the set* $\{z_j\}_{1\le j\le N}$ *clusters into groups of points, then the corresponding rational algebro-geometric potential* q *is necessarily of the form*

$$q(z) = q_\infty - \sum_{\ell=1}^{M} s_\ell(s_\ell+1)(z-\zeta_\ell)^{-2}, \tag{3.2}$$

where for some $g \in \mathbb{N}$,

$$\{z_j\}_{1\le j\le N} = \{\zeta_\ell\}_{1\le \ell\le M} \subset \mathbb{C}, \text{ with } \zeta_\ell \text{ pairwise distinct}, \tag{3.3a}$$

$$s_\ell \in \mathbb{N}, \ 1 \le \ell \le M,$$

$$\sum_{\ell=1}^{M} s_\ell(s_\ell+1) = 2N \text{ for some } N \in \mathbb{N} \text{ of the type } N = g(g+1)/2. \tag{3.3b}$$

(*iii*) *The extreme case of all* z_j *colliding into one point, say* ζ_1, *that is,* $\{z_j\}_{1\le j\le N} = \{\zeta_1\} \subset \mathbb{C}$ *yields an algebro-geometric* KdV *potential of the elementary form*

$$q(z) = q_\infty - g(g+1)(z-\zeta_1)^{-2}, \quad g \in \mathbb{N} \tag{3.4}$$

and no additional constraints on $\zeta_1 \in \mathbb{C}$.

(*iv*) *In all cases* (*i*)–(*iii*), *if* q *is a rational* KdV *potential* (*i.e., if* $g \in \mathbb{N}$ *and the points* z_j (*resp.* ζ_ℓ) *satisfy appropriate restrictions, cf. Theorem 3.5*), *the underlying rational hyperelliptic curve* $\mathcal{K}_g$ *is of the especially simple form*

$$\mathcal{K}_g \colon y^2 = (E-q_\infty)^{2g+1}. \tag{3.5}$$

In particular, the potentials (3.1), (3.2), *and* (3.4) *are all isospectral* (*assuming* (3.1) *and* (3.2) *are algebro-geometric* KdV *potentials, of course*).

(*v*) (*Weikard* [38]) q *is a rational* KdV *potential if and only if* $\psi'' + (q-E)\psi = 0$ *has a meromorphic fundamental solutions* (*w.r.t.* z) *for all values of the spectral parameter* $E \in \mathbb{C}$.

(*vi*) *If* q *is a rational KdV potential of the form* (3.2), *then* $y'' + qy = Ey$ *has linearly independent solutions of the Baker-Akhiezer-type*

$$\psi_\pm(E,z) = \big(\pm E^{1/2}\big)^{-g}\bigg(\prod_{j=1}^{g}\big(\pm E^{1/2} - \nu_j(z)\big)\bigg)e^{\pm E^{1/2}z}, \tag{3.6}$$
$$E \in \mathbb{C}\backslash\{q_\infty\}, \ z \in \mathbb{C},$$

with $\mu_j(z) = \nu_j(z)^2$, $1 \le j \le g$ *the zeros of* $F_g(z,x)$ *as defined in* (A.12).

(To avoid annoying case distinctions we will in almost all circumstances exclude the trivial case $N = g = 0$ in this section.)

Remark 3.2. (i) It must be emphasized that for $N > 1$, not any potential q of the type (3.1) is an algebro-geometric KdV potential. In fact, for $N > 1$, there exist nontrivial constraints on the set $\{z_j\}_{1\le j\le N}$ for (3.1) to represent an algebro-geometric KdV potential. For instance, if the z_j in (3.1) are pairwise distinct, then Airault, MacKean, and Moser [5] proved that

$$\sum_{\substack{j'=1\\ j'\ne j}}^{N} \frac{1}{(z_j - z_{j'})^3} = 0 \quad \text{for } j = 1,\dots,N \tag{3.7}$$

are necessary conditions for q in (3.1) to be a stationary KdV potential. In the case of collisions (i.e., if $s_{\ell_0} > 1$ for some $1 \leq \ell_0 \leq M$) the necessary constraints on $\{\zeta_\ell\}_{1\leq\ell\leq M}$ are more involved than in the nondegenerate case above and a complete description of all constraints were originally obtained by Duistermaat and Grünbaum [10] in 1986. An alternative proof of their result will be given in Theorem 3.5 below.
(ii) In connection with Theorem 3.1 (ii) one might naively expect that any decomposition of $g(g+1) = \sum_{\ell=1}^{M} s_\ell(s_\ell+1)$ can actually be realized for some choice of $\{\zeta_\ell\}_{1\leq\ell\leq M}$ with $\zeta_\ell \neq \zeta_{\ell'}$ for $\ell \neq \ell'$. However, the simple counterexample $q(z) = -6(z-\zeta_1)^{-2} - 6(z-\zeta_2)^{-2}$, which satisfies $\mathrm{KdV}_3(q) = -5670(\zeta_1-\zeta_2)^2(\zeta_1+\zeta_2-2z)(z-\zeta_1)^{-6}(z-\zeta_2)^{-6}$, quickly destroys such hopes.
(iii) Strictly speaking, the version of Theorem 3.1 (v) proven in [38] assumes in addition to q being rational, that q is bounded at infinity. However, assuming that

$$q(z) \underset{z\to\infty}{=} \alpha z^k + O(z^{k-1}) \text{ for some } \alpha \neq 0 \text{ and } k \in \mathbb{N},$$

a simple inductive argument using (A.1) proves

$$\hat f_j'(z) = \frac{k\alpha^j}{2}\Bigg(\prod_{\ell=1}^{j-1}\frac{2\ell+1}{2\ell}\Bigg) z^{jk-1} + O(z^{jk-2}), \quad j \geq 1,$$

using the usual convention (for $j=1$) that products over empty sets are put equal to one. Thus, since $\hat f_j'$ cannot vanish in this case, a rational q unbounded at infinity cannot satisfy any of the stationary KdV equations (cf. (A.8)).

Before we discuss additional facts, we briefly pause and mention some of the ingredients entering the proof of items (i)–(v) in Theorem 3.1. We start with a fairly complete treatment of item (iii) and for simplicity of notation put $q_\infty = \zeta_1 = 0$ and

$$q_g(z) = -g(g+1)z^{-2}, \quad g \in \mathbb{N}, \ z \in \mathbb{C}\backslash\{0\}. \tag{3.8}$$

From [2, Ch. 10] one infers that ($E \in \mathbb{C}\backslash\{0\}$, $z \in \mathbb{C}$)

$$\psi_\pm(E,z) = \Bigg(\sum_{k=0}^{g}\frac{(g+k)!}{k!(g-k)!}(\pm 2E^{1/2}z)^{-k}\Bigg)e^{\mp E^{1/2}z}, \tag{3.9}$$

are linearly independent solutions of $\psi'' + (q_g - E)\psi = 0$, $E \in \mathbb{C}\backslash\{0\}$. Thus, one concludes that

$$\psi_+(E,z)\psi_-(E,z) = \prod_{j=1}^{g}\Big(1 - \frac{\kappa_j}{Ez^2}\Big) \text{ for some } \kappa_j \in \mathbb{C}, \ 1 \leq j \leq g. \tag{3.10}$$

Hence a comparison with (A.12)–(A.15), (A.19)–(A.24) yields

$$\hat F_g(E,z) = \prod_{j=1}^{g}\big(E - \mu_j(z)\big), \quad \mu_j(z) = \kappa_j z^{-2}, \ 1 \leq j \leq g, \tag{3.11}$$

where $\hat F_g(E,z)$ denotes the polynomial of degree g with respect to E associated with $q_g(z)$ in (3.8), as introduced in the appendix. Thus, $q_g(z)$ is a KdV potential satisfying $\widehat{\mathrm{KdV}}_g(q_g) = 0$ for a particular set of constants $\{c_\ell\}_{1\leq\ell\leq g}$ in (A.10). However, taking into account the simple form of $q_g(z)$ in (3.8), homogeneity considerations in connection with the corresponding $\hat f_j$ and (A.25) then yield in the special case $q(z) = q_g(z)$,

$$c_\ell = 0, \quad 1 \leq \ell \leq g, \tag{3.12}$$

$$\hat{F}_g(E,z) = F_g(E,z), \quad \hat{f}_j(z) = f_j(z), \quad 1 \leq j \leq g, \tag{3.13}$$

$$f_j(z) = d_j x^{-2j} \text{ for some } d_j \in \mathbb{C}\backslash\{0\}, \ 1 \leq j \leq g, \tag{3.14}$$

$$f_{k+1}(z) = 0, \quad \text{s-KdV}_k(q_g) = 0, \quad k \geq g, \tag{3.15}$$

$$y^2 = E^{2g+1}, \text{ that is, } \hat{E}_m = 0, \quad 0 \leq m \leq 2g \tag{3.16}$$

(and of course $c_0 = \hat{f}_0(z) = f_0(z) = 1$). This yields item (iii) and part of item (iv). Since q in (3.1) and (3.2) in the special case $q_\infty = 0$ satisfies $q(z) \underset{|z|\to\infty}{=} 2Nz^{-2}\big(1 + O\big(|z|^{-1}\big)\big)$, one infers that $f_{k+1} = 0$ for some $k \in \mathbb{N}$ can only happen if $N = k(k+1)/2$ for some $k \in \mathbb{N}$. This illustrates $N = g(g+1)/2$ and (3.3b). Item (v) in [38] follows from a careful combination of Frobenius theory for second-order linear ordinary differential equations in the complex domain, Halphen's theorem, Theorem 1.1 (for $n = 2$), and some of the algebro-geometric formalism briefly sketched in the appendix. As a by-product of a proof of item (v) one shows that $\psi''(z) - cz^{-2}\psi(z) = E\psi(z)$, $z \in \mathbb{C}\backslash\{0\}$ has a meromorphic fundamental system of solutions for all $E \in \mathbb{C}$ if and only if $c \in \mathbb{C}$ is of the special form $c = s(s+1)$ for some $s \in \mathbb{N}_0$. This illustrates why collisions necessarily must happen as described in (3.3a). This fact was already known to Kruskal [28] in 1974. That q in (3.1), (3.2), and (3.4) are all isospectral KdV potentials, that is, they all belong to the same algebraic curve (3.5) (assuming (3.1) and (3.2) satisfy the additional restrictions to make them algebro-geometric KdV potentials, of course) can be shown by several methods. Either by invoking time-dependent KdV flows as in [5], or by commutation techniques (i.e., Darboux-type transformations) as in [3], [11], [30], [31] (cf. also [14]). This fact also follows from the results in [38]. Finally, identifying $\psi_\pm(E,z)/\psi_\pm(E,z_0)$ with the two branches of the Baker-Akhiezer function $\psi(P,z,z_0)$, $P = (E,y)$ in (A.20), a combination of (A.12), (A.23), and the normalizations

$$\lim_{|z|\to\infty} \psi_\pm(E,z)\exp(\mp E^{1/2}z) = 1, \qquad \lim_{|E|\to\infty} \psi_\pm(E,z)\exp(\mp E^{1/2}z) = 1,$$

then proves $\psi_+(E,z)\psi_-(E,z) = E^{-g}F_g(E,z) = \prod_{j=1}^g \Big(1 - \frac{\mu_j(z)}{E}\Big)$, and hence (3.6).

Finally, we study the precise restrictions on the set of poles $\{z_j\}_{1\leq j\leq N} = \{\zeta_\ell\}_{1\leq \ell\leq M}$ for q in (3.2) to be a KdV potential.

Lemma 3.3. *Suppose the function q has a Laurent expansion about the point $z_0 \in \mathbb{C}$ of the type*

$$q(z) = \sum_{j=0}^\infty q_j(z - z_0)^{j-2}, \tag{3.17}$$

where $q_0 = -s(s+1)$ and, without loss of generality, $\mathrm{Re}(2s+1) \geq 0$. *Define for $\sigma \in \mathbb{C}$,*

$$f_0(\sigma) = -\sigma(\sigma - 1) - q_0 = (s+\sigma)(s+1-\sigma), \tag{3.18}$$

$$c_0(\sigma) = \prod_{j=1}^{2s+1} f_0(\sigma + j), \ c_j(\sigma) = \frac{\sum_{m=0}^{j-1} q_{j-m}c_m(\sigma)}{f_0(\sigma+j)}, \ j \in \mathbb{N}, \tag{3.19}$$

$$w(\sigma,z)=\sum_{j=0}^{\infty}c_j(\sigma)(z-z_0)^{\sigma+j}, \tag{3.20}$$

$$v(\sigma,z)=\frac{\partial w}{\partial\sigma}(\sigma,z)=\sum_{j=0}^{\infty}\left(\frac{\partial c_j}{\partial\sigma}+c_j\log(z-z_0)\right)(z-z_0)^{\sigma+j}. \tag{3.21}$$

If $2s+1$ is not an integer, then $y''+qy=0$ has the linearly independent solutions $y_1=w(s+1,\cdot)$ and $y_2=w(-s,\cdot)$. If $2s+1$ is an integer, then $y''+qy=0$ has the linearly independent solutions $y_1=w(s+1,\cdot)$ and $y_2=v(-s,\cdot)$.

Moreover, $y''+qy=0$ has a meromorphic fundamental system of solutions near z_0 if and only if $s\in\mathbb{N}_0$ and $c_{2s+1}(-s)=0$.

This is a classical result in ordinary differential equations (cf., e.g., [22], Chs. XV, XVI). A recent proof can be found in Section 3 of [38].

Definition 3.4. *Let q be a rational function. Then q is called a Halphen potential if it is bounded near infinity and if $y''+qy=Ey$ has a meromorphic fundamental system of solutions (w.r.t. z) for each value of the complex spectral parameter $E\in\mathbb{C}$.*

Of course every constant is a Halphen potential. Moreover, by Theorem 3.1 (v), q is a Halphen potential if and only if it is a rational KdV potential (i.e., if and only if it satisfies one and hence infinitely many of the equations of the stationary KdV hierarchy).

Theorem 3.5. *Let q be a nonconstant rational function. Then q is a Halphen potential if and only if there are $M\in\mathbb{N}$, $s_\ell\in\mathbb{N}$, $1\le\ell\le M$, $q_\infty\in\mathbb{C}$, and pairwise distinct $\zeta_\ell\in\mathbb{C}$, $\ell=1,\dots,M$, such that*

$$q(z)=q_\infty-\sum_{\ell=1}^{M}s_\ell(s_\ell+1)(z-\zeta_\ell)^{-2} \tag{3.22}$$

and

$$\sum_{\substack{\ell'=1\\ \ell'\neq\ell}}^{M}\frac{s_{\ell'}(s_{\ell'}+1)}{(\zeta_\ell-\zeta_{\ell'})^{2k+1}}=0 \quad \text{for } k=1,...,s_\ell \text{ and } \ell=1,\dots,M. \tag{3.23}$$

Moreover, q is a rational KdV potential if and only if q is of the type (3.22) and the constraints (3.23) hold. In particular, for fixed g, the constraints (3.23) characterize the isospectral class of all rational KdV potentials associated with the curve $y^2=(E-q_\infty)^{2g+1}$, where $g(g+1)=\sum_{\ell=1}^{M}s_\ell(s_\ell+1)$.

Proof. By Theorem 3.1 (v), it suffices to prove the characterization of Halphen potentials. Suppose that q is a nonconstant Halphen potential. Then a pole z_0 of q is a regular singular point of $y''+qy=Ey$ and hence

$$q(z)-E=\sum_{j=0}^{\infty}Q_j(z-z_0)^{j-2}$$

in a sufficiently small neighborhood of z_0, where Q_2 is a first order polynomial in E, while Q_j for $j\neq2$ are independent of E. The indices associated with z_0, defined as the roots of $\sigma(\sigma-1)+Q_0=0$ (hence they are E-independent), must be distinct integers whose sum equals one. We denote them by $-s$ and $s+1$ where $s>0$ and note that $Q_0=-s(s+1)$. We intend to prove that $Q_{2j+1}=0$ whenever $j\in\{0,...,s\}$ by applying Lemma 3.3. Proceeding by way of contradiction, we thus assume that for some nonnegative integer $k\in\{0,...,s\}$, $Q_{2k+1}\neq 0$ and k is the smallest such integer.

We note that $f_0(\cdot+j)$ are positive in $(-s-1,-s+1)$ for $j=1,...,2s$, whereas $f_0(\cdot+2s+1)$ has a simple zero at $-s$ and its derivative is negative at $-s$. Next one defines

$$\gamma_0(\sigma)=\prod_{j=1}^{2s+1} f_0(\sigma+j) \quad\text{and}\quad \gamma_1(\sigma)=\prod_{j=2}^{2s+1} f_0(\sigma+j). \tag{3.24}$$

γ_0 and γ_1 have simple zeros at $-s$ and and $\gamma_0'(-s)$ and $\gamma_1'(-s)$ are negative.

The functions $c_0=\gamma_0$ and $c_1=Q_1\gamma_1$ are polynomials with respect to E. Actually, c_0 has degree zero in E and c_1 is constant but might equal zero. Hence the relations (3.25), (3.26), and (3.27) below are satisfied for $j=1$. Next we assume that for some integer $\ell\in\{1,...,s\}$, the functions c_0, ..., $c_{2\ell-1}$ are polynomials in E and that the relations

$$c_{2j-2}(\sigma)=\gamma_{2j-2}(\sigma)Q_2^{j-1}+O(E^{j-2}), \tag{3.25}$$

$$c_{2j-1}(\sigma)=\begin{cases}\gamma_{2j-1}(\sigma)Q_{2k+1}Q_2^{j-k-1}+O(E^{j-k-2}), & j-1\geq k,\\ 0, & j-1<k,\end{cases} \tag{3.26}$$

$$\gamma_{2j-2}(-s)=\gamma_{2j-1}(-s)=0, \quad \gamma_{2j-2}'(-s),\gamma_{2j-1}'(-s)<0 \tag{3.27}$$

are satisfied for $1\leq j\leq\ell$. Using the recursion relation (3.19) we then obtain that $c_{2\ell}(\sigma)$ and $c_{2\ell+1}(\sigma)$ are polynomials in E and that

$$c_{2\ell}(\sigma)=\frac{\gamma_{2\ell-2}(\sigma)}{f_0(\sigma+2\ell)}Q_2^\ell+O(E^{\ell-1}),$$

$$c_{2\ell+1}(\sigma)=\begin{cases}\frac{\gamma_{2\ell-1}(\sigma)+\gamma_{2(\ell-k)}(\sigma)}{f_0(\sigma+2\ell+1)}Q_{2k+1}Q_2^{\ell-k}+O(E^{\ell-k-1}), & \ell\geq k,\\ 0, & \ell<k.\end{cases}$$

Letting $\gamma_{2\ell}=\gamma_{2\ell-2}/f_0(\cdot+2\ell)$ and $\gamma_{2\ell+1}=(\gamma_{2\ell-1}+\gamma_{2(\ell-k)})/f_0(\cdot+2\ell+1)$ we find that the relations (3.25), (3.26), and (3.27) are satisfied for $j=\ell+1$. Hence an inductive argument proves that c_{2s+1} is a polynomial in E and that

$$\begin{aligned}c_{2s+1}(\sigma)&=\frac{\gamma_{2s-1}(\sigma)+\gamma_{2(s-k)}(\sigma)}{f_0(\sigma+2s+1)}Q_{2k+1}Q_2^{s-k}+O(E^{s-k-1})\\ &=\gamma_{2s+1}(\sigma)Q_{2k+1}Q_2^{s-k}+O(E^{s-k-1}).\end{aligned}$$

But both $\gamma_{2s-1}+\gamma_{2(s-k)}$ and $f_0(\cdot+2s+1)$ have simple zeros at $-s$ so that $\gamma_{2s+1}(-s)$ is different from zero. Lemma 3.3 then shows that $y''+qy=Ey$ has a solution which is not meromorphic whenever E is not a root of the polynomial $c_{2s+1}(-s)$. This contradiction proves our assumption $Q_{2k+1}\neq 0$ wrong.

Since $Q_1=0$, we proved that if q is a Halphen potential with pairwise distinct poles $\zeta_1,\dots,\zeta_M$, then the principal part of q about any pole ζ_ℓ is of the form $-s_\ell(s_\ell+1)/(z-\zeta_\ell)^2$ for an appropriate positive integer s_ℓ. Since q is bounded at infinity a partial fraction expansion then proves (3.22). This immediately implies that for $z_0 = \zeta_\ell$,

$$Q_{2k+1} = 2k \sum_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{M} \frac{s_{\ell'}(s_{\ell'}+1)}{(\zeta_\ell - \zeta_{\ell'})^{2k+1}}. \tag{3.28}$$

This proves necessity of the conditions (3.22) and (3.23) for q to be a Halphen potential. To prove their sufficiency we now assume that (3.22) and (3.23) hold. Then, if z_0 denotes any of the points ζ_ℓ, one infers that the corresponding $c_{2s_\ell+1}(-s_\ell) = 0$. Lemma 3.3 then guarantees that all solutions of $y'' + qy = Ey$ are meromorphic and hence that q is a Halphen potential. □

Remark 3.6. (i) We emphasize again that the necessary and sufficient conditions on ζ_ℓ for q in (3.22) to be a rational KdV potential were first obtained by Duistermaat and Grünbaum [10] in their analysis of bispectral pairs of differential operators. Our approach based on Halphen's theorem and a direct Frobenius-type analysis is a bit more streamlined since we aim directly at rational KdV solutions (and do not cover the case of the Airy equation) but there are undoubtedly some similarities in both approaches.
(ii) We note that the restrictions (3.23) simplify in the absence of collisions, where $s_\ell = 1$, $1 \leq \ell \leq N$. In this case (3.23) reduces to $\sum_{j'=1, j'\neq j}^{N} (z_j - z_{j'})^{-3} = 0$, $1 \leq j \leq N$, which represents the well-known locus introduced by Airault, McKean, and Moser [5]. This locus generated considerable interest, and especially its generalizations to elliptic KdV potentials and (elliptic) KP potentials, were intensively studied (cf., e.g., [4], [6], [7], [8], [12], [13], [23], [24]–[28], [32], [33], [34], [40]). The current derivation of (3.23) properly extends this locus to the case of collisions (i.e., to cases where some of the $s_\ell > 1$). Moreover, this appears to be the first systematic derivation of this locus (with or without collisions) within a purely stationary approach (i.e., without involving special time-dependent KdV flows, etc.).
(iii) For $k = 1$, conditions (3.23) coincide with the necessary conditions at collision points found by Airault, McKean, and Moser [5] in their Remark 1 on p. 113. However, since there are additional necessary conditions in (3.23) corresponding to $k \geq 2$, this disproves the conjecture made at the end of the proof of their Remark 1.
(iv) The genus $g = 2$ ($N = 3$) example, $\tilde{q}_2(z,t) = -6z(z^3 + 6t)(z^3 - 3t)^{-2}$, $t \in \mathbb{C}$, with $z_j = (3t)^{1/3}\omega_j$, $\omega_j = \exp(2\pi i j/3)$, $1 \leq j \leq 3$, explicitly illustrates the locus in (3.23). One verifies that $\tilde{q}_2(t)$ satisfies the kth stationary KdV equation, s-KdV$_k(\tilde{q}_2(t)) = 0$ for all $k \geq 2$ and all $t \in \mathbb{R}$, as well as the 1st time-dependent KdV equation $\tilde{q}_{2,t} = 4^{-1}\tilde{q}_{2,xxx} + 2^{-1}3\tilde{q}_2\tilde{q}_{2,x}$ (see, e.g., [4], [10]).

Extensions of the stationary formalism described in this section to elliptic KdV potentials are in preparation.

The stationary KdV hierarchy

In this section we review basic facts on the stationary KdV hierarchy. Since this material is well-known, we confine ourselves to a brief account. Assuming q to be meromorphic in $\mathbb{C}$, consider the recursion relation

$$\hat{f}_0(z) = 1, \quad \hat{f}'_{j+1}(z) = 4^{-1}\hat{f}'''_j(z) + q(z)\hat{f}'_j(z) + 2^{-1}q'(z)\hat{f}_j(z) \tag{A.1}$$

for $j \in \mathbb{N}_0$ (with $'$ denoting differentiation with respect to z and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) and the associated differential expressions (Lax pair)

$$L_2 = \frac{d^2}{dz^2} + q(z), \tag{A.2}$$

$$\hat{P}_{2g+1} = \sum_{j=0}^{g} \left[-\frac{1}{2}\hat{f}_j'(z) + \hat{f}_j(z)\frac{d}{dz} \right] L_2^{g-j}, \quad g \in \mathbb{N}_0. \tag{A.3}$$

One can show that

$$\left[\hat{P}_{2g+1}, L_2\right] = 2\hat{f}_{g+1}' \tag{A.4}$$

($[\cdot,\cdot]$ the commutator symbol) and explicitly computes from (A.1),

$$\hat{f}_0 = 1, \ \hat{f}_1 = 2^{-1}q + c_1, \ \hat{f}_2 = 8^{-1}q'' + 8^{-1}3q^2 + c_1 2^{-1}q + c_2, \ \text{etc.}, \tag{A.5}$$

where $c_j \in \mathbb{C}$ are integration constants. Using the convention that the corresponding homogeneous quantities obtained by setting $c_\ell = 0$ for $\ell = 1, 2, \ldots$ are denoted by f_j, that is,

$$f_j = \hat{f}_j\big|_{c_\ell=0,\, 1\le\ell\le j}, \quad j \in \mathbb{N}, \tag{A.6}$$

one obtains

$$\hat{f}_j = \sum_{\ell=0}^{j} c_\ell f_{j-\ell}, \quad 0 \le j \le g. \tag{A.7}$$

The (homogeneous) stationary KdV hierarchy is then defined as the sequence of equations

$$\text{s-KdV}_g(q) = 2f_{g+1}' = 0, \quad g \in \mathbb{N}_0. \tag{A.8}$$

Explicitly, this yields

$$\text{s-KdV}_0(q) = q' = 0, \quad \text{s-KdV}_1(q) = 4^{-1}q''' + 2^{-1}3qq' = 0, \ \text{etc.} \tag{A.9}$$

The corresponding nonhomogeneous version of $\text{s-KdV}_g(q) = 0$ is then defined by

$$\widehat{\text{s-KdV}}_g(q) = 2\hat{f}_{g+1}' = 2\sum_{j=0}^{g} c_{g-j} f_{j+1}' = 0, \tag{A.10}$$

where $c_0 = 1$ and $c_1, ..., c_g$ are arbitrary complex constants.

If one assigns to $q^{(\ell)} = d^\ell q/dz^\ell$ the degree $\deg(q^{(\ell)}) = \ell + 2, \ \ell \in \mathbb{N}_0$, then the homogeneous differential polynomial f_j with respect to q turns out to have degree $2j$, that is,

$$\deg(f_j) = 2j, \quad j \in \mathbb{N}_0. \tag{A.11}$$

Next, introduce the polynomial $\hat{F}_g(E, z)$ in $E \in \mathbb{C}$,

$$\hat{F}_g(E, z) = \sum_{j=0}^{g} \hat{f}_{g-j}(z)E^j = \prod_{j=1}^{g}(E - \mu_j(z)). \tag{A.12}$$

Since $\hat{f}_0(z) = 1$,

$$\begin{aligned} &-2^{-1}\hat{F}_g''(E,z)\hat{F}_g(E,z) + 4^{-1}\hat{F}_g'(E,z)^2 + (E - q(z))\hat{F}_g(E,z)^2 \\ &= \hat{R}_{2g+1}(E,z) \end{aligned} \tag{A.13}$$

is a monic polynomial in E of degree $2g+1$. However, equations (A.1) and (A.10) imply that

$$2^{-1}\hat{F}_g''' - 2(E-q)\hat{F}_g' + q'\hat{F}_g = 0 \tag{A.14}$$

and this shows that $\hat{R}_{2g+1}(E,z)$ is in fact independent of z. Hence it can be written as

$$\hat{R}_{2g+1}(E) = \prod_{m=0}^{2g}(E-\hat{E}_m), \quad \{\hat{E}_m\}_{0\leq m\leq 2g} \subset \mathbb{C}. \tag{A.15}$$

By (A.4) the nonhomogeneous KdV equation (A.10) is equivalent to the commutativity of L_2 and $\hat{P}_{2g+1}$. This shows that

$$[\hat{P}_{2g+1}, L_2] = 0, \tag{A.16}$$

and therefore, if $L_2\psi = E\psi$, this implies that $\hat{P}^2_{2g+1}\psi = \hat{R}_{2g+1}(E)\psi$. Thus $[\hat{P}_{2g+1}, L_2] = 0$ implies $\hat{P}^2_{2g+1} = \hat{R}_{2g+1}(L_2)$ by the Burchnall and Chaundy theorem. This illustrates the intimate connection between the stationary KdV equation $\hat{f}'_{g+1} = 0$ in (A.10) and the compact (possibly singular) hyperelliptic curve $\hat{\mathcal{K}}_g$ of (arithmetic) genus g obtained upon one-point compactification of the curve

$$\hat{\mathcal{K}}_g \colon y^2 = \hat{R}_{2g+1}(E) = \prod_{m=0}^{2g}(E-\hat{E}_m) \tag{A.17}$$

by joining the point at infinity, denoted by P_∞. Points $P \in \hat{\mathcal{K}}_g\backslash\{P_\infty\}$ will be denoted by $P = (E,y)$, moreover, the involution (hyperelliptic sheet exchange map) $*$ on $\hat{\mathcal{K}}_g$ is defined by

$$*\colon \hat{\mathcal{K}}_g \to \hat{\mathcal{K}}_g, \quad P = (E,y) \mapsto P^* = (E,-y),\ P^*_{\pm\infty} = P_{\mp\infty}. \tag{A.18}$$

Introducing the meromorphic function $\phi(\cdot,z)$ on $\hat{\mathcal{K}}_g$,

$$\phi(P,z) = \big[y(P) + (1/2)\hat{F}'_g(E,z)\big]/\hat{F}_g(E,z), \quad P = (E,y) \in \hat{\mathcal{K}}_g \tag{A.19}$$

and the stationary Baker-Akhiezer function $\psi(\cdot,z,z_0)$ by

$$\psi(P,z,z_0) = \exp\bigg(\int_{z_0}^{z} dz'\,\phi(P,z')\bigg), \quad P \in \hat{\mathcal{K}}_g\backslash\{P_\infty\}, \tag{A.20}$$

one infers (for $P = (E,y) \in \hat{\mathcal{K}}_g\backslash\{P_\infty\}$, $(z,z_0) \in \mathbb{C}^2$)

$$L_2\psi(P,\cdot,z_0) = E\psi(P,\cdot,z_0), \tag{A.21}$$

$$P_{2g+1}\psi(P,\cdot,z_0) = y\psi(P,\cdot,z_0), \tag{A.22}$$

$$\psi(P,z,z_0)\psi(P^*,z,z_0) = \hat{F}_g(E,z)/\hat{F}_g(E,z_0), \tag{A.23}$$

$$W(\psi(P,\cdot,z_0),\psi(P^*,\cdot,z_0)) = -2y(P)/\hat{F}_g(E,z_0), \tag{A.24}$$

where $W(f,g)(z) = f(z)g'(z) - f'(z)g(z)$ denotes the Wronskian of f and g. Thus, $\psi(P,z,z_0)$ and $\psi(P^*,z,z_0)$ are linearly independent solutions of $L_2\psi = E\psi$ as long as $E \in \mathbb{C}\backslash\{\hat{E}_m\}_{0\leq m\leq 2g}$. The two branches of $\psi(P,z,z_0)$ will be denoted by $\psi_\pm(E,z,z_0)$, respectively.

The above formalism leads to the following standard definition.

Definition .1. *Any solution q of one of the stationary KdV equations* (A.10) *is called an* **algebro-geometric KdV potential**.

For brevity of notation we will occasionally call such q simply KdV potentials.

Finally, denoting $\underline{\hat{E}} = (\hat{E}_0, \dots, \hat{E}_{2g})$, consider

$$\left(\prod_{m=0}^{2g} \left(1 - \frac{\hat{E}_m}{z} \right) \right)^{1/2} = \sum_{k=0}^{\infty} c_k(\underline{\hat{E}}) z^{-k},$$

$$\text{where } c_0(\underline{\hat{E}}) = 1, \quad c_1(\underline{\hat{E}}) = -\frac{1}{2} \sum_{m=0}^{N} \hat{E}_m, \quad \text{etc.}$$

Assuming that q satisfies the gth stationary (nonhomogeneous) KdV equation (A.10), the integration constants c_ℓ in (A.7) become a functional of the $\hat{E}_m$ in the underlying curve (A.17) and one verifies

$$c_\ell = c_\ell(\underline{\hat{E}}), \quad \ell = 0, \dots, g. \tag{A.25}$$

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