A NEW APPROACH TO THE BOUSSINESQ HIERARCHY

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Abstract. We develop a new systematic approach to the Boussinesq (Bsq) hierarchy based on elementary algebraic methods. In particular, we recursively construct Lax pairs for the Bsq hierarchy by introducing a fundamental polynomial formalism and establish the basic algebro-geometric setting including associated Burchnall-Chaundy curves, Baker-Akhiezer functions, trace formulas, and Dubrovin-type equations for Dirichlet and Neumann divisors.

1. Introduction

The principal purpose of this paper is to lay the foundation for an elementary algebraic approach to the entire Boussinesq (Bsq) hierarchy in the spirit of previous treatments of the Korteweg-de Vries (KdV) and Toda hierarchies. More precisely, we introduce a fundamental polynomial formalism to recursively construct Lax pairs for the Bsq hierarchy, that is, pairs(L_3, P_r) of differential expressions of order three (i.e., L_3) and $r \in \mathbb{N}$ (i.e., P_r) with $r \not\equiv 0 \pmod 3$, which closely resemble the corresponding Lax pairs (L_2, P_r), $r \not\equiv 0 \pmod 2$ for the KdV hierarchy. In addition, we establish the basic algebro-geometric setup for special classes of solutions of the Bsq hierarchy such as solitons, rational solutions, algebro-geometric quasi-periodic solutions, and limiting cases thereof. Our treatment includes a systematic approach to Burchnall-Chaundy curves, Baker-Akhiezer functions, trace formulas, and Dubrovin-type equations describing the dynamics of Dirichlet and Neumann divisors.

Before we enter into a description of the content of each section, it seems appropriate to describe some of the existing literature on the subject and its relation to our approach in order to justify the addition of yet another extensive account on this topic. Despite a fair number of papers on the Boussinesq system, the current status of research has not yet reached the high level of the KdV hierarchy, or more generally, that of the AKNS hierarchy. From the perspective of completely integrable systems, the reasons for this discrepancy are easily traced back to the enormously increased complexity when making the step from the second-order operator L_2 associated with the KdV hierarchy to the third-order operator L_3 in connection with the Bsq hierarchy. On an algebro-geometrical level this difference amounts to hyperelliptic curves in the KdV (and AKNS) context as opposed to non-hyperelliptic ones in the Bsq, and more generally, higher-order Gelfand-Dickey case (for genus $g > 2$ curves). The reader will get a first hand impression of these differences in complexity when comparing Sections 2–4 and 5–7 in the KdV and Bsq cases, respectively.

The classical paper on the Bsq equation, or perhaps more appropriately, the nonlinear string equation, is due to Zakharov [46]. In particular, he introduced the basic Lax pair (L_3, P_2) and discussed the infinite set of polynomial integrals of motion. In many ways closest in spirit to our approach is the seminal paper by McKean [37] (see also

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[36]) describing spatially periodic solutions of the Bsq equation. In contrast to [37] though, we concentrate here on the algebro-geometric (i.e., finite-genus) case and make no assumptions of periodicity in order to make our formalism equally applicable to solitons, rational solutions, as well as certain classes of quasi-periodic solutions. The application of inverse scattering techniques for the third-order differential expression L_3 to the initial value problem of the Bsq equation is discussed in great detail by Deift, Tomei, and Trubowitz [15] and Beals, Deift, and Tomei [7]. General existence theorems (local and global in time) for solutions of the Bsq equation can also be found, e.g., in Craig [14] and Bona and Sachs [8] and the references therein. In particular, [7], [8], [14], [15], and [37] further discuss and contrast the blow-up mechanism for solutions of the nonlinear string equation obtained by Kalantarov and Ladyzhenskaya [31]. Other special classes of solutions have been considered by a variety of authors. For instance, certain classes of rational Bsq solutions are treated by Airault [3], Airault, McKean, and Moser [4], Chudnovsky [13], and Latham and Previato [32]. Moreover, certain algebro-geometric Bsq solutions, obtained as special solutions of the Kadomtsev-Petviashvili (KP) equation or by the reduction theory of Riemann theta functions, are briefly discussed by Dubrovin [19], Matveev and Smirnov [33], [34], Previato [40], [41], and Smirnov [44].

Next we describe the content of this paper. Since the inevitable complexity of the Bsq formalism tends to cloud the simplicity of the basic ideas involved, we decided to include a corresponding treatment of the KdV hierarchy in Section 2–4, especially since the latter case is by far the most transparent one within the Gelfand-Dickey hierarchy. Following Al'ber [5], [6] (see also [16], Ch. 12, [21]) we describe a recursive approach to Lax pairs of the KdV hierarchy in Section 2 and establish its connection with the Burchnall-Chaundy theory [10], [11], [12] and hence with hyperelliptic curves branched at infinity. Combining the recursive formalism of Section 2 with a polynomial approach to represent positive divisors of degree n on a hyperelliptic curve of genus n originally developed by Jacobi [30] and applied to the KdV case by Mumford [39], Section III.a.1 and McKean [38], a detailed analysis of the stationary KdV hierarchy is provided in Section 3. The corresponding time-dependent formalism of the KdV hierarchy is then developed in Section 4. Our presentation of Sections 2–4 follows the one in [28]. In Section 5 we develop a recursive approach to Lax pairs of the Bsq hierarchy and establish its connection with Burchnall-Chaundy curves (which are non-hyperelliptic for genus larger than one). The recursive approach of Section 5 is then combined with a fundamental polynomial approach (in the spirit of Jacobi's treatment of the hyperelliptic case in Section 3) to represent positive divisors of degree n on Bsq curves of genus n in order to analyze the stationary Bsq hierarchy in Section 6. Section 7 then extends this analysis to the time-dependent Bsq hierarchy. The content of Sections 5–7 represents our principal new contribution in this paper. Finally, Section 8 collects a variety of explicit examples illustrating the KdV and Bsq formalisms.

In order to keep this paper within a reasonable length we have refrained from including explicit theta-function representations of algebro-geometric Bsq solutions (in the case where the corresponding Burchnall-Chaundy curves are nonsingular). We shall return to this task in a subsequent paper [17].

It should perhaps be noted at this point that our elementary algebraic approach to the KdV and Bsq hierarchies is in fact universally applicable to 1+1-dimensional hierarchies such as the AKNS hierarchy (see, [22]) and the Toda and Kac-van Moerbeke hierarchies (cf. [9]).

Finally, we mention that a combination of the Bsq formalism developed in this paper and the Picard-type techniques introduced in a recent explicit characterization of all elliptic solutions of the KdV hierarchy in [26] (see also [25]) are expected to yield a similar characterization of all elliptic solutions of the Bsq hierarchy, a topic that continues to attract considerable interest (see, e.g., [33], [34], [40], [41], [44]).

2. The Recursive Approach to the KdV Hierarchy and Hyperelliptic Curves

Following the treatment in [28] we present in this section the recursive approach to Lax pairs of the KdV hierarchy and its connection with the Burchnall-Chaundy theory [10], [11], [12] and hence with hyperelliptic curves branched at infinity. Originally, this approach was advocated by Al'ber [5], [6] (see also [16], Ch. 12, [21], [22]–[23]).

Suppose $q_0 \in C^\infty(\mathbb{R})$ (or q_0 meromorphic on $\mathbb{C}$) and introduce the second-order differential expression

$$L_2 = \frac{d^2}{d\,x^2} + q_0(x), \quad x \in \mathbb{R} \ \ (\text{or} \ \ \mathbb{C}). \tag{2.1}$$

In order to explicitly construct odd-order differential expressions $P_r, r \not\equiv 0 \pmod 2$ commuting with L_2, that will be used later to define the stationary KdV hierarchy, one proceeds as follows.

Pick $n \in \mathbb{N}_0 (= \mathbb{N} \cup \{0\})$ and define $\{f_\ell(x)\}_{0 \le \ell \le n+1}$ recursively by

$$\begin{aligned} f_0 &= 1, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.2)\\ 2f_{\ell,x}(x) &= \frac{1}{2}\, f_{\ell-1,xxx}(x) + 2\, q_0(x)\, f_{\ell-1,x}(x) + q_{0,x}(x)\, f_{\ell-1}(x), \quad 1 \le \ell \le n+1. \end{aligned}$$

Explicitly, one computes

$$f_0 = 1, \quad f_1 = \frac{1}{2}\, q_0 + c_1, \quad f_2 = \frac{1}{8}\, q_{0,xx} + \frac{3}{8}\, q_0^2 + c_1\, \frac{1}{2}\, q_0 + c_2, \quad \text{etc.,} \tag{2.3}$$

where $\{c_\ell\}_{1 \le \ell \le n}$ are integration constants. Given (2.2), one defines the differential expression of order r by

$$\begin{aligned} P_r = \sum_{\ell=0}^{n} \Big[-\frac{1}{2}\, f_{n-\ell,x} + f_{n-\ell} \frac{d}{d\,x} \Big] L_2^\ell + \sum_{\ell=0}^{n} k_{r,\ell} L_2^\ell, \quad k_{r,\ell} \in \mathbb{C}, \ \ 0 \le \ell \le n,&\\ r = 2\,n+1, \ \ n \in \mathbb{N}_0, \qquad (2.4)& \end{aligned}$$

and verifies

$$[P_r, L_2] = 2f_{n+1,x}, \quad r = 2\,n+1, \ n \in \mathbb{N}_0 \tag{2.5}$$

(where $[\ .\ ,\ .\ ]$ denotes the commutator symbol). The pair (L_2, P_r) represents the celebrated Lax pair for the KdV hierarchy. Varying $n \in \mathbb{N}_0$, the stationary KdV hierarchy is then defined by the vanishing of the commutators of P_r and L_2 in (2.5), that is, by

$$[P_r, L_2] = 0, \quad r = 2\,n+1, \ n \in \mathbb{N}_0, \tag{2.6}$$

or equivalently, by

$$f_{n+1,x} = 0, \quad n \in \mathbb{N}_0. \tag{2.7}$$

Explicitly, one obtains for the first few equations of the stationary KdV hierarchy

$$q_{0,x} = 0,$$

$$\frac{1}{4}\,q_{0,xxx} + \frac{3}{2}\,q_0 q_{0,x} + c_1\, q_{0,x} = 0,$$
$$\frac{1}{16}\,q_{0,xxxxx} + \frac{5}{8}\,q_0\, q_{0,xxx} + \frac{5}{4}\,q_{0,x}\, q_{0,xx} + \frac{15}{8}\,q_0^2\, q_{0,x} + c_1\big(\frac{1}{4}\,q_{0,xxx} + \frac{3}{2}\,q_0\, q_{0,x}\big) + c_2\, q_{0,x} = 0, \tag{2.8}$$
etc.

By definition, solutions $q_0(x)$ of any of the stationary KdV equations (2.8) are called **algebro-geometric finite-gap potentials** associated with the KdV hierarchy. If $f_{n+1,x} = 0$, one also calls q_0 a stationary n-gap solution.

Next, we introduce the polynomial F_r of degree n with respect to $z \in \mathbb{C}$,

$$F_r(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}(x) z^{\ell}, \quad f_0 = 1, \quad r = 2\,n + 1. \tag{2.9}$$

Explicitly, the first few polynomials F_r read

$$F_1 = 1,$$
$$F_3 = z + \big(\frac{1}{2}\,q_0 + c_1\big), \tag{2.10}$$
$$F_5 = z^2 + \big(\frac{1}{2}\,q_0 + c_1\big)z + \big(\frac{1}{8}\,q_{0,xx} + \frac{3}{8}\,q_0^2 + c_1\,\frac{1}{2}\,q_0 + c_2\big),$$
etc.

Given (2.9), (2.6) respectively, (2.7) becomes

$$\frac{1}{2}\,F_{r,xxx} - 2(z - q_0)\, F_{r,x} + q_{0,x}\, F_r = 0. \tag{2.11}$$

Multiplying (2.11) by F_r and integrating once results in

$$R_r(z) = -\frac{1}{2}\,F_{r,xx}\,F_r + \frac{1}{4}F_{r,x}^2 + (z - q_0)\,F_r^2, \tag{2.12}$$

where the integration constant $R_r(z)$ is seen to be a monic polynomial in z of degree $2\,n + 1$. Thus we may write

$$R_r(z) = \prod_{m=0}^{2n} (z - E_m), \qquad \{E_m\}_{0 \le m \le 2n} \subset \mathbb{C}. \tag{2.13}$$

Next, we consider the kernel (i.e., the formal null space in a purely algebraic sense) of $(L_2 - z), z \in \mathbb{C}$,

$$(L_2 - z)\psi = 0, \quad \psi = \psi(z,x), \quad z \in \mathbb{C} \tag{2.14}$$

and, taking into account (2.6), that is, $[P_r, L_2] = 0$, compute the restriction of P_r to the $\ker(L_2 - z)$. Using

$$\psi_{xx} = (z - q_0)\psi, \quad \psi_{xxx} = (z - q_0)\psi_x - q_{0,x}\psi, \quad \text{etc.}, \tag{2.15}$$

to eliminate higher-order derivatives of ψ, one obtains from (2.2), (2.4), (2.7), (2.9), and (2.11),

$$P_r\Big|_{\ker(L_2 - z)} = \Big[F_r(z,x)\frac{d}{dx} + G_r(z,x)\Big]\Big|_{\ker(L_2 - z)}, \tag{2.16}$$

where

$$G_r(z,x) = -\frac{1}{2}F_{r,x} + k_r(z), \tag{2.17}$$

and (cf. (2.4))

$$k_r(z) = \sum_{\ell=0}^{n} k_{r,\ell} z^\ell. \tag{2.18}$$

The construction of P_r in (2.4) and (2.16) should be contrasted with the one based on formal pseudo-differential expressions originally developed by Gel'fand-Dickey [20] and further refined by Adler [2] (see also [16], Ch. 1).

Still assuming $f_{n+1,x} = 0$ as in (2.7), $[P_r, L_2] = 0$ in (2.6) yields an algebraic relationship between P_r and L_2 by a celebrated result of Burchnall and Chaundy [10], [11], [12] (see also [45]). The following theorem gives a detailed account of this relationship.

Theorem 2.1. *Assume $f_{n+1,x} = 0$, that is $[P_r, L_2] = 0$ for some $r = 2n+1$, $n \in \mathbb{N}_0$. Then the Burchnall-Chaundy polynomial $\mathcal{F}_{(r-1)/2}(L_2, P_r)$ of the pair (L_2, P_r) explicitly reads (cf. (2.13) and (2.18))*

$$\mathcal{F}_{(r-1)/2}(L_2, P_r) = \Big(P_r - k_r(L_2)\Big)^2 - R_r(L_2) = 0, \tag{2.19}$$

$$k_r(z) = \sum_{\ell=0}^{n} k_{r,\ell} z^\ell, \quad R_r(z) = \prod_{m=0}^{2n} (z - E_m), \quad z \in \mathbb{C}, \quad r = 2n+1, \ n \in \mathbb{N}_0.$$

Proof. Let $\psi_j \in \ker(L_2 - z)$, $j = 1,2$ be linearly independent. Since $[P_r, L_2] = 0$, one can represent P_r as a 2×2 matrix $\mathcal{P}_r(z)$ on $\ker(L_2 - z)$,

$$P_r \psi_j = \sum_{k=1}^{2} \mathcal{P}_{r,j,k} \psi_k, \tag{2.20}$$

$$\mathcal{P}_r \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \mathcal{P}_{r,1,1} & \mathcal{P}_{r,1,2} \\ \mathcal{P}_{r,2,1} & \mathcal{P}_{r,2,2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\mathcal{P}_{r,1,j} = \frac{W(P_r\psi_j, \psi_2)}{W(\psi_1, \psi_2)}, \quad \mathcal{P}_{r,2,j} = \frac{W(\psi_1, P_r\psi_j)}{W(\psi_1, \psi_2)}, \quad 1 \le j \le 2. \tag{2.21}$$

Using (2.11) and (2.15)–(2.18) one verifies

$$\mathrm{tr}(\mathcal{P}_r(z)) = 2\, k_r(z), \tag{2.22}$$

$$\det(\mathcal{P}_r(z)) = \frac{W(P_r\psi_1(z), P_r\psi_2(z))}{W(\psi_1(z), \psi_2(z))} = k_r(z)^2 - R_r(z). \tag{2.23}$$

(Here tr(.) and det(.) denote the trace and determinant, respectively and $W(f,g) = fg' - f'g$ denotes the Wronskian of f and g. The characteristic polynomial $\det(y - \mathcal{P}_r(z)) = 0$ of $\mathcal{P}_r(z)$ then yields

$$\mathcal{F}_{(r-1)/2}(z,y) = y^2 - y\ \mathrm{tr}(\mathcal{P}_r(z)) + \det(\mathcal{P}_r(z)) = \big(y - k_r(z)\big)^2 - R_r(z) = 0. \tag{2.24}$$

The result (2.19) then follows from the Cayley-Hamilton theorem, since $z \in \mathbb{C}$ is arbitrary. ☐

Remark 2.2. *Equation (2.24) naturally leads to the (possibly singular) hyperelliptic curve $\mathcal{K}_{(r-1)/2}$,*

$$\mathcal{K}_{(r-1)/2}: \quad \mathcal{F}_{(r-1)/2}(z,y) = \big(y - k_r(z)\big)^2 - R_r(z) = 0, \tag{2.25}$$

$$k_r(z) = \sum_{\ell=0}^{n} k_{r,\ell} z^\ell, \quad R_r(z) = \prod_{m=0}^{2n} (z - E_m), \quad r = 2n+1, \ n \in \mathbb{N}_0$$

of (arithmetic) genus $n = (r-1)/2$. In the nonsingular case, where $E_m \neq E_{m'}$ for $m \neq m'$, the Riemann theta function associated with (the one-point compactification of) $\mathcal{K}_{(r-1)/2}$ then yields an explicit expression for $q_0(x)$ originally derived by Its and Matveev [29].

For specific examples illustrating the preceding formalism we refer to Section 8.

Finally, introducing a deformation parameter $t_r \in \mathbb{R}$ in q_0 (i.e., $q_0(x) \to q_0(x, t_r)$), the time-dependent KdV hierarchy is defined as the collection of evolution equations (varying $r \in 2\,\mathbb{N}_0 + 1$),

$$\frac{d}{dt_r} L_2(t_r) - [P_r(t_r), L_2(t_r)] = 0, \quad (x, t_r) \in \mathbb{R}^2, \quad r = 2\,n+1, \quad n \in \mathbb{N}_0, \tag{2.26}$$

or equivalently, by

$$\mathrm{KdV}_r(q_0) = q_{0,t_r} - 2 f_{n+1,x} = 0, \quad (x, t_r) \in \mathbb{R}^2, \quad r = 2\,n+1, \quad n \in \mathbb{N}_0, \tag{2.27}$$

that is, by

$$\begin{aligned} \mathrm{KdV}_r(q_0) = q_{0,t_r} - \frac{1}{2} F_{r,xxx} + 2\,(z - q_0)\,F_{r,x} - q_{0,x}\,F_r = 0,& \\ (x, t_r) \in \mathbb{R}^2, \quad r = 2\,n+1, \quad n \in \mathbb{N}_0.& \end{aligned} \tag{2.28}$$

Explicitly, one obtains for the first few equations in (2.27),

$$\begin{aligned} &\mathrm{KdV}_1(q_0) = q_{0,t_1} - q_{0,x} = 0, \\ &\mathrm{KdV}_3(q_0) = q_{0,t_3} - \frac{1}{4}\, q_{0,xxx} - \frac{3}{2}\, q_0\, q_{0,x} - c_1\, q_{0,x} = 0, \\ &\mathrm{KdV}_5(q_0) = q_{0,t_5} - \frac{1}{16}\, q_{0,xxxxx} - \frac{5}{8}\, q_0\, q_{0,xxx} - \frac{5}{4}\, q_{0,x}\, q_{0,xx} - \frac{15}{8}\, q_0^2\, q_{0,x} \\ &\qquad -c_1\,\Big(\frac{1}{4}\, q_{0,xxx} + \frac{3}{2}\, q_0\, q_{0,x}\Big) - c_2\, q_{0,x} = 0, \end{aligned} \tag{2.29}$$

etc.

Remark 2.3. *We chose to start by postulating the recursion relation (2.2) and then developed the whole formalism based on (2.2), (2.4)–(2.6). Alternatively, one could have started from*

$$(L_2 - z)\psi(P) = 0, \quad (P_r - y(P))\psi(P) = 0, \quad P = (z, y(P)) \in \mathcal{K}_{(r-1)/2} \backslash \{P_\infty\} \tag{2.30}$$

and obtained the recursion relation (2.2) and the remaining stationary results of this section as a consequence of (2.9) and (2.16). Similarly, starting with

$$(L_2 - z)\psi(P, t_r) = 0, \quad \Big(\frac{\partial}{\partial t_r} - P_r\Big)\psi(P, t_r) = 0, \quad t_r \in \mathbb{R}, \tag{2.31}$$

one infers the time-dependent results (2.26)–(2.29).

3. The Stationary KdV Formalism

In this section we continue our discussion of the KdV hierarchy and focus our attention on the stationary case. Following [28] we outline the connections between the polynomial approach described in Section 2 and a fundamental meromorphic function $\phi(P,x)$ defined on the hyperelliptic curve $\mathcal{K}_{(r-1)/2}$ in (2.25). Moreover, we discuss in some detail the associated stationary Baker-Akhiezer function $\psi(P,x,x_0)$, the common eigenfunction of L_2 and P_r (we recall that $[P_r, L_2] = 0$), and associated positive (Dirichlet and Neumann) divisors of degree $(r-1)/2$ on $\mathcal{K}_{(r-1)/2}$. The latter topic was originally developed by Jacobi [30] and applied to the KdV case by Mumford [39], Section III.a.1 and McKean [38].

We recall the hyperelliptic curve $\mathcal{K}_{(r-1)/2}$ in (2.25),

$$\mathcal{K}_{(r-1)/2}: \quad \mathcal{F}_{(r-1)/2}(z,y) = \big(y - k_r(z)\big)^2 - R_r(z) = 0, \tag{3.1}$$
$$k_r(z) = \sum_{\ell=0}^{n} k_{r,\ell} z^\ell, \quad R_r(z) = \prod_{m=0}^{2n} (z - E_m),$$

where $r \in 2\,\mathbb{N}_0 + 1$ will be fixed throughout this section and denote its one-point compactification (joining the branch point P_∞) by the same symbol $\mathcal{K}_{(r-1)/2}$. (In the following $\mathcal{K}_{(r-1)/2}$ will always denote the compactified curve.) Thus $\mathcal{K}_{(r-1)/2}$ becomes a (possibly singular) two-sheeted hyperelliptic Riemann surface of arithmetic genus $(r-1)/2$ in a standard manner. We now introduce a bit more notation in this context. Points P on $\mathcal{K}_{(r-1)/2}$ are represented as pairs $P = (z, y(P))$ satisfying (3.1) together with $P_\infty = (\infty, \infty)$, the point at infinity. The complex structure on $\mathcal{K}_{(r-1)/2}$ is defined in the usual way by introducing local coordinates $\zeta_{P_0}: P \to (z - z_0)$ near points $P_0 \in \mathcal{K}_{(r-1)/2}$ which are neither branch nor singular points of $\mathcal{K}_{(r-1)/2}$, $\zeta_{P_\infty}: P \to 1/z^{1/2}$ near the branch point $P_\infty \in \mathcal{K}_{(r-1)/2}$ (with an appropriate determination of the branch of $z^{1/2}$) and similarly at branch and/or singular points of $\mathcal{K}_{(r-1)/2}$. The holomorphic sheet exchange map (involution) $*$ is defined by

$$*: \begin{cases} \mathcal{K}_{(r-1)/2} \to \mathcal{K}_{(r-1)/2} \\ P = (z, y_j(z)) \to P^* = (z, y_{j+1 \ (\mathrm{mod}\ 2)}(z)), \quad j = 1,2 \end{cases}, \tag{3.2}$$

where $y_j(z)$, $j = 1,2$ denote the two branches of $y(P)$ satisfying $\mathcal{F}_{(r-1)/2}(z,y) = 0$, that is,

$$(y - y_1(z))\,(y - y_2(z)) = (y - k_r(z))^2 - R_r(z) = 0. \tag{3.3}$$

Finally, positive divisors on $\mathcal{K}_{(r-1)/2}$ of degree $n = (r-1)/2$ are denoted by

$$\mathcal{D}_{P_1,\dots,P_n}: \begin{cases} \mathcal{K}_{(r-1)/2} \to \mathbb{N}_0 \\ P \to \mathcal{D}_{P_1,\dots,P_n}(P) = \begin{cases} m \text{ if } P \text{ occurs } m \text{ times in } \{P_1,\dots,P_n\} \\ 0 \text{ if } P \notin \{P_1,\dots,P_n\} \end{cases} \end{cases}. \tag{3.4}$$

Given these preliminaries, let $\psi(P,x,x_0)$ denote the common normalized (cf. (3.8)) eigenfunction of L_2 and P_r, whose existence is guaranteed by the commutativity of L_2 and P_r (cf., e.g., [10], [11]), that is, by

$$[P_r, L_2] = 0, \qquad r = 2\,n + 1 \tag{3.5}$$

for a given $n \in \mathbb{N}_0$, or equivalently, by the requirement,

$$f_{n+1,x} = 0. \tag{3.6}$$

Explicitly, this yields

$$L_2\psi(P,x,x_0) = z\,\psi(P,x,x_0), \quad P_r\psi(P,x,x_0) = y(P)\,\psi(P,x,x_0), \tag{3.7}$$
$$P=(z,y(P))\in\mathcal{K}_{(r-1)/2}\setminus\{P_\infty\}, \quad x\in\mathbb{R}$$

for some fixed $x_0\in\mathbb{R}$ with the assumed normalization,

$$\psi(P,x_0,x_0)=1, \qquad P\in\mathcal{K}_{(r-1)/2}\setminus\{P_\infty\}. \tag{3.8}$$

$\psi(P,x,x_0)$ is called the stationary Baker-Akhiezer (BA) function of the KdV hierarchy. Closely related to $\psi(P,x,x_0)$ is the following meromorphic function $\phi(P,x)$ on $\mathcal{K}_{(r-1)/2}$ defined by

$$\phi(P,x)=\frac{\psi_x(P,x,x_0)}{\psi(P,x,x_0)}, \quad P\in\mathcal{K}_{(r-1)/2},\ x\in\mathbb{R}, \tag{3.9}$$

such that

$$\psi(P,x,x_0)=\exp\Big[\int_{x_0}^{x} d\,x'\phi(P,x')\Big], \qquad P\in\mathcal{K}_{(r-1)/2}\setminus\{P_\infty\}. \tag{3.10}$$

Since $\phi(P,x)$ is a fundamental object for the stationary KdV hierarchy we next seek its connection with the recursion formalism of Section 2. Recalling (2.16) and (2.17), one infers

$$P_r\psi = F_r\psi_x + \big(-\frac{1}{2}F_{r,x}+k_r\big)\psi = y\,\psi \tag{3.11}$$

and

$$\big(P_r\psi\big)_x \;=\; \big(\frac{1}{2}F_{r,x}+k_r\big)\psi_x + \big((z-q_0)F_r-\frac{1}{2}F_{r,xx}\big)\psi = y\,\psi_x \tag{3.12}$$

using (2.15). Thus

$$\phi=\frac{\psi_x}{\psi}=\frac{y-k_r+\frac{1}{2}F_{r,x}}{F_r}=\frac{(z-q_0)F_r-\frac{1}{2}F_{r,xx}}{y-k_r-\frac{1}{2}F_{r,x}}. \tag{3.13}$$

Introducing

$$D_n(z,x) \;=\; F_r(z,x), \tag{3.14}$$
$$N_{n+1}(z,x) \;=\; (z-q_0)F_r(z,x)-\frac{1}{2}F_{r,xx}(z,x) \tag{3.15}$$

then yields

$$\phi(P,x) \;=\; \frac{y(P)-k_r(z)+\frac{1}{2}D_{n,x}(z,x)}{D_n(z,x)} \tag{3.16}$$
$$\;=\; \frac{N_{n+1}(z,x)}{y(P)-k_r(z)-\frac{1}{2}D_{n,x}(z,x)}, \tag{3.17}$$
$$P=(z,y(P))\in\mathcal{K}_{(r-1)/2}$$

and

$$D_n(z,x)\,N_{n+1}(z,x)=\big(y(P)-k_r(z)\big)^2-\frac{1}{4}D_{n,x}(z,x)^2. \tag{3.18}$$

In order to motivate our introduction of the basic quantity $\phi(P,x)$ we started with the common eigenfunction $\psi(P,x,x_0)$ of L_2 and P_r. However, given (2.12) and the definitions (3.14), (3.15), we could have defined $\phi(P,x)$ as in (3.16) and then verified that $\psi(P,x,x_0)$ in (3.10) satisfies (3.7) and (3.8). Since by (2.9) D_n and N_{n+1} are monic polynomials with respect to z of degree n and $n+1$ respectively, we may write

$$D_n(z,x) = \prod_{j=1}^{n}[z-\mu_j(x)], \tag{3.19}$$

$$N_{n+1}(z,x) = \prod_{\ell=0}^{n}[z-\nu_\ell(x)]. \tag{3.20}$$

Defining

$$\hat{\mu}_j(x) = \big(\mu_j(x), y(\hat{\mu}_j(x))\big) = \big(\mu_j(x), k_r(\mu_j(x)) + \frac{1}{2}D_{n,x}(\mu_j(x),x)\big) \in \mathcal{K}_{(r-1)/2}, \quad 1 \le j \le n, \quad x \in \mathbb{R}, \tag{3.21}$$

$$\hat{\nu}_\ell(x) = \big(\nu_\ell(x), y(\hat{\nu}_\ell(x))\big) = \big(\nu_\ell(x), k_r(\nu_\ell(x)) - \frac{1}{2}D_{n,x}(\nu_\ell(x),x)\big) \in \mathcal{K}_{(r-1)/2}, \quad 0 \le \ell \le n, \quad x \in \mathbb{R}, \tag{3.22}$$

one infers from (3.16) and (3.17) that the divisor $\big(\phi(P,x)\big)$ of $\phi(P,x)$ is given by

$$\big(\phi(P,x)\big) = \mathcal{D}_{\hat{\nu}_0(x),\dots,\hat{\nu}_n(x)}(P) - \mathcal{D}_{P_\infty,\hat{\mu}_1(x),\dots,\hat{\mu}_n(x)}(P). \tag{3.23}$$

Here we used our convention (3.4) and the additive notation for divisors. Equivalently, $\hat{\nu}_0(x),\dots,\hat{\nu}_n(x)$ are the $n+1$ zeros of $\phi(P,x)$ and $P_\infty, \hat{\mu}_1(x),\dots,\hat{\mu}_n(x)$ its $n+1$ poles.

Further properties of $\phi(P,x)$ and $\psi(P,x,x_0)$ are summarized in

Lemma 3.1. *Assume (3.5)–(3.9), $P=(z,y(P)) \in \mathcal{K}_{(r-1)/2} \setminus \{P_\infty\}$, $r = 2n+1$, and let $(z,x,x_0) \in \mathbb{C}\times\mathbb{R}^2$. Then*

(i). $\phi(P,x)$ *satisfies the Riccati-type equation*

$$\phi_x(P,x) + \phi(P,x)^2 = z - q_0(x). \tag{3.24}$$

(ii). $$\phi(P,x)\,\phi(P^*,x) = -\frac{N_{n+1}(z,x)}{D_n(z,x)}. \tag{3.25}$$

(iii). $$\phi(P,x) + \phi(P^*,x) = \frac{D_{n,x}(z,x)}{D_n(z,x)}. \tag{3.26}$$

(iv). $$\phi(P,x) - \phi(P^*,x) = \frac{2\,(y(P)-k_r(z))}{D_n(z,x)}, \tag{3.27}$$

$$\big(y(P)-k_r(z)\big)\phi(P,x) + \big(y(P^*)-k_r(z)\big)\phi(P^*,x) = \frac{2\,R_r(z)}{D_n(z,x)}. \tag{3.28}$$

(v). $$\psi(P,x,x_0)\,\psi(P^*,x,x_0) = \frac{D_n(z,x)}{D_n(z,x_0)}. \tag{3.29}$$

(vi). $$\psi_x(P,x,x_0)\psi_x(P^*,x,x_0) = -\frac{N_{n+1}(z,x)}{D_n(z,x_0)}. \tag{3.30}$$

(vii). $$\psi(P,x,x_0) = \left[\frac{D_n(z,x)}{D_n(z,x_0)}\right]^{1/2} \exp\Big\{\big[y(P)-k_r(z)\big]\int_{x_0}^{x} d\,x' D_n(z,x')^{-1}\Big\}. \tag{3.31}$$

(viii). $$N_{n+1,x}(z,x) = -[z-q_0(x)]D_{n,x}(z,x). \tag{3.32}$$

Proof. (3.24) follows from $\phi = \psi_x/\psi$ and $\psi_{xx} = (z - q_0)\psi$. (3.25)–(3.28) follow from (3.16), (3.18), and

$$y(P) + y(P^*) = 2\, k_r(z), \qquad y(P)\, y(P^*) = k_r(z)^2 - R_r(z). \tag{3.33}$$

(3.29) follows from (3.31) and (3.25) and (3.30) from (3.29) and (3.25). In order to prove (3.31) it suffices to insert (3.17) into (3.9). (3.32) finally follows by differentiating (3.18) with respect to x (using (2.12) and (3.14)) and checking the resulting equation at the $n+1$ zeros $\nu_\ell(x)$ of $N_{n+1}(z,x)$. $\square$

A comparison of (3.19), (3.20) and (3.29), (3.30) reveals that $\mathcal{D}_{P_\infty,\hat\mu_1(x),\dots,\hat\mu_n(x)}$ and $\mathcal{D}_{\hat\nu_0(x),\dots,\hat\nu_n(x)}$ in (3.23) are the Dirichlet and Neumann divisors associated with $L_2 = \frac{d}{dx^2} + q_0(x)$ (see [28] for further spectral interpretations in this context). In particular, (3.25), (3.29), and (3.30) clarify the role played by D_n and N_{n+1}. Up to normalizations, D_n represents the product of the two branches of ψ and N_{n+1} the product of the two branches of ψ_x, their zeros represent Dirichlet and Neumann eigenvalues of L_2 with the corresponding boundary conditions imposed at the point $x \in \mathbb{R}$.

The reader puzzled by our definition (3.14) might compare with (6.16) in the Bsq case where F_r and D_n considerably differ from each other but the analogs of (3.25), (3.29), and (3.30) remain valid as can be seen from (6.41), (6.44), and (6.45). Using the hyperelliptic curve (3.1) we could have replaced $y - k_r(z)$ by $R_r(z)^{1/2}$ in (3.13), (3.16), (3.17) and (3.18). However, a quick look at (6.33) reveals that the polynomial behavior of the numerator and denominator of $\phi(P,x)$ with respect to y in (3.13), (3.16), and (3.17) is the key in generalizing this formalism from the KdV to the Bsq case.

Returning to $D_n(z,x)$ and $N_{n+1}(z,x)$ we note that (2.2), (2.9), (3.14), and (3.15) yield

$$\begin{aligned}
&D_0 = 1,\\
&D_1 = z + \frac{1}{2}\, q_0 + c_1,\\
&D_2 = z^2 + \big(\frac{1}{2}\, q_0 + c_1\big) z + \frac{1}{8}\, q_{0,xx} + \frac{3}{8}\, q_0^2 + c_1 \frac{1}{2}\, q_0 + c_2,\\
&\text{etc.}
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
&N_1 = z - q_0,\\
&N_2 = z^2 + \big(-\frac{1}{2}\, q_0 + c_1\big) z - \frac{1}{4}\, q_{0,xx} - \frac{1}{2}\, q_0^2 - c_1\, q_0,\\
&N_3 = z^3 + \big(-\frac{1}{2}\, q_0 + c_1\big) z^2 + \big(-\frac{1}{8}\, q_{0,xx} - \frac{1}{8}\, q_0^2 - c_1 \frac{1}{2}\, q_0 + c_2\big) z\\
&\qquad - \frac{1}{16}\, q_{0,xxxx} - \frac{3}{8}\, q_0^3 - \frac{3}{8}\, q_{0,x}^2 - \frac{1}{2}\, q_0 q_{0,xx} - c_1 \frac{1}{4}\, q_{0,xx} - c_1 \frac{1}{2}\, q_0^2 - c_2 q_0,\\
&\text{etc.}
\end{aligned} \tag{3.35}$$

Concerning the dynamics of the zeros $\mu_j(x)$ and $\nu_\ell(x)$ of $D_n(z,x)$ and $N_{n+1}(z,x)$ one obtains the following equations first derived by Dubrovin [18] in the Dirichlet case.

Lemma 3.2. *Assume (3.6), (3.19), (3.20) and let $x \in \mathbb{R}$. Then*

$$(i).\quad \mu_{j,x}(x) = \frac{-2\left(y(\hat{\mu}_j(x)) - k_r(\mu_j(x))\right)}{\prod\limits_{\substack{k=1\\k\neq j}}^{n}[\mu_j(x) - \mu_k(x)]}, \qquad 1 \le j \le n. \tag{3.36}$$

$$(ii).\quad \nu_{\ell,x}(x) = \frac{-2\left[\nu_\ell(x) - q_0(x)\right]\left(y(\hat{\nu}_j(x)) - k_r(\nu_j(x))\right)}{\prod\limits_{\substack{m=0\\m\neq \ell}}^{n}\left[\nu_\ell(x) - \nu_m(x)\right]}, \qquad 0 \le \ell \le n. \tag{3.37}$$

Proof. (3.36) is clear from (3.19) and (3.21), and (3.37) follows from (3.20), (3.22) and (3.32). □

We conclude this section with some hints concerning trace formulas for the KdV invariants in terms of Dirichlet and Neumann data.

Lemma 3.3. *Assume (3.6) and let $x \in \mathbb{R}$. Then*

$$(i).\quad \frac{1}{2}\, q_0(x) + c_1 = -\sum_{j_1=1}^{n} \mu_{j_1}(x),$$

$$\frac{1}{8}\, q_{0,xx}(x) + \frac{3}{8}\, q_0(x)^2 + c_1 \frac{1}{2}\, q_0(x) + c_2 = \sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{n} \mu_{j_1}(x)\, \mu_{j_2}(x), \tag{3.38}$$

etc.

$$(ii).\quad \frac{1}{2}\, q_0(x) - c_1 = \sum_{\ell_1=0}^{n} \nu_{\ell_1}(x),$$

$$\frac{1}{8}\, q_{0,xx}(x) + \frac{1}{8}\, q_0(x)^2 + c_1\, \frac{1}{2}\, q_0(x) - c_2 = -\sum_{\substack{\ell_1,\ell_2=0\\ \ell_1<\ell_2}}^{n} \nu_{\ell_1}(x)\, \nu_{\ell_2}(x), \tag{3.39}$$

etc.

Here

$$c_1 = -\frac{1}{2}\sum_{m_1=0}^{2n} E_{m_1}, \qquad c_2 = \frac{1}{2}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{2n} E_{m_1}E_{m_2} - \frac{1}{8}\Big(\sum_{m_1=0}^{2n} E_{m_1}\Big)^2, \tag{3.40}$$

etc.

Proof. (3.38) and (3.40) follow by comparison of powers of z substituting (3.19) into (2.9) (taking into account (2.3)) and (2.12) (taking into account(2.13)). (3.39) is proven similarly using (3.1), (3.18)–(3.20), and the fact that $D_{n,x}(z,x)^2 = O(z^{2n-2})$ as $z \to \infty$. □

For a systematic approach to trace formulas based on a second-order nonlinear differential equation satisfied by the diagonal Green's function of L_2 in the Dirichlet case (3.38) and an analogous treatment of the Neumann case (3.39), see [28]. (The latter approach goes far beyond the special algebro-geometric situation presented in this section.)

Explicit examples illustrating the formalism in this section are provided in Section 8.

4. The Time-Dependent KdV Formalism

In our final KdV section we indicate how to generalize the polynomial approach of Sections 2 and 3 to the time-dependent KdV hierarchy. Again we lean on the material presented in [28].

Our starting point is a stationary n-gap solution $q_0^{(0)}(x)$ associated with $\mathcal{K}_n$ satisfying

$$\mathrm{KdV}_{2n+1}(q_0^{(0)}) = -2\, f_{n+1,x} = 0, \quad x \in \mathbb{R} \tag{4.1}$$

for some fixed $n \in \mathbb{N}_0$ and a given set of integration constants $\{c_\ell\}_{1\leq \ell \leq n}$. Our aim is to construct the r-th KdV flow

$$\mathrm{KdV}_r(q_0) = 0, \quad q_0(x, t_{0,r}) = q_0^{(0)}(x), \quad x \in \mathbb{R} \tag{4.2}$$

for some fixed $r \in 2\,\mathbb{N}_0 + 1$ and $t_{0,r} \in \mathbb{R}$. In terms of Lax pairs this amounts to solving

$$\begin{aligned} \frac{d}{d\,t_r} L_2(t_r) - [\tilde{P}_r(t_r), L_2(t_r)] &= 0, \quad t_r \in \mathbb{R}, & (4.3)\\ [P_{2n+1}(t_{0,r}), L_2(t_{0,r})] &= 0. & (4.4)\end{aligned}$$

As a consequence one obtains

$$\begin{aligned} [P_{2n+1}(t_r), L_2(t_r)] &= 0, \qquad t_r \in \mathbb{R}, & (4.5)\\ \Big(P_{2n+1}(t_r) - k_{2n+1}(L_2(t_r))\Big)^2 &= R_{2n+1}(L_2(t_r)) \\ &= \prod_{m=0}^{2n} \big(L_2(t_r) - E_m\big), \qquad t_r \in \mathbb{R} & (4.6)\end{aligned}$$

since the KdV flows are isospectral deformations of $L_2(t_{0,r})$.

We emphasize that the integration constants $\{\tilde{c}_\ell\}$ in $\tilde{P}_r$ and $\{c_\ell\}$ in P_{2n+1} are independent of each other (even if $r = 2n+1$). Hence we shall employ the notation $\tilde{P}_r, \tilde{k}_r, \tilde{F}_r, \tilde{G}_r$, etc. in order to distinguish them from $P_{2n+1}, k_{2n+1}, F_{2n+1}, G_{2n+1}$, etc. In addition, we followed a more elaborate notation inspired by Hirota's τ-function approach and indicated the individual r-th KdV flow by a separate time variable $t_r \in \mathbb{R}$. (The latter notation suggests considering all KdV flows simultaneously by introducing $\underline{t} = (t_1, t_3, t_5, \dots)$.)

Instead of working directly with (4.3) and (4.5), we find it more convenient to take the following equations as our point of departure,

$$q_{0,t_r} = \frac{1}{2}\, \tilde{F}_{r,xxx} - 2(z - q_0)\, \tilde{F}_{r,x} + q_{0,x}\, \tilde{F}_r, \quad (x, t_r) \in \mathbb{R}^2, \tag{4.7}$$

$$-\frac{1}{2}\, F_{2n+1}\, F_{2n+1,xx} + \frac{1}{4}\, F_{2n+1,x}^2 + (z - q_0)\, F_{2n+1}^2 = R_{2n+1}, \quad (x, t_r) \in \mathbb{R}^2, \tag{4.8}$$

where (cf. (2.9))

$$\begin{aligned} F_{2n+1}(z, x, t_r) &= \sum_{\ell=0}^{n} f_{n-\ell}(x, t_r) z^\ell, \\ F_{2n+1}(z, x, t_{0,r}) &= F_{2n+1}^{(0)}(z, x) = \sum_{\ell=0}^{n} f_{n-\ell}^{(0)}(x) z^\ell & (4.9)\end{aligned}$$

for fixed $t_{0,r}\in\mathbb{R},\ n\in\mathbb{N}_0,\ r\in 2\,\mathbb{N}_0+1$. Here $f_\ell(x,t_r)$ and $f_\ell^{(0)}(x)$ are defined as in (2.2) with $q_0(x)$ replaced by $q_0(x,t_r)$ and $q_0^{(0)}(x)$, respectively.

In analogy to (3.14), (3.15), (3.19), and (3.20), we introduce

$$\begin{aligned} D_n(z,x,t_r) &= F_{2\,n+1}(z,x,t_r)=\prod_{j=1}^{n}[z-\mu_j(x,t_r)], && (4.10)\\ N_{n+1}(z,x,t_r) &= \big(z-q_0(x,t_r)\big)F_{2\,n+1}(z,x,t_r)-\frac{1}{2}\,F_{2\,n+1,xx}(z,x,t_r)\\ &= \prod_{\ell=0}^{n}[z-\nu_\ell(x,t_r)], && (4.11)\end{aligned}$$

such that

$$D_n(z,x,t_r)\,N_{n+1}(z,x,t_r)=R_{2\,n+1}(z)-\frac{1}{4}D_{n,x}(z,x,t_r)^2. \tag{4.12}$$

Hence we can define, in analogy to (3.16) and (3.17), the following meromorphic function $\phi(P,x,t_r)$ on $\mathcal{K}_n$ the fundamental ingredient for the construction of algebro-geometric solutions of the time-dependent KdV hierarchy,

$$\begin{aligned} \phi(P,x,t_r) &= \frac{y(P)-k_{2\,n+1}(z)+\frac{1}{2}D_{n,x}(z,x,t_r)}{D_n(z,x,t_r)} && (4.13)\\ &= \frac{N_{n+1}(z,x,t_r)}{y(P)-k_{2\,n+1}(z)-\frac{1}{2}D_{n,x}(z,x,t_r)}, && (4.14)\\ &\qquad\qquad P = (z,y(P))\in\mathcal{K}_n. \end{aligned}$$

As in (3.21) and (3.22) one introduces Dirichlet and Neumann data by

$$\begin{aligned} \hat\mu_j(x,t_r) &= \big(\mu_j(x,t_r),y(\hat\mu_j(x,t_r))\big)\\ &= \big(\mu_j(x,t_r),k_{2\,n+1}(\mu_j(x,t_r))+\frac{1}{2}D_{n,x}(\mu_j(x,t_r),x,t_r)\big)\in\mathcal{K}_n, && (4.15)\\ &\qquad 1\le j\le n,\quad (x,t_r)\in\mathbb{R}^2,\\ \hat\nu_\ell(x,t_r) &= \big(\nu_\ell(x,t_r),y(\hat\nu_\ell(x,t_r))\big)\\ &= \big(\nu_\ell(x,t_r),k_{2\,n+1}(\nu_\ell(x,t_r))-\frac{1}{2}D_{n,x}(\nu_\ell(x,t_r),x,t_r)\big)\in\mathcal{K}_n, && (4.16)\\ &\qquad 0\le \ell\le n,\quad (x,t_r)\in\mathbb{R}^2, \end{aligned}$$

and infers that the divisor $\big(\phi(P,x,t_r)\big)$ of $\phi(P,x,t_r)$ is given by

$$\big(\phi(P,x,t_r)\big)=\mathcal{D}_{\hat\nu_0(x,t_r),\dots,\hat\nu_n(x,t_r)}(P)-\mathcal{D}_{P_\infty,\hat\mu_1(x,t_r),\dots,\hat\mu_n(x,t_r)}(P). \tag{4.17}$$

Next we define the time-dependent BA-function $\psi(P,x,x_0,t_r,t_{0,r})$ by

$$\begin{aligned}\psi(P,x,x_0,t_r,t_{0,r}) = \exp\Big\{\int_{x_0}^{x}d\,x'\phi(P,x',t_r)+\int_{t_{0,r}}^{t_r}d\,s\big[\tilde F_r(z,x_0,s)\phi(P,x_0,s)\\ -\frac{1}{2}\tilde F_{r,x}(z,x_0,s)+\tilde k_r(z)\big]\Big\},\quad P\in\mathcal{K}_n\setminus\{P_\infty\},\quad (x,t_r)\in\mathbb{R}^2, \quad (4.18)\end{aligned}$$

with fixed $(x_0,t_{0,r})\in\mathbb{R}^2$. The following lemma records some properties of $\phi(P,x,t_r)$ and $\psi(P,x,x_0,t_r,t_{0,r})$ (see [28] for the original result).

Lemma 4.1. *Assume (4.7)–(4.11), $P = (z, y(P)) \in \mathcal{K}_n \backslash \{P_\infty\}$, and let $(z, x, x_0, t_r, t_{0,r}) \in \mathbb{C} \times \mathbb{R}^4$. Then*

(i). $\phi(P, x, t_r)$ satisfies

$$\phi_x(P, x, t_r) + \phi(P, x, t_r)^2 = z - q_0(x, t_r), \tag{4.19}$$

$$\phi_{t_r}(P, x, t_r) = \partial_x\big[\tilde{F}_r(z, x, t_r)\phi(P, x, t_r) - \frac{1}{2}\tilde{F}_{r,x}(z, x, t_r) + \tilde{k}_r(z)\big]. \tag{4.20}$$

(ii). $\psi(P, x, x_0, t_r, t_{0,r})$ satisfies

$$\psi_{xx}(P, x, x_0, t_r, t_{0,r}) + [q_0(x, t_r) - z]\psi(P, x, x_0, t_r, t_{0,r}) = 0, \tag{4.21}$$

$$\begin{aligned}\psi_{t_r}(P, x, x_0, t_r, t_{0,r}) = \big[&\tilde{F}_r(z, x, t_r)\phi(P, x, t_r) - \frac{1}{2}\tilde{F}_{r,x}(z, x, t_r) \\ &+ \tilde{k}_r(z)\big]\psi(P, x, x_0, t_r, t_{0,r})\end{aligned} \tag{4.22}$$

(i.e., $(L_2 - z)\psi = 0$, $(P_{2n+1} - y)\psi = 0$, $\psi_{t_r} = \tilde{P}_r\psi$).

(iii). $$\phi(P, x, t_r)\,\phi(P^*, x, t_r) = -\frac{N_{n+1}(z, x, t_r)}{D_n(z, x, t_r)}. \tag{4.23}$$

(iv). $$\phi(P, x, t_r) + \phi(P^*, x, t_r) = \frac{D_{n,x}(z, x, t_r)}{D_n(z, x, t_r)}. \tag{4.24}$$

(v). $$\phi(P, x, t_r) - \phi(P^*, x, t_r) = \frac{2\,(y(P) - k_{2n+1}(z))}{D_n(z, x, t_r)}, \tag{4.25}$$

$$\big(y(P) - k_{2n+1}(z)\big)\phi(P, x, t_r) + \big(y(P^*) - k_{2n+1}(z)\big)\phi(P^*, x, t_r) = \frac{2\,R_{2n+1}(z)}{D_n(z, x, t_r)}. \tag{4.26}$$

Proof. (i). (4.19) follows from (4.8), and (4.13). In order to prove (4.20) one first derives from (4.7), (4.8) and (4.13) that

$$(\partial_x + 2\,\phi)\big[\phi_{t_r} - \partial_x\big(\tilde{F}_r\phi - \frac{1}{2}\tilde{F}_{r,x} + \tilde{k}_r(z)\big)\big] = 0.$$

Thus

$$\phi_{t_r} - \partial_x\big(\tilde{F}_r\phi - \frac{1}{2}\tilde{F}_{r,x} + \tilde{k}_r(z)\big) = Ce^{-\int^x d\,x' 2\phi}, \tag{4.27}$$

where C is independent of x (but may depend on P and t_r). The high-energy behavior $\phi(P, x, t_r) \underset{|z|\to\infty}{=} O(|z|^{1/2})$ (cf. (4.13)) then proves $C = 0$ since the left-hand side of (4.27) is meromorphic on $\mathcal{K}_n$ (and hence especially near P_∞).
(ii). (4.21) is clear from (4.18) ($\phi = \psi_x/\psi$) and (4.19). (4.22) follows from (4.18) and (4.20).
(iii)–(v) follow as in Lemma 3.1 (ii)–(iv). □

Next we introduce

$$\tilde{N}_{r+1}(z, x, t_r) = [z - q_0(x, t_r)]\tilde{F}_r(z, x, t_r) - \frac{1}{2}\tilde{F}_{r,xx}(z, x, t_r) \tag{4.28}$$

and note that by (4.7),

$$\tilde{N}_{r+1,x}(z, x, t_r) = -q_0(x, t_r) - [z - q_0(x, t_r)]\tilde{F}_{r,x}(z, x, t_r). \tag{4.29}$$

In analogy to (3.32) one also obtains

$$N_{n+1,x}(z,x,t_r) = -[z - q_0(x,t_r)]D_{n,x}(z,x,t_r). \tag{4.30}$$

We recall (cf. [28]),

Lemma 4.2. *Assume (4.7)–(4.11) and let* $(z,x,t_r) \in \mathbb{C}\times\mathbb{R}^2$. *Then*

$$(i).\ D_{n,t_r}(z,x,t_r) = \tilde F_r(z,x,t_r)D_{n,x}(z,x,t_r) - \tilde F_{r,x}(z,x,t_r)D_n(z,x,t_r). \tag{4.31}$$

$$(ii).\ D_{n,x\,t_r}(z,x,t_r) = 2\big[\tilde N_{r+1}(z,x,t_r)D_n(z,x,t_r) - N_{n+1}(z,x,t_r)\tilde F_r(z,x,t_r)\big]. \tag{4.32}$$

$$(iii).\ N_{n+1,t_r}(z,x,t_r) = \tilde F_{r,x}(z,x,t_r)N_{n+1}(z,x,t_r) - D_{n,x}(z,x,t_r)\tilde N_{r+1}(z,x,t_r). \tag{4.33}$$

Proof. In order to prove (4.31) consider $\phi_{t_r}(P) - \phi_{t_r}(P^*)$ and combine (4.13) and (4.20). (4.32) follows from (4.31) using (4.11). (4.33) is a consequence of (4.11), (4.31), and (4.32). ☐

The remaining analogs of Lemma 3.1 (v)–(vii) then read (cf. again [28])

Lemma 4.3. *Assume (4.7)–(4.11),* $P = (z, y(P)) \in \mathcal{K}_n\backslash\{P_\infty\}$, *and let* $(z,x,x_0,t_r,t_{0,r}) \in \mathbb{C}\times\mathbb{R}^4$. *Then*

$$(i).\ \psi(P,x,x_0,t_r,t_{0,r})\psi(P^*,x,x_0,t_r,t_{0,r}) = e^{2\,\tilde k_r(z)(t_r - t_{0,r})}\frac{D_n(z,x,t_r)}{D_n(z,x_0,t_{0,r})}. \tag{4.34}$$

$$(ii).\ \psi_x(P,x,x_0,t_r,t_{0,r})\psi_x(P^*,x,x_0,t_r,t_{0,r}) = -e^{2\,\tilde k_r(z)(t_r - t_{0,r})}\frac{N_{n+1}(z,x,t_r)}{D_n(z,x_0,t_{0,r})}. \tag{4.35}$$

$$(iii).\ \psi(P,x,x_0,t_r,t_{0,r}) = \bigg[\frac{D_n(z,x,t_r)}{D_n(z,x_0,t_{0,r})}\bigg]^{1/2}\exp\bigg\{\tilde k_r(z)(t_r - t_{0,r}) + \big[y(P) - k_{2\,n+1}(z)\big]\Big[\int_{x_0}^{x} d\,x'\frac{1}{D_n(z,x',t_r)} + \int_{t_{0,r}}^{t_r} d\,s\frac{\tilde F_r(z,x_0,s)}{D_n(z,x_0,s)}\Big]\bigg\}. \tag{4.36}$$

Proof. (4.34) follows from (4.18), (4.24) and (4.31). (4.35) is clear from $\psi_x(P)\,\psi_x(P^*) = \phi(P)\,\phi(P^*)\,\psi(P)\,\psi(P^*)$, (4.23), and (4.34). (4.36) finally is a consequence of (4.18), (4.24), (4.25), and (4.31) by splitting up $\phi(P) = \frac{1}{2}[\phi(P)+\phi(P^*)] + \frac{1}{2}[\phi(P)-\phi(P^*)]$ in (4.18). ☐

The dynamics of the zeros $\mu_j(x,t_r)$ and $\nu_\ell(x,t_r)$ of $D_n(z,x,t_r)$ and $N_{n+1}(z,x,t_r)$, in analogy to Lemma 3.2, are then described by Dubrovin's equations as follows.

Lemma 4.4. *Assume (4.7)–(4.11) and let* $(x,t_r) \in \mathbb{R}^2$. *Then*

$$(i).\quad \mu_{j,x}(x,t_r) = \frac{-2\,\big(y(\hat\mu_j(x,t_r)) - k_{2\,n+1}(\mu_j(x,t_r))\big)}{\prod_{\substack{k=1\\k\neq j}}^{n}[\mu_j(x,t_r) - \mu_k(x,t_r)]}, \qquad 1 \le j \le n, \tag{4.37}$$

$$\mu_{j,t_r}(x,t_r) = \frac{-2\,\tilde F_r(\mu_j(x,t_r),x,t_r)\big(y(\hat\mu_j(x,t_r)) - k_{2\,n+1}(\mu_j(x,t_r))\big)}{\prod_{\substack{k=1\\k\neq j}}^{n}[\mu_j(x,t_r) - \mu_k(x,t_r)]}, \quad 1 \le j \le n.$$

$$(4.38)$$

$$(ii).\quad \nu_{\ell,x}(x,t_r) = \frac{-2\left[\nu_\ell(x,t_r)-q_0(x,t_r)\right]\big(y(\hat\nu_\ell(x,t_r))-k_{2n+1}(\nu_\ell(x,t_r))\big)}{\prod\limits_{\substack{m=0\\ m\neq\ell}}^{n}\left[\nu_\ell(x,t_r)-\nu_m(x,t_r)\right]},$$

$$0\le \ell\le n, \tag{4.39}$$

$$\nu_{\ell,t_r}(x,t_r) = \frac{-2\,\tilde N_r(\nu_\ell(x,t_r),x,t_r)\big(y(\hat\nu_\ell(x,t_r))-k_{2n+1}(\nu_\ell(x,t_r))\big)}{\prod\limits_{\substack{m=0\\ m\neq\ell}}^{n}\left[\nu_\ell(x,t_r)-\nu_m(x,t_r)\right]},\qquad 0\le\ell\le n. \tag{4.40}$$

Proof. (4.37) and (4.39) are completely analogous to (3.36) and (3.37). (4.38) (respectively, (4.40)) follows from (4.10), (4.15), and (4.31) (respectively, (4.11), (4.16), and (4.33)). ☐

The initial condition

$$q_0(x,t_{0,r}) = q_0^{(0)}(x),\quad x\in\mathbb{R} \tag{4.41}$$

in (4.2) is taken care of by

$$\hat\mu_j(x,t_{0,r}) = \hat\mu_j^{(0)}(x),\quad 1\le j\le n,\quad x\in\mathbb{R} \tag{4.42}$$

(cf. (4.9) and (4.10)). There is, of course, an analogous condition

$$\hat\nu_\ell(x,t_{0,r}) = \hat\nu_\ell^{(0)}(x),\quad 0\le\ell\le n,\quad x\in\mathbb{R}. \tag{4.43}$$

Finally, the trace relations in Lemma 3.3 extend in a one-to-one manner to the present time-dependent setting by substituting,

$$\begin{aligned} &q_0(x)\to q_0(x,t_r), \\ &\mu_j(x)\to\mu_j(x,t_r),\quad 1\le j\le n,\qquad \nu_\ell(x)\to\nu_\ell(x,t_r),\quad 0\le\ell\le n, \end{aligned} \tag{4.44}$$

keeping $\{c_\ell\}_{1\le\ell\le n}$ as in (3.40) since $\mathcal{K}_n$ is t_r-independent.

5. The Recursive Approach to the Bsq Hierarchy and associated algebraic curves

In analogy to the KdV case in Section 2 we now develop the recursive approach to the Bsq hierarchy. These results are new.

Suppose $q_0, q_1 \in C^\infty(\mathbb{R})$ (or q_0, q_1 meromorphic on $\mathbb{C}$) and introduce the third-order differential expression

$$L_3 = \frac{d^3}{d\,x^3} + q_1\frac{d}{d\,x} + \frac{1}{2}\,q_{1,x} + q_0,\quad x\in\mathbb{R}\ \ (\text{or}\ \ \mathbb{C}). \tag{5.1}$$

(For computational reasons we found L_3 as in (5.1) more convenient to work with than its alternative $\tilde L_3 = \frac{d^3}{d\,x^3} + q_1\frac{d}{d\,x} + q_0$.)

In order to explicitly construct differential expressions $P_r, r\not\equiv 0 \pmod 3$ commuting with L_3, which will be used later to define the stationary Bsq hierarchy, we proceed as in Section 2.

Pick $n \in \mathbb{N}_0$ and define $\{f_\ell(x)\}_{0\le \ell\le n+1}$ and $\{g_\ell(x)\}_{0\le \ell\le n+1}$ recursively by

$$\begin{aligned}
f_0 &= 0, \quad g_0 = 1, \quad \text{if} \quad r \equiv 1 \pmod 3,\\
f_0 &= 1, \quad g_0 = d_0, \quad \text{if} \quad r \equiv 2 \pmod 3,\\
3f_{\ell,x} &= 2g_{\ell-1,xxx} + 2q_1 g_{\ell-1,x} + q_{1,x} g_{\ell-1} + 3q_0 f_{\ell-1,x} + 2q_{0,x} f_{\ell-1}, \qquad (5.2)\\
3g_{\ell,x} &= 3q_0 g_{\ell-1,x} + q_{0,x} g_{\ell-1} - \frac{1}{6} f_{\ell-1,xxxxx} - \frac{5}{6} q_1 f_{\ell-1,xxx} - \frac{5}{4} q_{1,x} f_{\ell-1,xx}\\
&\quad -\big(\frac{3}{4} q_{1,xx} + \frac{2}{3} q_1^2\big) f_{\ell-1,x} - \big(\frac{1}{6} q_{1,xxx} + \frac{2}{3} q_1 q_{1,x}\big) f_{\ell-1}, \quad 1 \le \ell \le n+1.
\end{aligned}$$

Explicitly, one computes

(i). $r \equiv 1 \pmod 3$:

$$\begin{aligned}
f_0 &= 0, \qquad g_0 = 1,\\
3f_1 &= q_1 + 3c_1, \qquad 3g_1 = q_0 + 3d_1,\\
3f_2 &= \frac{2}{3} q_{0,xx} + \frac{4}{3} q_0 q_1 + c_1\, 2q_0 + d_1\, q_1 + 3c_2,\\
3g_2 &= -\frac{1}{18} q_{1,xxxx} - \frac{1}{6} q_{1,x}^2 - \frac{4}{27} q_1^3 - \frac{1}{3} q_1 q_{1,xx} + \frac{2}{3} q_0^2\\
&\quad + c_1\big(-\frac{1}{6} q_{1,xx} - \frac{1}{3} q_1^2\big) + d_1 q_0 + 3d_2, \qquad (5.3)\\
3f_3 &= -\frac{1}{27} q_{1,xxxxxx} - \frac{7}{27} q_1 q_{1,xxxx} - \frac{35}{54} q_{1,x} q_{1,xxx} - \frac{49}{108} q_{1,xx}^2 - \frac{14}{27} q_1^2 q_{1,xx}\\
&\quad -\frac{35}{54} q_1 q_{1,x}^2 - \frac{7}{81} q_1^4 + \frac{14}{9} q_0 q_{0,xx} + \frac{7}{9} q_{0,x}^2 + \frac{14}{9} q_0^2 q_1\\
&\quad + c_1\,\big(-\frac{1}{9} q_{1,xxxx} - \frac{5}{9} q_1 q_{1,xx} - \frac{5}{12} q_{1,x}^2 - \frac{5}{27} q_1^3 + \frac{5}{3} q_0^2\big)\\
&\quad + d_1\,\big(\frac{2}{3} q_{0,xx} + \frac{4}{3} q_0 q_1\big) + c_2\, 2q_0 + d_2\, q_1 + 3c_3,\\
3g_3 &= \frac{14}{27} q_0^3 - \frac{28}{81} q_0 q_1^3 - \frac{7}{9} q_{0,x} q_1 q_{1,x} - \frac{7}{18} q_0 q_{1,x}^2 - \frac{14}{27} q_1^2 q_{0,xx}\\
&\quad -\frac{7}{9} q_0 q_1 q_{1,xx} - \frac{14}{27} q_{0,xx} q_{1,xx} - \frac{7}{18} q_{1,x} q_{0,xxx} - \frac{7}{27} q_{0,x} q_{1,xxx}\\
&\quad -\frac{7}{27} q_1 q_{0,xxxx} - \frac{7}{54} q_0 q_{1,xxxx} - \frac{1}{27} q_{0,xxxxxx}\\
&\quad + c_1\,\big(-\frac{1}{9} q_{0,xxxx} - \frac{5}{18} q_0 q_{1,xx} - \frac{5}{9} q_1 q_{0,xx} - \frac{5}{18} q_{1,x} q_{0,x} - \frac{5}{9} q_0 q_1^2\big)\\
&\quad + d_1\,\big(-\frac{1}{18} q_{1,xxxx} - \frac{1}{6} q_{1,x}^2 - \frac{4}{27} q_1^3 - \frac{1}{3} q_1 q_{1,xx} + \frac{2}{3} q_0^2\big)\\
&\quad + c_2\big(-\frac{1}{6} q_{1,xx} - \frac{1}{3} q_1^2\big) + d_2 q_0 + 3d_3,
\end{aligned}$$

etc.

(ii). $r \equiv 2 \pmod 3$:

$$\begin{aligned}
f_0 &= 1, \qquad g_0 = d_0,\\
3f_1 &= 2q_0 + d_0 q_1 + 3c_1, \qquad 3g_1 = -\frac{1}{6} q_{1,xx} - \frac{1}{3} q_1^2 + d_0 q_0 + 3d_1,\\
3f_2 &= \big(-\frac{1}{9} q_{1,xxxx} - \frac{5}{9} q_1 q_{1,xx} - \frac{5}{27} q_1^3 - \frac{5}{12} q_{1,x}^2 + \frac{5}{3} q_0^2\big)
\end{aligned}$$

$$
\begin{aligned}
&+d_0\big(\frac{2}{3}\,q_{0,xx}+\frac{4}{3}\,q_0q_1\big)+c_1\,2q_0+d_1\,q_1+3c_2,\\
3g_2 \;=\;& \big(-\frac{1}{9}\,q_{0,xxxx}-\frac{5}{9}\,q_1^2q_0-\frac{5}{18}\,q_0q_{1,xx}-\frac{5}{9}\,q_1q_{0,xx}-\frac{5}{18}\,q_{0,x}q_{1,x}\big)\\
&+d_0\,\big(-\frac{1}{18}\,q_{1,xxxx}-\frac{1}{6}\,q_{1,x}^2-\frac{4}{27}\,q_1^3-\frac{1}{3}\,q_1q_{1,xx}+\frac{2}{3}\,q_0^2\big)\\
&+c_1\,\big(-\frac{1}{6}\,q_{1,xx}-\frac{1}{3}\,q_1^2\big)+d_1\,q_0+3d_2, \qquad (5.4)\\
3f_3 \;=\;& \frac{40}{27}\,q_0^3-\frac{40}{81}\,q_0q_1^3-\frac{10}{9}\,q_0q_{1,x}^2-\frac{20}{27}\,q_1^2q_{0,xx}-\frac{40}{27}\,q_0q_1q_{1,xx}\\
&-\frac{26}{27}\,q_{0,xx}q_{1,xx}-\frac{40}{27}\,q_1q_{1,x}q_{0,x}-\frac{8}{9}\,q_{1,x}q_{0,xxx}-\frac{14}{27}\,q_{0,x}q_{1,xxx}\\
&-\frac{4}{9}\,q_1q_{0,xxxx}-\frac{8}{27}\,q_0q_{1,xxxx}-\frac{2}{27}\,q_{0,xxxxxx}\\
&+d_0\,\big(-\frac{1}{27}\,q_{1,xxxxxx}-\frac{7}{24}\,q_1q_{1,xxxx}-\frac{35}{54}\,q_{1,x}q_{1,xxx}-\frac{49}{108}\,q_{1,xx}^2\\
&-\frac{14}{27}\,q_1^2q_{1,xx}-\frac{35}{54}\,q_1q_{1,x}^2-\frac{7}{81}\,q_1^4+\frac{14}{9}\,q_0q_{0,xx}+\frac{7}{9}\,q_{0,x}^2+\frac{14}{9}\,q_0^2q_1\big)\\
&+c_1\,\big(-\frac{1}{9}\,q_{1,xxxx}-\frac{5}{9}\,q_1q_{1,xx}-\frac{5}{12}\,q_{1,x}^2-\frac{5}{27}\,q_1^3+\frac{5}{3}\,q_0^2\big)\\
&+d_1\,\big(\frac{2}{3}\,q_{0,xx}+\frac{4}{3}\,q_0q_1\big)+c_2\,2q_0+d_2\,q_1+3c_3,\\
3g_3 \;=\;& \frac{8}{243}\,q_1^5-\frac{20}{27}\,q_0^2q_1^2-\frac{20}{27}\,q_{0,x}^2q_1-\frac{20}{27}\,q_0q_{0,x}q_{1,x}+\frac{35}{81}\,q_1^2q_{1,x}^2\\
&-\frac{40}{27}\,q_0q_1q_{0,xx}-\frac{16}{27}\,q_{0,xx}^2+\frac{70}{243}\,q_1^3q_{1,xx}+\frac{11}{18}\,q_{1,x}^2q_{1,xx}\\
&+\frac{17}{27}\,q_1q_{1,xx}^2-\frac{2}{3}\,q_{0,x}q_{0,xxx}+\frac{68}{81}\,q_1q_{1,x}q_{1,xxx}+\frac{7}{27}\,q_{1,xxx}^2\\
&-\frac{10}{27}\,q_0^2q_{1,xx}-\frac{8}{27}\,q_0q_{0,xxxx}+\frac{17}{81}\,q_1^2q_{1,xxxx}+\frac{67}{162}\,q_{1,xx}q_{1,xxxx}\\
&+\frac{5}{27}\,q_{1,x}q_{1,xxxxx}+\frac{5}{81}\,q_1q_{1,xxxxxx}+\frac{1}{162}\,q_{1,xxxxxxxx}\\
&+d_0\,\big(\frac{14}{27}\,q_0^3-\frac{28}{81}\,q_0q_1^3-\frac{7}{9}\,q_{0,x}q_1q_{1,x}-\frac{7}{18}\,q_0q_{1,x}^2-\frac{14}{27}\,q_1^2q_{0,xx}\\
&-\frac{7}{9}\,q_0q_1q_{1,xx}-\frac{14}{27}\,q_{0,xx}q_{1,xx}-\frac{7}{18}\,q_{1,x}q_{0,xxx}-\frac{7}{27}\,q_{0,x}q_{1,xxx}\\
&-\frac{7}{27}\,q_1q_{0,xxxx}-\frac{7}{54}\,q_0q_{1,xxxx}-\frac{1}{27}\,q_{0,xxxxxx}\big)\\
&+c_1\,\big(-\frac{1}{9}\,q_{0,xxxx}-\frac{5}{18}\,q_0q_{1,xx}-\frac{5}{9}\,q_1q_{0,xx}-\frac{5}{18}\,q_{1,x}q_{0,x}-\frac{5}{9}\,q_0q_1^2\big)\\
&+d_1\,\big(-\frac{1}{18}\,q_{1,xxxx}-\frac{1}{6}\,q_{1,x}^2-\frac{4}{27}\,q_1^3-\frac{1}{3}\,q_1q_{1,xx}+\frac{2}{3}\,q_0^2\big)\\
&+c_2\big(-\frac{1}{6}q_{1,xx}-\frac{1}{3}\,q_1^2\big)+d_2q_0+3d_3,
\end{aligned}
$$

etc.,

where $\{c_\ell\}_{1\leq \ell\leq n}$, $\{d_\ell\}_{0\leq \ell\leq n}$ are integration constants. Given (5.2) one defines the differential expression P_r of order r by

$$\begin{aligned} P_r &= \sum_{\ell=0}^{n}\Big[f_{n-\ell}\,\frac{d^2}{d\,x^2}+\big(g_{n-\ell}-\frac{1}{2}\,f_{n-\ell,x}\big)\frac{d}{d\,x} \\ &\quad +\big(\frac{1}{6}\,f_{n-\ell,xx}-g_{n-\ell,x}+\frac{2}{3}\,q_1 f_{n-\ell}\big)\Big]L_3^\ell+\sum_{\ell=0}^{n}k_{r,\ell}L_3^\ell, \\ &\qquad k_{r,\ell}\in\mathbb{C},\quad 0\leq\ell\leq n,\quad r=3\,n+1 \text{ or } r=3\,n+2,\quad n\in\mathbb{N}_0 \end{aligned} \tag{5.5}$$

and verifies

$$\begin{aligned} [P_r,L_3] &= 3\,f_{n+1,x}\,\frac{d}{d\,x}+\frac{3}{2}\,f_{n+1,xx}+3\,g_{n+1,x}, \\ &\qquad r=3\,n+1 \text{ or } r=3\,n+2,\quad n\in\mathbb{N}_0. \end{aligned} \tag{5.6}$$

The pair (L_3,P_r) represents the Lax pair for the Bsq hierarchy. Varying $n\in\mathbb{N}_0$, the stationary Bsq hierarchy is then defined by the vanishing of the commutator of P_r and L_3 in (5.6), that is, by

$$[P_r,L_3]=0,\qquad r=3\,n+1 \text{ or } r=3\,n+2,\quad n\in\mathbb{N}_0, \tag{5.7}$$

or equivalently, by

$$f_{n+1,x}=0,\quad g_{n+1,x}=0,\qquad n\in\mathbb{N}_0. \tag{5.8}$$

Explicitly, one obtains for the first few equations of the stationary Bsq hierarchy,

$$\begin{aligned} & r=1,n=0: \\ &\qquad q_{0,x}=0, \\ &\qquad q_{1,x}=0. \\ & r=2,n=0: \\ &\qquad -\frac{1}{6}\,q_{1,xxx}-\frac{2}{3}\,q_1q_{1,x}+d_0q_{0,x}=0, \\ &\qquad 2\,q_{0,x}+d_0q_{1,x}=0. \\ & r=4,n=1: \\ &\qquad -\frac{1}{18}\,q_{1,xxxxx}-\frac{1}{3}\,q_1q_{1,xxx}-\frac{2}{3}\,q_{1,x}q_{1,xx}-\frac{4}{9}\,q_1^2q_{1,x}+\frac{4}{3}\,q_0q_{0,x} \\ &\qquad\quad +c_1\big(-\frac{1}{6}\,q_{1,xxx}-\frac{2}{3}\,q_1q_{1,x}\big)+d_1q_{0,x}=0, \\ &\qquad \frac{2}{3}\,q_{0,xxx}+\frac{4}{3}\,q_1q_{0,x}+\frac{4}{3}\,q_{1,x}q_0+c_12q_{0,x}+d_1q_{1,x}=0. \\ & r=5,n=1: \\ &\qquad \frac{1}{9}\,q_{0,xxxxx}+\frac{5}{18}\,q_0q_{1,xxx}+\frac{5}{9}\,q_1q_{0,xxx}+\frac{5}{9}\,q_{1,xx}q_{0,x}+\frac{5}{6}\,q_{1,x}q_{0,xx} \\ &\qquad\quad +\frac{5}{9}\,q_1^2q_{0,x}+\frac{10}{9}\,q_0q_1q_{1,x}+d_0\big(\frac{1}{18}\,q_{1,xxxxx}+\frac{1}{3}\,q_1q_{1,xxx}+\frac{2}{3}\,q_{1,x}q_{1,xx} \\ &\qquad\quad +\frac{4}{9}\,q_1^2q_{1,x}-\frac{4}{3}\,q_0q_{0,x}\big)+c_1\,\big(\frac{1}{6}\,q_{1,xxx}+\frac{2}{3}\,q_1q_{1,x}\big)-d_1q_{0,x}=0, \\ &\qquad -\frac{1}{9}\,q_{1,xxxxx}-\frac{5}{9}\,q_1q_{1,xxx}-\frac{25}{18}\,q_{1,x}q_{1,xx}-\frac{5}{9}\,q_1^2q_{1,x}+\frac{10}{3}\,q_0q_{0,x} \end{aligned} \tag{5.9}$$

$$+ d_0 \big(\frac{2}{3}\, q_{0,xxx} + \frac{4}{3}\, q_1 q_{0,x} + \frac{4}{3}\, q_{1,x} q_0\big) + c_1\, 2q_{0,x} + d_1\, q_{1,x} = 0.$$

etc.

By definition, solutions $(q_1(x), q_0(x))$ of any of the stationary Bsq equations (5.9) are called **algebro-geometric finite-gap potentials** associated with the Bsq hierarchy. If (q_0, q_1) satisfies $f_{n+1,x} = g_{n+1,x} = 0$, one also calls (q_0, q_1) a stationary $(r-1)$-gap solution, $r = 3\,n+1$ or $r = 3\,n+2$, $n \in \mathbb{N}_0$.

Next, we introduce the analogs of (2.9), that is,

$$F_r(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}(x) z^{\ell}, \tag{5.10}$$

$$G_r(z,x) = \sum_{\ell=0}^{n} g_{n-\ell}(x) z^{\ell}, \qquad r = 3\,n+1 \text{ or } r = 3\,n+2, \quad n \in \mathbb{N}_0. \tag{5.11}$$

Explicitly, the first few polynomials F_r, G_r read

$$F_1 = 0,$$
$$G_1 = 1,$$

$$F_2 = 1,$$
$$G_2 = d_0,$$

$$F_4 = \frac{1}{3}\, q_1 + c_1,$$
$$G_4 = z + \frac{1}{3}\, q_0 + d_1,$$

$$F_5 = z + \frac{2}{3}\, q_0 + d_0\, \frac{1}{3}\, q_1 + c_1,$$
$$G_5 = d_0 z - \frac{1}{18}\, q_{1,xx} - \frac{1}{9}\, q_1^2 + d_0\, \frac{1}{3}\, q_0 + d_1,$$

$$F_7 = z\, \big(\frac{1}{3}\, q_1 + c_1\big) + \frac{2}{9}\, q_{0,xx} + \frac{4}{9}\, q_0 q_1 + c_1\, \frac{2}{3}\, q_0 + d_1\, \frac{1}{3}\, q_1 + c_2,$$
$$G_7 = z^2 + z\, \big(\frac{1}{3}\, q_0 + d_1\big) + \frac{1}{3}\, \big(- \frac{1}{18}\, q_{1,xxxx} - \frac{1}{6}\, q_{1,x}^2 - \frac{4}{27} q_1^3$$
$$- \frac{1}{3}\, q_1 q_{1,xx} + \frac{2}{3}\, q_0^2\big) + c_1\, \frac{1}{3}\, \big(- \frac{1}{6} q_{1,xx} - \frac{1}{3}\, q_1^2\big) + d_1\, \frac{1}{3}\, q_0 + d_2,$$

$$F_8 = z^2 + z\big(\frac{2}{3}\, q_0 + d_0\, \frac{1}{3}\, q_1 + c_1\big)$$
$$+ \frac{1}{3}\, \big(- \frac{1}{9}\, q_{1,xxxx} - \frac{5}{9}\, q_1 q_{1,xx} - \frac{5}{27}\, q_1^3 - \frac{5}{12}\, q_{1,x}^2 + \frac{5}{3}\, q_0^2\big)$$
$$+ d_0 \frac{1}{3}\, \big(\frac{2}{3}\, q_{0,xx} + \frac{4}{3}\, q_0 q_1\big) + c_1\, \frac{2}{3}\, q_0 + d_1\, \frac{1}{3}\, q_1 + c_2,$$
$$G_8 = d_0 z^2 + z\big(- \frac{1}{18}\, q_{1,xx} - \frac{1}{9}\, q_1^2 + d_0\, \frac{1}{3}\, q_0 + d_1\big)$$

$$+\frac{1}{3}\big(-\frac{1}{9}\,q_{0,xxxx}-\frac{5}{9}\,q_1^2q_0-\frac{5}{18}\,q_0q_{1,xx}-\frac{5}{9}\,q_1q_{0,xx}-\frac{5}{18}\,q_{0,x}q_{1,x}\big)$$
$$+d_0\,\frac{1}{3}\big(-\frac{1}{18}\,q_{1,xxxx}-\frac{1}{6}\,q_{1,x}^2-\frac{4}{27}\,q_1^3-\frac{1}{3}\,q_1q_{1,xx}+\frac{2}{3}\,q_0^2\big)$$
$$+c_1\,\frac{1}{3}\big(-\frac{1}{6}\,q_{1,xx}-\frac{1}{3}\,q_1^2\big)+d_1\,\frac{1}{3}\,q_0+d_2, \tag{5.12}$$

etc.

Given (5.10), (5.11), and (5.7) respectively, (5.8) becomes

$$2\,G_{r,xxx}+2\,q_1G_{r,x}+q_{1,x}G_r-3\,(z-q_0)F_{r,x}+2\,q_{0,x}F_r \;=\; 0, \tag{5.13}$$
$$\frac{1}{6}\,F_{r,xxxxx}+\frac{5}{6}\,q_1F_{r,xxx}+\frac{5}{4}\,q_{1,x}F_{r,xx}+\big(\frac{3}{4}\,q_{1,xx}+\frac{2}{3}\,q_1^2\big)F_{r,x}$$
$$+\big(\frac{1}{6}\,q_{1,xxx}+\frac{2}{3}\,q_1q_{1,x}\big)F_r+3(z-q_0)G_{r,x}-q_{0,x}G_r \;=\; 0. \tag{5.14}$$

Multiplying (5.13) by G_r and (5.14) by F_r and taking the difference one can integrate the resulting expression to get

$$S_r(z)=-\frac{1}{6}\,F_{r,xxxx}F_r+\frac{1}{6}\,F_{r,xxx}F_{r,x}-\frac{1}{12}\,F_{r,xx}^2-\frac{5}{6}\,q_1F_{r,xx}F_r-\frac{5}{12}\,q_{1,x}F_{r,x}F_r$$
$$+\frac{5}{12}\,q_1F_{r,x}^2-\frac{1}{3}\,\big(\frac{1}{2}\,q_{1,xx}+q_1^2\big)F_r^2+2\,G_{r,xx}G_r-G_{r,x}^2+q_1G_r^2-3(z-q_0)F_rG_r, \tag{5.15}$$

where the integration constant $S_r(z)$ is seen to be a polynomial in z of degree $\leq 2\,n+1$, that is

$$S_r(z)=\sum_{p=0}^{2\,n+1-s}s_{r,p}z^p, \qquad 0\leq s\leq 2\,n+1. \tag{5.16}$$

Similarly, multiplying (5.14) by $\big(\frac{1}{3}\,F_{r,xx}F_r-\frac{1}{4}\,F_{r,x}^2+\frac{1}{3}\,q_1F_r^2+G_r^2\big)$ and (5.13) by $\big(\frac{2}{3}\,q_1F_rG_r-(z-q_0)F_r^2\big)$ and summing one can integrate the resulting expression to get the second integration constant $T_r(z)$,

$$\begin{aligned}T_r(z)\;&=\frac{1}{18}\,F_{r,xxxx}F_{r,xx}F_r-\frac{1}{24}\,F_{r,xxxx}F_{r,x}^2\\
&+\frac{1}{36}\,F_{r,xxx}F_{r,xx}F_{r,x}-\frac{1}{108}\,F_{r,xx}^3-\frac{1}{36}F_rF_{r,xxx}^2+\frac{1}{18}\,q_1F_{r,xxxx}F_r^2\\
&-\frac{1}{18}\,q_{1,x}F_{r,xxx}F_r^2-\frac{1}{9}\,q_1F_{r,xxx}F_{r,x}F_r+\frac{1}{18}\,q_{1,xx}F_{r,xx}F_r^2\\
&+\frac{2}{9}\,q_{1,x}F_{r,xx}F_{r,x}F_r-\frac{7}{72}\,q_1F_{r,xx}F_{r,x}^2+\frac{7}{36}\,q_1F_{r,xx}^2F_r\\
&+\frac{5}{18}\,q_1^2F_{r,xx}F_r^2-\frac{1}{24}\,q_{1,xx}F_{r,x}^2F_r-\frac{7}{48}\,q_{1,x}F_{r,x}^3+\frac{1}{12}\,q_{1,x}q_1F_{r,x}F_r^2\\
&-\frac{1}{6}\,q_1^2F_{r,x}^2F_r+\big(\frac{2}{27}\,q_1^3-\frac{1}{36}\,q_{1,x}^2+\frac{1}{18}\,q_{1,xx}q_1+(z-q_0)^2\big)F_r^3\\
&+(z-q_0)G_r^3+\frac{1}{6}\,F_{r,xxxx}G_r^2-\frac{1}{3}\,F_{r,xxx}G_{r,x}G_r+F_rG_{r,xx}^2\\
&+\frac{1}{3}\,F_{r,xx}\big(G_{r,x}^2+G_{r,xx}G_r\big)-F_{r,x}G_{r,xx}G_{r,x}-q_1(z-q_0)F_r^2G_r\\
&+\frac{2}{3}\,q_1^2F_rG_r^2+\frac{5}{6}\,q_1F_{r,xx}G_r^2-\frac{4}{3}\,q_1F_{r,x}G_{r,x}G_r+\frac{7}{12}\,q_{1,x}F_{r,x}G_r^2\end{aligned}$$

$$+\frac{1}{3}\,q_1F_rG_{r,x}^2+\frac{4}{3}\,q_1F_rG_{r,xx}G_r+\frac{1}{6}\,q_{1,xx}F_rG_r^2-\frac{1}{3}\,q_{1,x}F_rG_{r,x}G_r$$
$$+(z-q_0)F_{r,x}F_rG_{r,x}-\frac{1}{4}\,(z-q_0)F_{r,x}^2G_r-2(z-q_0)F_r^2G_{r,xx}. \tag{5.17}$$

By inspection, $T_r(z)$ is a monic polynomial of degree r, that is,

$$T_r(z)=\sum_{q=0}^{r}t_{r,q}z^q,\quad t_{r,r}=1,\quad r=3\,n+1 \text{ or } r=3\,n+2,\quad n\in\mathbb{N}_0. \tag{5.18}$$

Next, we consider the kernel of (L_3-z), $z\in\mathbb{C}$, and, taking into account (5.7), that is, $[P_r,L_3]=0$, compute the restriction of P_r to $\ker(L_3-z)$. Using

$$\psi_{xxx}=-q_1\psi_x+\big(z-\frac{1}{2}\,q_{1,x}-q_0\big)\psi,\qquad \text{etc.} \tag{5.19}$$

to eliminate higher-order derivatives of ψ, one obtains from (5.2), (5.5), (5.8), (5.10), (5.11), (5.13), and (5.14)

$$P_r\Big|_{\ker(L_3-z)}=\bigg[F_r\frac{d^2}{d\,x^2}+\big(G_r-\frac{1}{2}F_{r,x}\big)\frac{d}{dx}+H_r\bigg]_{\ker(L_3-z)}, \tag{5.20}$$

where

$$H_r(z,x)=\frac{1}{6}\,F_{r,xx}(z,x)+\frac{2}{3}\,q_1(x)F_r(z,x)-G_{r,x}(z,x)+k_r(z) \tag{5.21}$$

and (cf. (5.5))

$$k_r(z)=\sum_{\ell=0}^{n}k_{r,\ell}z^\ell. \tag{5.22}$$

Again the reader might want to contrast our construction of P_r in (5.5) and (5.20) with the one based on formal pseudo-differential expressions in [2], [16], Ch. 1, and [20].

Still assuming $f_{n+1,x}=g_{n+1,x}=0$ as in (5.8), $[P_r,L_3]=0$ in (5.5) yields an algebraic relationship between P_r and L_3 by appealing to the result of Burchnall and Chaundy [10], [11], [12] (see also [45]). In fact, one can prove

Theorem 5.1. *Assume $f_{n+1,x}=g_{n+1,x}=0$ that is, $[P_r,L_3]=0$, $r=3\,n+1$ or $r=3\,n+2$, $n\in\mathbb{N}_0$. Then the Burchnall-Chaundy polynomial $\mathcal{F}_{r-1}(L_3,P_r)$ of the pair (L_3,P_r)explicitly reads (cf. (5.16), (5.18), and (5.22))*

$$\begin{aligned}
\mathcal{F}_{r-1}(L_3,P_r)&=\Big(P_r-k_r(L_3)\Big)^3+\Big(P_r-k_r(L_3)\Big)S_r(L_3)-T_r(L_3)=0,\\
k_r(z)&=\sum_{\ell=0}^{n}k_{r,\ell}z^\ell,\quad S_r(z)=\sum_{p=0}^{2\,n+1-s}s_{r,p}z^p,\quad 0\le s\le 2\,n+1,\\
T_r(z)&=\sum_{q=0}^{r}t_{r,q}z^q,\quad t_{r,r}=1,\quad r=3\,n+1 \text{ or } r=3\,n+2,\quad n\in\mathbb{N}_0.
\end{aligned} \tag{5.23}$$

Proof. Let $\psi_j(x)\in\ker(L_3-z)$, $j=1,2,3$ be linearly independent. Since $[P_r,L_3]=0$ one can represent P_r as a 3×3 matrix $\mathcal{P}_r$ on $\ker(L_3-z)$,

$$P_r\psi_j=\sum_{k=1}^{3}\mathcal{P}_{r,j,k}\psi_k, \tag{5.24}$$

$$\mathcal{P}_r \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \mathcal{P}_{r,1,1} & \mathcal{P}_{r,1,2} & \mathcal{P}_{r,1,3} \\ \mathcal{P}_{r,2,1} & \mathcal{P}_{r,2,2} & \mathcal{P}_{r,2,3} \\ \mathcal{P}_{r,3,1} & \mathcal{P}_{r,3,2} & \mathcal{P}_{r,3,3} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{5.25}$$

$$\begin{aligned} \mathcal{P}_{r,1,j}(z) &= \frac{W(P_r\psi_j, \psi_2, \psi_3)}{W(\psi_1,\psi_2,\psi_3)}, \quad \mathcal{P}_{r,2,j}(z) = \frac{W(\psi_1, P_r\psi_j, \psi_3)}{W(\psi_1,\psi_2,\psi_3)}, \\ \mathcal{P}_{r,3,j}(z) &= \frac{W(\psi_1, \psi_2, P_r\psi_j)}{W(\psi_1,\psi_2,\psi_3)}, \qquad 1 \le j \le 3. \end{aligned}$$

(Here $W(f,g,h)$ denotes the Wronski determinant of f, g, and h.) Using (5.13), (5.14) and (5.19)–(5.22) one verifies

$$\begin{aligned} \operatorname{tr}(\mathcal{P}_r(z)) &= 3\,k_r(z), \qquad\qquad (5.26) \\ \mathrm{M}_1(\mathcal{P}_r(z)) &= \mathcal{P}_{r,1,1}\mathcal{P}_{r,2,2} + \mathcal{P}_{r,1,1}\mathcal{P}_{r,3,3} + \mathcal{P}_{r,2,2}\mathcal{P}_{r,3,3} \\ &\quad - \mathcal{P}_{r,2,3}\mathcal{P}_{r,3,2} - \mathcal{P}_{r,3,1}\mathcal{P}_{r,1,3} - \mathcal{P}_{r,1,2}\mathcal{P}_{r,2,1} \\ &= \frac{W(P_r\psi_1, P_r\psi_2, \psi_3) + W(\psi_1, P_r\psi_2, P_r\psi_3) + W(P_r\psi_1, \psi_2, P_r\psi_3)}{W(\psi_1,\psi_2,\psi_3)} \\ &= 3\,k_r(z)^2 + S_r(z), \qquad\qquad (5.27) \\ \det(\mathcal{P}_r(z)) &= \frac{W(P_r\psi_1, P_r\psi_2, P_r\psi_3)}{W(\psi_1,\psi_2,\psi_3)} = k_r(z)^3 + k_r(z)S_r(z) + T_r(z). \qquad (5.28) \end{aligned}$$

The characteristic polynomial $\det(y - \mathcal{P}_r(z)) = 0$ of $\mathcal{P}_r(z)$ then yields

$$\begin{aligned} \mathcal{F}_{r-1}(z,y) &= y^3 - y^2 \operatorname{tr}(\mathcal{P}_r(z)) + y\,\mathrm{M}_1(\mathcal{P}_r(z)) - \det(\mathcal{P}_r(z)) \\ &= \big(y - k_r(z)\big)^3 + \big(y - k_r(z)\big)S_r(z) - T_r(z) = 0. \end{aligned} \tag{5.29}$$

Since $z \in \mathbb{C}$ is arbitrary, the result (5.23) then follows from the Cayley-Hamilton theorem (as in the proof of Theorem 2.1). $\square$

Remark 5.2. *Equation (5.29) naturally leads to the plane algebraic curve* $\mathcal{K}_{r-1}$,

$$\mathcal{K}_{r-1} : \mathcal{F}_{r-1}(z,y) = (y - k_r(z))^3 + (y - k_r(z))\, S_r(z) - T_r(z) = 0, \tag{5.30}$$

$$k_r(z) = \sum_{\ell=0}^{n} k_{r,\ell} z^\ell, \quad S_r(z) = \sum_{p=0}^{2n+1-s} s_{r,p} z^p, \quad 0 \le s \le 2n+1,$$

$$T_r(z) = \sum_{q=0}^{r} t_{r,q} z^q, \quad t_{r,r} = 1, \quad r = 3n+1 \ \textit{or}\ r = 3n+2, \quad n \in \mathbb{N}_0$$

of (arithmetic) genus $r-1$. *For* $r \ge 4$ *these curves are non-hyperelliptic.*

Examples illustrating this formalism can be found in our final Section 8.

Finally, introducing a deformation parameter t_r in (q_0, q_1) (i.e., $q_\ell(x) \to q_\ell(x, t_r)$, $\ell = 0, 1$), the time-dependent Bsq hierarchy is defined as a collection of evolution equations (varying $r = 3n+1$ or $r = 3n+2$, $n \in \mathbb{N}_0$)

$$\frac{d}{dt_r} L_3(t_r) - [P_r(t_r), L_3(t_r)] = 0, \quad (x, t_r) \in \mathbb{R}^2, \quad r = 3n+1 \text{ or } r = 3n+2, \quad n \in \mathbb{N}_0, \tag{5.31}$$

or equivalently, by

$$\mathrm{Bsq}_r(q_0,q_1) \;=\; \begin{cases} q_{0,t_r} - 3\,g_{n+1,x} = 0 \\ q_{1,t_r} - 3\,f_{n+1,x} = 0 \end{cases},$$

$$(x,t_r)\in\mathbb{R}^2, \quad r=3\,n+1 \text{ or } r=3\,n+2, \quad n\in\mathbb{N}_0, \tag{5.32}$$

that is, by

$$\mathrm{Bsq}_r(q_0,q_1) \;=\; \begin{cases} q_{0,t_r} + \frac{1}{6}\,F_{r,xxxxx} + \frac{5}{6}\,q_1 F_{r,xxx} + \frac{5}{4}\,q_{1,x}F_{r,xx} \\ \qquad +(\frac{3}{4}\,q_{1,xx} + \frac{2}{3}\,q_1^2)F_{r,x} + (\frac{1}{6}\,q_{1,xxx} + \frac{2}{3}\,q_1 q_{1,x})F_r \\ \qquad +3(z-q_0)G_{r,x} - q_{0,x}G_r = 0 \\ q_{1,t_r} - 2G_{r,xxx} - 2q_1 G_{r,x} - q_{1,x}G_r \\ \qquad +3(z-q_0)F_{r,x} - 2q_{0,x}F_r = 0 \end{cases}, \tag{5.33}$$

$$(x,t_r)\in\mathbb{R}^2, \quad r=3\,n+1 \text{ or } r=3\,n+2, \quad n\in\mathbb{N}_0.$$

Explicitly, one obtains for the first few equations in (5.32),

$$\mathrm{Bsq}_1(q_0,q_1) = \begin{cases} q_{0,t_1} - q_{0,x} = 0 \\ q_{1,t_1} - q_{1,x} = 0 \end{cases},$$

$$\mathrm{Bsq}_2(q_0,q_1) = \begin{cases} q_{0,t_2} + \frac{1}{6}\,q_{1,xxx} + \frac{2}{3}\,q_1 q_{1,x} - d_0 q_{0,x} = 0 \\ q_{1,t_2} - 2\,q_{0,x} - d_0 q_{1,x} = 0 \end{cases},$$

$$\mathrm{Bsq}_4(q_0,q_1) = \begin{cases} q_{0,t_4} + \frac{1}{18}\,q_{1,xxxxx} + \frac{1}{3}\,q_1 q_{1,xxx} + \frac{2}{3}\,q_{1,x}q_{1,xx} + \frac{4}{9}\,q_1^2 q_{1,x} \\ \qquad -\frac{4}{3}\,q_0 q_{0,x} + c_1\big(\frac{1}{6}\,q_{1,xxx} + \frac{2}{3}\,q_1 q_{1,x}\big) - d_1 q_{0,x} = 0 \\ q_{1,t_4} - \frac{2}{3}\,q_{0,xxx} - \frac{4}{3}\,q_1 q_{0,x} - \frac{4}{3}\,q_{1,x}q_0 - c_1 2q_{0,x} - d_1 q_{1,x} = 0 \end{cases}, \tag{5.34}$$

$$\mathrm{Bsq}_5(q_0,q_1) = \begin{cases} q_{0,t_5} + \frac{1}{9}\,q_{0,xxxxx} + \frac{5}{18}\,q_0 q_{1,xxx} + \frac{5}{9}\,q_1 q_{0,xxx} + \frac{5}{9}\,q_{1,xx}q_{0,x} \\ \qquad +\frac{5}{6}\,q_{1,x}q_{0,xx} + \frac{5}{9}\,q_1^2 q_{0,x} + \frac{10}{9}\,q_0 q_1 q_{1,x} + d_0\big(\frac{1}{18}\,q_{1,xxxxx} \\ \qquad +\frac{1}{3}\,q_1 q_{1,xxx} + \frac{2}{3}\,q_{1,x}q_{1,xx} + \frac{4}{9}\,q_1^2 q_{1,x} - \frac{4}{3}\,q_0 q_{0,x}\big) \\ \qquad +c_1\left(\frac{1}{6}\,q_{1,xxx} + \frac{2}{3}\,q_1 q_{1,x}\right) - d_1 q_{0,x} = 0 \\ q_{1,t_5} + \frac{1}{9}\,q_{1,xxxxx} + \frac{5}{9}\,q_1 q_{1,xxx} + \frac{25}{18}\,q_{1,x}q_{1,xx} + \frac{5}{9}\,q_1^2 q_{1,x} \\ \qquad -\frac{10}{3}\,q_0 q_{0,x} - d_0\left(\frac{2}{3}\,q_{0,xxx} + \frac{4}{3}\,q_1 q_{0,x} + \frac{4}{3}\,q_{1,x}q_0\right) \\ \qquad -c_1\,2q_{0,x} - d_1\,q_{1,x} = 0 \end{cases},$$

etc.

Remark 5.3. *Due to our choice of L_3 in (5.1) (as opposed to $\tilde{L}_3$ mentioned immediately after (5.1)) our* Bsq_2 *system in (5.34) differs slightly from the standard* Bsq *system discussed, for instance, in* [8], [14], [27], *and* [46]. *In fact, the simple transformation (put $d_0 = 0$ for simplicity),*

$$q_0 \to \tilde{q}_0 = q_0 + \frac{1}{2}\, q_{1,x}, \quad q_1 \to \tilde{q}_1 = q_1 \tag{5.35}$$

transforms Bsq_2 *into*

$$\mathrm{Bsq}_2(\tilde{q}_0, \tilde{q}_1) = \begin{cases} \tilde{q}_{0,t_2} - \tilde{q}_{0,xx} + \frac{2}{3}\,(\tilde{q}_{1,xxx} + \tilde{q}_1 \tilde{q}_{1,x}) = 0 \\ \\ \tilde{q}_{1,t_2} - 2\,\tilde{q}_{0,x} + \tilde{q}_{1,xx} = 0 \end{cases}, \tag{5.36}$$

which in turn transforms into the nonlinear string equation

$$u_{tt} = b u_{xx} + a(u^2)_{xx} - \frac{1}{3}\, u_{xxxx}, \tag{5.37}$$

where

$$q_1(x,t) = -\frac{1}{4}\big(6\, a u(x,t) + 3\, b\big), \quad t = t_2, \tag{5.38}$$

with $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$ arbitrary constants. Moreover, we should emphasize that our Bsq_2 *system in (5.34) or (5.36) differs from the Kaup-Boussinesq system (see, e.g.,* [42] *and the references therein), whose algebro-geometric quasi-periodic solutions can be derived from an associated hyperelliptic curve (not branched at infinity)* [35], [43] *as opposed to the non-hyperelliptic case typical in our paper (for genus larger than 2).*

Remark 5.4. *As in Section 2 (cf. Remark 2.3) we decided to start by postulating the recursion relation (5.2) as the point of departure for developing our formalism. Alternatively, we could have started with*

$$(L_3 - z)\psi(P) = 0, \quad (P_r - y(P))\psi(P) = 0, \quad P = (z, y(P)) \in \mathcal{K}_{r-1} \setminus \{P_\infty\} \tag{5.39}$$

in the stationary case, respectively by

$$(L_3 - z)\psi(P, t_r) = 0, \quad (\frac{\partial}{\partial t_r} - P_r)\psi(P, t_r) = 0, \quad t_r \in \mathbb{R} \tag{5.40}$$

in the time-dependent case. This then yields (5.2) as a consequence of (5.10), (5.11), and (5.20) and analogously one infers (5.31)–(5.34).

6. The Stationary Bsq Formalism

We continue our study of the Bsq hierarchy and focus, in particular, on the stationary case. Our main strategy will be to develop the Bsq material in close analogy to the KdV discussion in Section 3 and establish the connections between the polynomial approach described in Section 5 and a fundamental meromorphic function $\phi(P, x)$ defined on the Boussinesq curve $\mathcal{K}_{r-1}$ in (5.30). Moreover, we discuss in some detail the associated stationary Baker-Akhiezer function $\psi(P, x, x_0)$, the common eigenfunction of L_3 and P_r, and associated positive divisors of degree $(r-1)$ on $\mathcal{K}_{r-1}$.

Before we enter any further details we should perhaps stress one important point. In spite of the considerable complexity of the formulas displayed at various places in Sections 5–7, the basic underlying formalism is a recursive one as described in depth in Section 5 (cf. (5.2), (5.10), and (5.11)). Consequently, almost every formula in this paper can be generated quite comfortably by using symbolic programs (such as Mathematica or Maple).

We recall the Bsq curve $\mathcal{K}_{r-1}$ in (5.30)

$$\begin{aligned} \mathcal{K}_{r-1} : \mathcal{F}_{r-1}(z,y) &= (y-k_r(z))^3 + (y-k_r(z))\, S_r(z) - T_r(z) = 0, \\ k_r(z) &= \sum_{\ell=0}^{n} k_{r,\ell} z^\ell, \quad S_r(z) = \sum_{p=0}^{2n+1-s} s_{r,p} z^p, \quad 0 \le s \le 2n+1, \\ T_r(z) &= \sum_{q=0}^{r} t_{r,q} z^q, \quad t_{r,r}=1, \quad r = 3n+1 \text{ or } r = 3n+2, \quad n \in \mathbb{N}_0, \end{aligned} \tag{6.1}$$

where $r \in 3\,\mathbb{N}_0 + 1$ or $3\,\mathbb{N}_0 + 2$ will be fixed throughout this section and denote its compactification (adding the branch point P_∞) by the same symbol $\mathcal{K}_{r-1}$. (In the following $\mathcal{K}_{r-1}$ will always denote the compactified curve.) Thus $\mathcal{K}_{r-1}$ becomes a (possibly singular) three-sheeted Riemann surface of arithmetic genus $(r-1)$ in a standard manner. We will need a bit more notation in this context. Points P on $\mathcal{K}_{r-1}$ are represented as pairs $P = (z, y(P))$ satisfying (6.1) together with $P_\infty = (\infty, \infty)$, the point at infinity. The complex structure on $\mathcal{K}_{r-1}$ is defined in the usual way by introducing local coordinates $\zeta_{P_0} : P \to (z - z_0)$ near points $P_0 \in \mathcal{K}_{r-1}$ which are neither branch nor singular points of $\mathcal{K}_{r-1}$, $\zeta_{P_\infty} : P \to 1/z^{1/3}$ near the branch point $P_\infty \in \mathcal{K}_{r-1}$ (with an appropriate determination of the branch of $z^{1/3}$) and similar at branch and/or singular points of $\mathcal{K}_{r-1}$. The holomorphic map $*$, changing sheets, is defined by

$$\begin{aligned} * : & \quad \begin{cases} \mathcal{K}_{r-1} \to \mathcal{K}_{r-1} \\ P = (z, y_j(z)) \to P^* = (z, y_{j+1 \,(\mathrm{mod}\, 3)}(z), \quad j = 1,2,3 \end{cases}, \\ P^{**} & := (P^*)^*, \quad \text{etc.}, \end{aligned} \tag{6.2}$$

where $y_j(z)$, $j = 1, 2, 3$ denote the three branches of $y(P)$ satisfying $\mathcal{F}_{r-1}(z, y) = 0$, that is,

$$(y - y_1(z))(y - y_2(z))(y - y_3(z)) = (y - k_r(z))^3 + (y - k_r(z))\, S_r(z) - T_r(z) = 0. \tag{6.3}$$

Finally, positive divisors on $\mathcal{K}_{r-1}$ of degree $r - 1$ are denoted by

$$\mathcal{D}_{P_1,\dots,P_{r-1}} : \begin{cases} \mathcal{K}_{r-1} \to \mathbb{N}_0 \\ P \to \mathcal{D}_{P_1,\dots,P_{r-1}}(P) = \begin{cases} k \text{ if } P \text{ occurs } k \\ \qquad \text{times in } \{P_1, \dots, P_{r-1}\} \\ 0 \text{ if } P \notin \{P_1, \dots, P_{r-1}\} \end{cases} \end{cases}. \tag{6.4}$$

Given these preliminaries, let $\psi(P, x, x_0)$ denote the common normalized eigenfunction of L_3 and P_r, whose existence is guaranteed by the commutativity of L_3 and P_r (cf., e.g., [10], [11]), that is, by

$$[P_r, L_3] = 0, \qquad r = 3n+1 \quad \text{or} \quad r = 3n+2 \tag{6.5}$$

for a given $n \in \mathbb{N}_0$, or equivalently, by the requirement

$$f_{n+1,x} = 0, \qquad g_{n+1,x} = 0. \tag{6.6}$$

Explicitly, this yields

$$\begin{aligned} L_3 \psi(P, x, x_0) = z\, \psi(P, x, x_0), \quad P_r \psi(P, x, x_0) = y(P)\, \psi(P, x, x_0), \\ P = (z, y(P)) \in \mathcal{K}_{r-1} \setminus \{P_\infty\}, \quad x \in \mathbb{R} \end{aligned} \tag{6.7}$$

for some fixed $x_0 \in \mathbb{R}$ with the assumed normalization,

$$\psi(P, x_0, x_0) = 1, \qquad P \in \mathcal{K}_{r-1} \setminus \{P_\infty\}. \tag{6.8}$$

$\psi(P, x, x_0)$ is called the stationary BA-function for the Bsq hierarchy. Closely related to $\psi(P, x, x_0)$ is the following meromorphic function $\phi(P, x)$ on $\mathcal{K}_{r-1}$ defined by

$$\phi(P, x) = \frac{\psi_x(P, x, x_0)}{\psi(P, x, x_0)}, \quad P \in \mathcal{K}_{r-1},\ x \in \mathbb{R}, \tag{6.9}$$

such that

$$\psi(P, x, x_0) = \exp\Big[\int_{x_0}^{x} d\,x' \phi(P, x')\Big], \qquad P \in \mathcal{K}_{r-1} \setminus \{P_\infty\}. \tag{6.10}$$

Since $\phi(P, x)$ is a fundamental object for the stationary Bsq hierarchy, we next intend to establish its connection with the recursion formalism of Section 5.

In order to derive the analog of (3.16), (3.17) , etc., it seems helpful to define a variety of further polynomials with respect to z,

$$A_r(z, x) = -G_r(z, x)^2 - \frac{1}{3}\, q_1(x)\, F_r(z, x)^2 + \frac{1}{4}\, F_{r,x}(z, x)^2 - \frac{1}{3}\, F_r(z, x)\, F_{r,xx}(z, x), \tag{6.11}$$

$$\begin{aligned}
B_r(z, x) ={}& (z - q_0(x))\big(-2\, F_r(z, x)^2\, G_r(z, x) + \frac{1}{2}\, F_r(z, x)^2\, F_{r,x}(z, x)\big)\\
&- G_r(z, x)^2\, G_{r,x}(z, x) + \frac{1}{4}\, F_{r,x}(z, x)^2\, G_{r,x}(z, x) - F_r(z, x)\, G_{r,x}(z, x)^2\\
&- \frac{1}{6}\, q_{1,x}(x)\, F_r(z, x)^2\, G_r(z, x) - \frac{1}{2}\, q_{1,x}(x)\, F_r(z, x)^2\, F_{r,x}(z, x)\\
&+ \frac{1}{6}\, G_r(z, x)^2\, F_{r,xx}(z, x) - \frac{11}{18}\, q_1(x)\, F_r(z, x)^2\, F_{r,xx}(z, x)\\
&- \frac{1}{24}\, F_{r,x}(z, x)^2\, F_{r,xx}(z, x) + \frac{1}{36}\, F_r(z, x)\, F_{r,xx}(z, x)^2\\
&+ \frac{2}{3}\, q_1(x)\, F_r(z, x)\, G_r(z, x)^2 - \frac{2}{9}\, q_1(x)^2\, F_r(z, x)^3\\
&- \frac{2}{3}\, q_1(x) F_r(z, x)\, G_r(z, x)\, F_{r,x}(z, x) + \frac{1}{6}\, q_1(x) F_r(z, x)\, F_{r,x}(z, x)^2\\
&+ F_r(z, x)\, G_r(z, x)\, G_{r,xx}(z, x) - \frac{1}{2}\, F_r(z, x)\, F_{r,x}(z, x)\, G_{r,xx}(z, x)\\
&- \frac{1}{6}\, q_{1,xx}(x)\, F_r(z, x)^3 - \frac{1}{6}\, F_r(z, x)\, G_r(z, x)\, F_{r,xxx}(z, x)\\
&+ \frac{1}{12}\, F_r(z, x)\, F_{r,x}(z, x)\, F_{r,xxx}(z, x) - \frac{1}{6}\, F_r(z, x)^2\, F_{r,xxxx}(z, x),
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
C_r(z, x) ={}& (z - q_0(x)) F_r(z, x)^2 - \frac{2}{3}\, q_1(x) F_r(z, x) G_r(z, x) + \frac{1}{6}\, q_{1,x}(x) F_r(z, x)^2\\
&+ G_r(z, x) G_{r,x}(z, x) + \frac{1}{2}\, F_{r,x}(z, x) G_{r,x}(z, x) - \frac{1}{12}\, F_{r,x}(z, x) F_{r,xx}(z, x)\\
&- \frac{1}{6}\, G_r(z, x) F_{r,xx}(z, x) - F_r(z, x) G_{r,xx}(z, x)\\
&+ \frac{1}{3}\, q_1(x) F_r(z, x) F_{r,x}(z, x) + \frac{1}{6}\, F_r(z, x) F_{r,xxx}(z, x),
\end{aligned} \tag{6.13}$$

$$
\begin{aligned}
E_r(z,x) = {} & (z-q_0)\Big(2F_r(z,x)G_r(z,x)^2 + \frac{1}{3}\, q_1(x)F_r(z,x)^3 \\
& + F_r(z,x)F_{r,x}(z,x)G_r(z,x) + \frac{1}{3}\, F_r(z,x)^2 F_{r,xx}(z,x)\Big) \\
& + \frac{1}{6}\, q_1(x)F_r(z,x)F_{r,x}(z,x)G_{r,x}(z,x) - \frac{1}{9}\, q_1(x)^2F_r(z,x)^2G_r(z,x) \\
& - \frac{1}{2}\, q_1(x)F_{r,x}(z,x)G_r(z,x)^2 + \frac{1}{6}\, q_1(x)^2F_r(z,x)^2F_{r,x}(z,x) \\
& - \frac{5}{12}\, q_1(x)F_{r,x}(z,x)^2G_r(z,x) - \frac{5}{24}\, q_1(x)F_{r,x}(z,x)^3 \\
& + \frac{1}{3}\, q_1(x)F_r(z,x)G_r(z,x)G_{r,x}(z,x) - q_1(x)G_r(z,x)^3 \\
& + \frac{1}{18}\, q_1(x)q_{1,x}(x)F_r(z,x)^3 - \frac{1}{6}\, q_{1,x}(x)F_r(z,x)F_{r,x}(z,x)G_r(z,x) \\
& - \frac{1}{12}\, q_{1,x}(x)F_r(z,x)F_{r,x}(z,x)^2 - \frac{1}{18}\, q_1(x)F_r(z,x)F_{r,xx}(z,x)G_r(z,x) \\
& + \frac{7}{36}\, q_1(x)F_r(z,x)F_{r,x}(z,x)F_{r,xx}(z,x) + \frac{1}{3}\, F_{r,xx}(z,x)G_r(z,x)G_{r,x}(z,x) \\
& + \frac{1}{6}\, F_{r,x}(z,x)F_{r,xx}(z,x)G_{r,x}(z,x) + \frac{1}{18}\, q_{1,x}(x)F_r(z,x)^2F_{r,xx}(z,x) \\
& + \frac{1}{18}\, F_{r,xx}(z,x)^2G_r(z,x) + \frac{1}{36}\, F_{r,x}(z,x)F_{r,xx}(z,x)^2 \\
& - 2\, G_r(z,x)^2G_{r,xx}(z,x) - \frac{1}{3}\, q_1(x)F_r(z,x)^2G_{r,xx}(z,x) \\
& - F_{r,x}(z,x)G_r(z,x)G_{r,xx}(z,x) - \frac{1}{3}\, F_r(z,x)F_{r,xx}(z,x)G_{r,xx}(z,x) \\
& + \frac{1}{18}\, q_1(x)F_r(z,x)^2F_{r,xxx}(z,x) - \frac{1}{6}\, F_{r,x}(z,x)F_{r,xxx}(z,x)G_r(z,x) \\
& - \frac{1}{12}\, F_{r,x}(z,x)^2F_{r,xxx}(z,x) + \frac{1}{18}\, F_r(z,x)F_{r,xx}(z,x)F_{r,xxx}(z,x),
\end{aligned} \tag{6.14}
$$

$$
J_r(z,x) = H_{r,x}(z,x) + \big(z - q_0(x) - \frac{1}{2}\, q_{1,x}(x)\big)\, F_r(z,x), \tag{6.15}
$$

$$
\begin{aligned}
D_{r-1}(z,x) = {} & \epsilon(r)\,\Big((z-q_0(x) - \frac{1}{6}\, q_{1,x})F_r(z,x)^3 - G_r(z,x)^3 + \frac{1}{4}\, G_r(z,x)\, F_{r,x}(z,x)^2 \\
& - q_1(x)\, F_r(z,x)^2\, G_r(z,x) + \frac{1}{2}\, G_r(z,x)^2\, F_{r,x}(z,x) - \frac{1}{8}\, F_{r,x}(z,x)^3 \\
& - \frac{1}{6}\, q_1(x)\, F_r(z,x)^2\, F_{r,x}(z,x) - F_r(z,x)\, G_r(z,x)\, G_{r,x}(z,x) \\
& + \frac{1}{2}\, F_r(z,x)\, F_{r,x}(z,x)\, G_{r,x}(z,x) - \frac{1}{2}\, F_r(z,x)\, G_r(z,x)\, F_{r,xx}(z,x) \\
& + \frac{1}{4}\, F_r(z,x)\, F_{r,x}(z,x)\, F_{r,xx}(z,x) - F_r(z,x)^2\, G_{r,xx}(z,x) \\
& - \frac{1}{6}\, F_r(z,x)^2\, F_{r,xxx}(z,x)\Big),
\end{aligned} \tag{6.16}
$$

$$
\begin{aligned}
N_r(z,x) = {} & \epsilon(r)\, \frac{1}{144}\Big(144\,(z-q_0(x))^2\, F_r(z,x)^3 - 144\,(z-q_0(x))\, G_r(z,x)^2\, F_{r,x}(z,x) \\
& - 144\,(z-q_0(x))\, q_1(x)\, F_r(z,x)^2\, G_r(z,x) - 144\,(z-q_0(x))\, G_r(z,x)^3
\end{aligned}
$$

$$
\begin{aligned}
&+ 120\,(z-q_0(x))\,q_1(x)\,F_r(z,x)^2\,F_{r,x}(z,x) - 24\,q_{1,xx}\,F_r(z,x)\,G_r(z,x)^2\\
&- 36\,(z-q_0(x))\,G_r(z,x)\,F_{r,x}(z,x)^2 + 16\,q_1(x)^2\,F_r(z,x)\,F_{r,x}(z,x)^2\\
&+ 288\,(z-q_0(x))\,F_r(z,x)\,G_r(z,x)\,G_{r,x}(z,x) - 144\,q_1(x)\,G_r(z,x)^2\,G_{r,x}(z,x)\\
&+ 48\,q_1(x)\,G_r(z,x)\,F_{r,x}(z,x)\,G_{r,x}(z,x) + 60\,q_1(x)\,F_{r,x}(z,x)^2\,G_{r,x}(z,x)\\
&+ 48\,(z-q_0(x))\,q_{1,x}\,F_r(z,x)^3 - 24\,q_1(x)\,q_{1,x}\,F_r(z,x)^2\,G_r(z,x)\\
&- 84\,q_{1,x}\,G_r(z,x)^2\,F_{r,x}(z,x) + 20\,q_1(x)\,q_{1,x}\,F_r(z,x)^2\,F_{r,x}(z,x)\\
&- 84\,q_{1,x}\,G_r(z,x)\,F_{r,x}(z,x)^2 + 48\,(z-q_0(x))\,F_r(z,x)^2\,F_{r,xxx}(z,x)\\
&+ 48\,q_{1,x}\,F_r(z,x)\,G_r(z,x)\,G_{r,x}(z,x) + 24\,q_{1,x}\,F_r(z,x)\,F_{r,x}(z,x)\,G_{r,x}(z,x)\\
&+ 20\,q_1(x)\,F_r(z,x)\,F_{r,x}(z,x)\,F_{r,xxx}(z,x) - 96\,q_1(x)\,G_r(z,x)^2\,F_{r,xx}(z,x)\\
&- 96\,q_1(x)\,G_r(z,x)\,F_{r,x}(z,x)\,F_{r,xx}(z,x) - 24\,q_1(x)\,F_{r,x}(z,x)^2\,F_{r,xx}(z,x)\\
&- 288\,(z-q_0)\,F_r(z,x)^2\,G_{r,xx}(z,x) + 144\,q_1(x)\,F_r(z,x)\,G_r(z,x)\,G_{r,xx}(z,x)\\
&- 120\,q_1(x)\,F_r(z,x)\,F_{r,x}(z,x)\,G_{r,xx}(z,x) - 288\,G_r(z,x)\,G_{r,x}(z,x)\,G_{r,xx}(z,x)\\
&- 144\,F_{r,x}(z,x)\,G_{r,x}(z,x)\,G_{r,xx}(z,x) - 48\ q_{1,x}F_r(z,x)^2\,G_{r,xx}(z,x)\\
&+ 144\,F_r(z,x)\,G_{r,xx}(z,x)^2 - 96\,q_1(x)^2\,F_r(z,x)\,G_r(z,x)\,F_{r,x}(z,x)\\
&- 24\,q_{1,xx}\,F_r(z,x)\,G_r(z,x)\,F_{r,x}(z,x) - 6\,q_{1,xx}\,F_r(z,x)\,F_{r,x}(z,x)^2\\
&- 21\,q_{1,x}\,F_{r,x}(z,x)^3 - 24\,q_1(x)\,F_r(z,x)\,G_r(z,x)\,F_{r,xxx}(z,x)\\
&- 6\,F_{r,x}(z,x)^2\,F_{r,xxxx}(z,x) + 48\,G_r(z,x)\,G_{r,x}(z,x)\,F_{r,xxx}(z,x)\\
&+ 24\,F_{r,x}(z,x)\,G_{r,x}(z,x)\,F_{r,xxx}(z,x) + 8\,q_{1,x}\,F_r(z,x)^2\,F_{r,xxx}(z,x)\\
&- 48\,F_r(z,x)\,G_{r,xx}(z,x)\,F_{r,xxx}(z,x) + 4\,F_r(z,x)\,F_{r,xxx}(z,x)^2\\
&- 24\,G_r(z,x)^2\,F_{r,xxxx}(z,x) - 24\,G_r(z,x)\,F_{r,x}(z,x)\,F_{r,xxxx}(z,x)\\
&+ 144\,(z-q_0(x))\,F_r(z,x)\,F_{r,x}(z,x)\,G_{r,x}(z,x) + 4\,q_{1,x}^2\,F_r(z,x)^3\Big),
\end{aligned} \tag{6.17}
$$

where

$$
\epsilon(r) = \begin{cases} 1, & r \equiv 2 \pmod 3 \\ -1, & r \equiv 1 \pmod 3 \end{cases}. \tag{6.18}
$$

The quantities $A_r, \dots, N_r$ in (6.11)–(6.17) are of course not independent of each other. There exist various interrelationships between them and S_r, T_r (cf. (6.1)), some of which are summarized below.

Lemma 6.1. *Let* $(z,x) \in \mathbb{C} \times \mathbb{R}$. *Then*

$$(i).\ A_r\,C_r - B_r(G_r + \frac{1}{2}\,F_{r,x}) + F_r\,E_r + S_r\,F_r\,(G_r + \frac{1}{2}\,F_{r,x}) = 0. \tag{6.19}$$

$$(ii).\ B_r\,C_r + A_r\,E_r + S_r\left[A_r\,(G_r + \frac{1}{2}\,F_{r,x}) - F_r\,C_r\right] - T_r\,F_r\,(G_r + \frac{1}{2}\,F_{r,x}) = 0. \tag{6.20}$$

$$(iii).\ C_r = F_r\,J_r - (G_r + \frac{1}{2}F_{r,x})(H_r - k_r). \tag{6.21}$$

$$(iv).\ B_r = \frac{2}{3}\,S_r\,F_r + \frac{1}{3}\,\epsilon(r)\,D_{r-1,x}. \tag{6.22}$$

$$(v).\ \epsilon(r)\,(G_r + \frac{1}{2}\,F_{r,x})\,D_{r-1} = F_r\,B_r - A_r^2 - S_r\,F_r^2. \tag{6.23}$$

$$(vi).\ \epsilon(r)\,C_r\,D_{r-1} = T_r\,F_r^2 - A_rB_r. \tag{6.24}$$

$$(vii).\ D_{r-1}N_r = B_r\,E_r - T_r[A_r\,(G_r + \frac{1}{2}\,F_{r,x}) - F_r\,C_r]. \tag{6.25}$$

$$(viii).\ \epsilon(r)\,A_r\,N_r = T_r\,(G_r + \frac{1}{2}\,F_{r,x})^2 - C_rE_r. \tag{6.26}$$

$$(ix).\ \epsilon(r)\,F_r\,N_r = C_r^2 + E_r\,(G_r + \frac{1}{2}\,F_{r,x}) + S_r(G_r + \frac{1}{2}\,F_{r,x})^2. \tag{6.27}$$

$$(x).\ N_{r,x}\big(G_r + \frac{1}{2}\,F_{r,x}\big) = N_r\big(q_1\,F_r + F_{r,xx}\big) - \epsilon(r)\,J_r\,\big[2\,\big(G_r + \frac{1}{2}\,F_{r,x}\big)\,S_r + 3\,E_r\big]. \tag{6.28}$$

Proof. This is a straightforward (but tedious) consequence of (5.15), (5.17), (6.11)–(6.17). ☐

Next we derive a first formula for $\phi(P,x)$. By (5.21) and (6.7) one infers

$$P_r\psi = F_r\psi_{xx} + (G_r - \frac{1}{2}\,F_{r,x})\psi_x + H_r\psi = y\psi \tag{6.29}$$

and hence

$$\begin{aligned}\big(P_r\psi\big)_x &= F_{r,x}\psi_{xx} + F_r\psi_{xxx} + (G_{r,x} - \frac{1}{2}\,F_{r,xx})\psi_x + (G_r - \frac{1}{2}\,F_{r,x})\psi_{xx}\\ &\qquad + H_{r,x}\psi + H_r\psi_x\\ &= F_{r,x}\psi_{xx} + (z - q_0 - \frac{1}{2}\,q_{1,x})F_r\psi - q_1F_r\psi_x + (G_{r,x} - \frac{1}{2}\,F_{r,xx})\psi_x\\ &\qquad + (G_r - \frac{1}{2}\,F_{r,x})\psi_{xx} + H_{r,x}\psi + H_r\psi_x = y\psi_x.\end{aligned} \tag{6.30}$$

Using (6.29) in (6.30) in order to eliminate ψ_{xx} in terms of $\phi = \psi_x/\psi$ one infers

$$\phi = \frac{(G_r + \frac{1}{2}\,F_{r,x})(y - H_r) + (z - q_0 - \frac{1}{2}\,q_{1,x})F_r^2 + H_{r,x}\,F_r}{(y - H_r)F_r - (G_{r,x} - \frac{1}{2}\,F_{r,xx})F_r + q_1F_r^2 + G_r^2 - \frac{1}{4}F_{r,x}^2}. \tag{6.31}$$

In fact, (6.31) is just one of three expressions one can derive linking ϕ and F_r, G_r.

Lemma 6.2. *Let $P = (z, y(P)) \in \mathcal{K}_{r-1}$ and $(z,x) \in \mathbb{C}\times\mathbb{R}$. Then*

$$\phi(P,x) = \frac{(G_r(z,x) + \frac{1}{2}\,F_{r,x}(z,x))\big(y(P) - k_r(z)\big) + C_r(z,x)}{F_r(z,x)\big(y(P) - k_r(z)\big) - A_r(z,x)} \tag{6.32}$$

$$= \frac{F_r(z,x)\big(y(P) - k_r(z)\big)^2 + A_r(z,x)\big(y(P) - k_r(z)\big) + B_r(z,x)}{\epsilon(r)D_{r-1}(z,x)} \tag{6.33}$$

$$= \frac{-\epsilon(r)N_r(z,x)}{(G_r(z,x) + \frac{1}{2}\,F_{r,x}(z,x))\big(y(P) - k_r(z)\big)^2 - C_r(z,x)\big(y(P) - k_r(z)\big) - E_r(z,x)}. \tag{6.34}$$

Proof. (6.32) follows from (5.21), (6.11), (6.13), and (6.31). (6.33) is a consequence of (6.1), (6.23), (6.24) and (6.32). Similarly, (6.34) follows from (6.1), (6.26), (6.27), and (6.32). ☐

By inspection of (5.10) and (5.11) one infers that D_{r-1} and N_r are monic polynomials with respect to z of degree $r-1$ and r, respectively. Hence we may write

$$D_{r-1}(z,x) = \prod_{j=1}^{r-1}[z - \mu_j(x)], \tag{6.35}$$

$$N_r(z,x) \;\; = \;\; \prod_{\ell=0}^{r-1}[z-\nu_\ell(x)]. \tag{6.36}$$

Defining Dirichlet and Neumann data in analogy to (3.21) and (3.22) by

$$\hat\mu_j(x) = \big(\mu_j(x), y(\hat\mu_j(x))\big) = \Big(\mu_j(x), k_r(\mu_j(x)) + \frac{A_r(\mu_j(x),x)}{F_r(\mu_j(x),x)}\Big) \in \mathcal{K}_{r-1}, \qquad 1 \le j \le r-1, \quad x \in \mathbb{R}, \tag{6.37}$$

$$\begin{aligned}\hat\nu_\ell(x) &= \big(\nu_\ell(x), y(\hat\nu_\ell(x))\big) \\ &= \Big(\nu_\ell(x), k_r(\nu_\ell(x)) - \frac{C_r(\nu_\ell(x),x)}{G_r(\nu_\ell(x),x) + \frac{1}{2}\, F_{r,x}(\nu_\ell(x),x)}\Big) \in \mathcal{K}_{r-1}, \\ &\qquad 0 \le \ell \le r-1, \quad x \in \mathbb{R},\end{aligned} \tag{6.38}$$

one infers from (6.32) that the divisor $\big(\phi(P,x)\big)$ of $\phi(P,x)$ is given by (cf. (6.4))

$$\big(\phi(P,x)\big) = \mathcal{D}_{\hat\nu_0(x),\dots,\hat\nu_{r-1}(x)}(P) - \mathcal{D}_{P_\infty,\hat\mu_1(x),\dots,\hat\mu_{r-1}(x)}(P). \tag{6.39}$$

That is, $\hat\nu_0(x),\dots,\hat\nu_{r-1}(x)$ are the r zeros of $\phi(P,x)$ and $P_\infty, \hat\mu_1(x),\dots,\hat\mu_{r-1}(x)$ its r poles.

Further properties of $\phi(P,x)$ and $\psi(P,x,x_0)$ are summarized in

Lemma 6.3. *Assume (6.5)–(6.9), $P = (z, y(P)) \in \mathcal{K}_{r-1} \setminus \{P_\infty\}$, and let $(z,x,x_0) \in \mathbb{C} \times \mathbb{R}^2$. Then*

(*i*). *$\phi(P,x)$ satisfies the second-order equation*

$$\phi_{xx}(P,x) + 3\,\phi_x(P,x)\phi(P,x) + \phi(P,x)^3 + q_1(x)\,\phi(P,x) = z - q_0(x) - \frac{1}{2}\,q_{1,x}(x). \tag{6.40}$$

$$(ii).\ \phi(P,x)\,\phi(P^*,x)\,\phi(P^{**},x) = \frac{N_r(z,x)}{D_{r-1}(z,x)}. \tag{6.41}$$

$$(iii).\ \phi(P,x) + \phi(P^*,x) + \phi(P^{**},x) = \frac{D_{r-1,x}(z,x)}{D_{r-1}(z,x)}. \tag{6.42}$$

$$\begin{aligned}(iv).\ &(y(P)-k_r(z))\,\phi(P,x) + (y(P^*)-k_r(z))\,\phi(P^*,x) \\ &\quad + (y(P^{**})-k_r(z))\,\phi(P^{**},x) = \frac{3\,T_r(z)\,F_r(z,x) - 2\,S_r(z)\,A_r(z,x)}{\epsilon(r)D_{r-1}(z,x)}.\end{aligned} \tag{6.43}$$

$$(v).\ \psi(P,x,x_0)\,\psi(P^*,x,x_0)\,\psi(P^{**},x,x_0) = \frac{D_{r-1}(z,x)}{D_{r-1}(z,x_0)}. \tag{6.44}$$

$$(vi).\ \psi_x(P,x,x_0)\,\psi_x(P^*,x,x_0)\,\psi_x(P^{**},x,x_0) = \frac{N_r(z,x)}{D_{r-1}(z,x_0)}. \tag{6.45}$$

$$\begin{aligned}(vii).\ \psi(P,x,x_0) &= \bigg[\frac{D_{r-1}(z,x)}{D_{r-1}(z,x_0)}\bigg]^{1/3} \exp\bigg\{\int_{x_0}^{x} d\,x'\epsilon(r)D_{r-1}(z,x')^{-1} \\ &\times \Big[F_r(z,x')\big(y(P)-k_r(z)\big)^2 + A_r(z,x')\big(y(P)-k_r(z)\big) + \frac{2}{3}\,F_r(z,x')\,S_r(z)\Big]\bigg\}.\end{aligned} \tag{6.46}$$

Proof. (6.40) is clear from $\phi = \psi_x/\psi$ and $\psi_{xxx} + q_1\psi_x + (q_0 + \frac{1}{2}\,q_{1,x} - z)\psi = 0$. (6.42) is a consequence of (6.22), (6.33), and

$$y(P) + y(P^*) + y(P^{**}) = 3\,k_r(z), \tag{6.47}$$

$$(y(P)-k_r(z))^2+(y(P^*)-k_r(z))^2+(y(P^{**})-k_r(z))^2=-2\,S_r(z). \tag{6.48}$$

Similarly, (6.43) follows from (6.1), (6.33), (6.47), and (6.48). In order to prove (6.41) one employs (6.32), (6.47), (6.48), and

$$(y(P)-k_r(z))\,(y(P^*)-k_r(z))\,(y(P^{**})-k_r(z))=T_r(z), \tag{6.49}$$

$$\begin{aligned}(y(P)-k_r(z))\,(y(P^*)-k_r(z))+(y(P^*)-k_r(z))\,(y(P^{**})-k_r(z))\\ +(y(P^{**})-k_r(z))\,(y(P)-k_r(z))=S_r(z)\end{aligned} \tag{6.50}$$

to get

$$\phi(P,x)\,\phi(P^*,x)\,\phi(P^{**},x)=\frac{T_r\,(G_r+\frac{1}{2}\,F_{r,x})^3+S_r\,C_r\,(G_r+\frac{1}{2}\,F_{r,x})^2+C_r^3}{T_r\,F_r^3-S_r\,F_r^2\,A_r-A_r^3}. \tag{6.51}$$

Using (6.16) and (6.17) one verifies that the numerator in (6.51) factors into $D^*_{r-1}\,N_r$ and the denominator into $D^*_{r-1}\,D_{r-1}$, where D^*_{r-1} is defined by

$$\begin{aligned}D^*_{r-1}(z,x)=\epsilon(r)\Big(&(z-q_0(x)+\frac{1}{6}\,q_{1,x}\,)F_r(z,x)^3-G_r(z,x)^3+\frac{1}{4}\,G_r(z,x)\,F_{r,x}(z,x)^2\\
&-q_1(x)\,F_r(z,x)^2\,G_r(z,x)-\frac{1}{2}\,G_r(z,x)^2\,F_{r,x}(z,x)+\frac{1}{8}\,F_{r,x}(z,x)^3\\
&+\frac{1}{6}\,q_1(x)\,F_r(z,x)^2\,F_{r,x}(z,x)+F_r(z,x)\,G_r(z,x)\,G_{r,x}(z,x)\\
&+\frac{1}{2}\,F_r(z,x)\,F_{r,x}(z,x)\,G_{r,x}(z,x)-\frac{1}{2}\,F_r(z,x)\,G_r(z,x)\,F_{r,xx}(z,x)\\
&-\frac{1}{4}\,F_r(z,x)\,F_{r,x}(z,x)\,F_{r,xx}(z,x)-F_r(z,x)^2\,G_{r,xx}(z,x)\\
&+\frac{1}{6}\,F_r(z,x)^2\,F_{r,xxx}(z,x)\Big).\end{aligned} \tag{6.52}$$

(6.44) immediately follows from (6.41) and (6.46) and (6.45) from (6.44) and (6.41). It remains to prove (6.46). The latter directly follows after inserting (6.33) into (6.10) and then replacing B_r according to (6.22). $\square$

Thus, up to normalizations, D_{r-1} represents the product of the three branches of ψ and N_r the product of the three branches of ψ_x.

Returning to $D_{r-1}(z,x)$ and $N_r(z,x)$ we note that (5.2), (5.10), (5.11), (6.16), and (6.17) yield

$$\begin{aligned}D_0&=1,\\
D_1&=z-q_0(x)-\frac{1}{6}\,q_{1,x}(x)-d_0\,q_1(x)-d_0^3,\\
D_3&=\frac{1}{648}\Big(648\,z^3+z^2\,\big(648\,q_0(x)-108\,q_{1,x}(x)\big)+z\,\big(216\,q_0(x)^2+48\,q_1(x)^3\\
&+72\,q_1(x)\,q_{0,x}(x)-72\,q_0(x)\,q_{1,x}(x)-18\,q_{1,x}(x)^2+36\,q_1(x)\,q_{1,xx}(x)\big)\\
&+24\,q_0(x)^3+48\,q_0(x)\,q_1(x)^3+24\,q_0(x)\,q_1(x)\,q_{0,x}(x)-12\,q_0(x)^2\,q_{1,x}(x)\\
&+8\,q_1(x)^3\,q_{1,x}(x)-12\,q_1(x)\,q_{0,x}(x)\,q_{1,x}(x)-6\,q_0(x)\,q_{1,x}(x)^2+3\,q_{1,x}(x)^3\\
&+24\,q_1(x)^2\,q_{0,xx}(x)+12\,q_0(x)\,q_1(x)\,q_{1,xx}(x)-6\,q_1(x)\,q_{1,x}(x)\,q_{1,xx}(x)\\
&+4\,q_1(x)^2\,q_{1,xxx}(x)+648\,{d_1}^3+1944\,d_1\,z^2+648\,{c_1}^3\,q_0(x)+648\,{d_1}^2\,q_0(x)\\
&+216\,d_1\,q_0(x)^2+648\,{c_1}^2\,d_1\,q_1(x)+864\,{c_1}^2\,q_0(x)\,q_1(x)+432\,c_1\,d_1\,q_1(x)^2\end{aligned}$$

$$
\begin{aligned}
&+ 360\, c_1\, q_0(x)\, q_1(x)^2 + 72\, d_1\, q_1(x)^3 + 216\, c_1\, d_1\, q_{0,x}(x) + 72\, c_1\, q_0(x)\, q_{0,x}(x) \\
&+ 72\, d_1\, q_1(x)\, q_{0,x}(x) + 108\, {c_1}^3\, q_{1,x}(x) - 108\, {d_1}^2\, q_{1,x}(x) - 72\, d_1\, q_0(x)\, q_{1,x}(x) \\
&+ 144\, {c_1}^2\, q_1(x)\, q_{1,x}(x) + 60\, c_1\, q_1(x)^2\, q_{1,x}(x) - 36\, c_1\, q_{0,x}(x)\, q_{1,x}(x) \\
&- 18\, d_1\, q_{1,x}(x)^2 + 216\, {c_1}^2\, q_{0,xx}(x) + 144\, c_1\, q_1(x)\, q_{0,xx}(x) + 108\, c_1\, d_1\, q_{1,xx}(x) \\
&+ 36\, c_1\, q_0(x)\, q_{1,xx}(x) + 36\, d_1\, q_1(x)\, q_{1,xx}(x) - 18\, c_1\, q_{1,x}(x)\, q_{1,xx}(x) \\
&+ z\, \big(- 648\, {c_1}^3 + 1944\, {d_1}^2 + 1296\, d_1\, q_0(x) + 216\, c_1\, q_1(x)^2 + 216\, c_1\, q_{0,x}(x) \\
&- 216\, d_1\, q_{1,x}(x) + 108\, c_1\, q_{1,xx}(x)\big) + 36\, {c_1}^2\, q_{1,xxx}(x) + 24\, c_1\, q_1(x)\, q_{1,xxx}(x)\Big),
\end{aligned}
$$

etc., (6.53)

and

$$
\begin{aligned}
N_1 &= z - q_0(x), \\
N_2 &= \big(z - q_0(x) + \frac{1}{6}\, q_{1,x}(x)\big)^2 - d_0\big((z - q_0(x))q_1(x) - \frac{1}{6}\, q_1(x) q_{1,x}(x)\big) \\
&\quad - d_0^2 \frac{1}{6}\, q_{1,xx}(x) - d_0^3 (z - q_0(x)), \qquad (6.54) \\
N_4 &= \Big(3888\, z^4 + 1296\, z^3\, q_{1,x}(x) + z^2\, \big(288\, q_1(x)^3 - 2592\, q_0(x)^2 + 432\, q_1(x)\, q_{0,x}(x) \\
&- 432\, q_0(x)\, q_{1,x}(x) + 864\, q_{1,x}(x)^2 + 1080\, q_1(x)\, q_{1,xx}(x) + 216\, q_{1,xxxx}(x)\big) \\
&+ z\, \big(1440\, q_0(x)\, q_1(x)\, q_{0,x}(x) - 1152\, q_0(x)^3 - 720\, q_0(x)^2\, q_{1,x}(x) \\
&+ 192\, q_1(x)^3\, q_{1,x}(x) - 432\, q_1(x)\, q_{0,x}(x)\, q_{1,x}(x) + 432\, q_0(x)\, q_{1,x}(x)^2 + 252\, q_{1,x}(x)^3 \\
&- 144\, q_1(x)^2\, q_{0,xx}(x) + 864\, q_{0,x}(x)\, q_{0,xx}(x) + 720\, q_0(x)\, q_1(x)\, q_{1,xx}(x) \\
&+ 360\, q_1(x)\, q_{1,x}(x)\, q_{1,xx}(x) + 24\, q_1(x)^2\, q_{1,xxx}(x) - 144\, q_{0,x}(x)\, q_{1,xxx}(x) \\
&+ 144\, q_0(x)\, q_{1,xxxx}(x) + 72\, q_{1,x}(x)\, q_{1,xxxx}(x)\big) - 144\, q_0(x)^4 - 288\, q_0(x)^2\, q_1(x)^3 \\
&+ 432\, q_0(x)^2\, q_1(x)\, q_{0,x}(x) - 144\, q_0(x)^3\, q_{1,x}(x) + 288\, q_0(x)\, q_1(x)^3\, q_{1,x}(x) \\
&+ 48\, q_0(x)\, q_1(x)\, q_{0,x}(x)\, q_{1,x}(x) + 48\, q_0(x)^2\, q_{1,x}(x)^2 - 40\, q_1(x)^3\, q_{1,x}(x)^2 \\
&- 84\, q_1(x)\, q_{0,x}(x)\, q_{1,x}(x)^2 + 84\, q_0(x)\, q_{1,x}(x)^3 - 432\, q_0(x)\, q_1(x)^2\, q_{0,xx}(x) \\
&+ 21\, q_{1,x}(x)^4 + 288\, q_0(x)\, q_{0,x}(x)\, q_{0,xx}(x) + 168\, q_1(x)^2\, q_{1,x}(x)\, q_{0,xx}(x) \\
&+ 144\, q_{0,x}(x)\, q_{1,x}(x)\, q_{0,xx}(x) - 144\, q_1(x)\, q_{0,xx}(x)^2 + 120\, q_0(x)^2\, q_1(x)\, q_{1,xx}(x) \\
&+ 120\, q_0(x)\, q_1(x)\, q_{1,x}(x)\, q_{1,xx}(x) + 30\, q_1(x)\, q_{1,x}(x)^2\, q_{1,xx}(x) \\
&+ 72\, q_0(x)\, q_1(x)^2\, q_{1,xxx}(x) - 48\, q_0(x)\, q_{0,x}(x)\, q_{1,xxx}(x) - 28\, q_1(x)^2\, q_{1,x}(x)\, q_{1,xxx}(x) \\
&- 24\, q_{0,x}(x)\, q_{1,x}(x)\, q_{1,xxx}(x) + 48\, q_1(x)\, q_{0,xx}(x)\, q_{1,xxx}(x) - 4\, q_1(x)\, q_{1,xxx}(x)^2 \\
&+ 24\, q_0(x)^2\, q_{1,xxxx}(x) + 24\, q_0(x)\, q_{1,x}(x)\, q_{1,xxxx}(x) + 6\, q_{1,x}(x)^2\, q_{1,xxxx}(x)\Big) \frac{1}{3888} \\
&+ \Big(1944\, d_1\, z^3 + z^2\, \big(1944\, {d_1}^2 - 648\, {c_1}^3 - 648\, d_1\, q_0(x) + 216\, c_1\, q_1(x)^2 \\
&- 432\, c_1\, q_{0,x}(x) + 432\, d_1\, q_{1,x}(x) + 108\, c_1\, q_{1,xx}(x)\big) + z\, \big(648\, {d_1}^3 + 1296\, {c_1}^3\, q_0(x) \\
&- 1296\, {d_1}^2\, q_0(x) - 1080\, d_1\, q_0(x)^2 + 648\, {c_1}^2\, d_1\, q_1(x) + 864\, {c_1}^2\, q_0(x)\, q_1(x) \\
&+ 432\, c_1\, d_1\, q_1(x)^2 + 144\, c_1\, q_0(x)\, q_1(x)^2 + 72\, d_1\, q_1(x)^3 - 432\, c_1\, d_1\, q_{0,x}(x)
\end{aligned}
$$

$$
\begin{aligned}
&+ 288\, c_1\, q_0(x)\, q_{0,x}(x) + 288\, d_1\, q_1(x)\, q_{0,x}(x) - 216\, {c_1}^3\, q_{1,x}(x) + 216\, {d_1}^2\, q_{1,x}(x) \\
&- 288\, d_1\, q_0(x)\, q_{1,x}(x) - 288\, {c_1}^2\, q_1(x)\, q_{1,x}(x) + 24\, c_1\, q_1(x)^2\, q_{1,x}(x) \\
&- 144\, c_1\, q_{0,x}(x)\, q_{1,x}(x) + 270\, d_1\, q_{1,x}(x)^2 + 432\, {c_1}^2\, q_{0,xx}(x) + 72\, c_1\, q_1(x)\, q_{0,xx}(x) \\
&+ 216\, c_1\, d_1\, q_{1,xx}(x) + 72\, c_1\, q_0(x)\, q_{1,xx}(x) + 360\, d_1\, q_1(x)\, q_{1,xx}(x) \\
&+ 36\, c_1\, q_{1,x}(x)\, q_{1,xx}(x) - 72\, {c_1}^2\, q_{1,xxx}(x) - 12\, c_1\, q_1(x)\, q_{1,xxx}(x) + 72\, d_1\, q_{1,xxxx}(x)\big) \\
&- 648\, {d_1}^3\, q_0(x) - 648\, {c_1}^3\, q_0(x)^2 - 648\, {d_1}^2\, q_0(x)^2 - 648\, {c_1}^2\, d_1\, q_0(x)\, q_1(x) \\
&- 216\, d_1\, q_0(x)^3 - 864\, {c_1}^2\, q_0(x)^2\, q_1(x) - 432\, c_1\, d_1\, q_0(x)\, q_1(x)^2 - 360\, c_1\, q_0(x)^2\, q_1(x)^2 \\
&- 72\, d_1\, q_0(x)\, q_1(x)^3 + 432\, c_1\, d_1\, q_0(x)\, q_{0,x}(x) + 144\, c_1\, q_0(x)^2\, q_{0,x}(x) \\
&+ 216\, {d_1}^2\, q_1(x)\, q_{0,x}(x) + 288\, d_1\, q_0(x)\, q_1(x)\, q_{0,x}(x) + 216\, {c_1}^3\, q_0(x)\, q_{1,x}(x) \\
&- 216\, {d_1}^2\, q_0(x)\, q_{1,x}(x) - 144\, d_1\, q_0(x)^2\, q_{1,x}(x) + 108\, {c_1}^2\, d_1\, q_1(x)\, q_{1,x}(x) \\
&+ 432\, {c_1}^2\, q_0(x)\, q_1(x)\, q_{1,x}(x) + 216\, c_1\, d_1\, q_1(x)^2\, q_{1,x}(x) + 264\, c_1\, q_0(x)\, q_1(x)^2\, q_{1,x}(x) \\
&+ 60\, d_1\, q_1(x)^3\, q_{1,x}(x) - 72\, c_1\, d_1\, q_{0,x}(x)\, q_{1,x}(x) + 48\, c_1\, q_0(x)\, q_{0,x}(x)\, q_{1,x}(x) \\
&- 48\, d_1\, q_1(x)\, q_{0,x}(x)\, q_{1,x}(x) - 18\, {c_1}^3\, q_{1,x}(x)^2 + 126\, {d_1}^2\, q_{1,x}(x)^2 + 66\, d_1\, q_0(x)\, q_{1,x}(x)^2 \\
&- 48\, {c_1}^2\, q_1(x)\, q_{1,x}(x)^2 - 34\, c_1\, q_1(x)^2\, q_{1,x}(x)^2 - 12\, c_1\, q_{0,x}(x)\, q_{1,x}(x)^2 + 42\, d_1\, q_{1,x}(x)^3 \\
&- 432\, {c_1}^2\, q_0(x)\, q_{0,xx}(x) - 216\, c_1\, d_1\, q_1(x)\, q_{0,xx}(x) - 360\, c_1\, q_0(x)\, q_1(x)\, q_{0,xx}(x) \\
&- 72\, d_1\, q_1(x)^2\, q_{0,xx}(x) + 144\, d_1\, q_{0,x}(x)\, q_{0,xx}(x) + 72\, {c_1}^2\, q_{1,x}(x)\, q_{0,xx}(x) \\
&+ 108\, c_1\, q_1(x)\, q_{1,x}(x)\, q_{0,xx}(x) - 72\, c_1\, q_{0,xx}(x)^2 + 108\, c_1\, {d_1}^2\, q_{1,xx}(x) \\
&+ 72\, c_1\, d_1\, q_0(x)\, q_{1,xx}(x) + 12\, c_1\, q_0(x)^2\, q_{1,xx}(x) + 180\, {d_1}^2\, q_1(x)\, q_{1,xx}(x) \\
&+ 120\, d_1\, q_0(x)\, q_1(x)\, q_{1,xx}(x) + 36\, c_1\, d_1\, q_{1,x}(x)\, q_{1,xx}(x) + 12\, c_1\, q_0(x)\, q_{1,x}(x)\, q_{1,xx}(x) \\
&+ 60\, d_1\, q_1(x)\, q_{1,x}(x)\, q_{1,xx}(x) + 3\, c_1\, q_{1,x}(x)^2\, q_{1,xx}(x) + 72\, {c_1}^2\, q_0(x)\, q_{1,xxx}(x) \\
&+ 36\, c_1\, d_1\, q_1(x)\, q_{1,xxx}(x) + 60\, c_1\, q_0(x)\, q_1(x)\, q_{1,xxx}(x) + 12\, d_1\, q_1(x)^2\, q_{1,xxx}(x) \\
&- 24\, d_1\, q_{0,x}(x)\, q_{1,xxx}(x) - 12\, {c_1}^2\, q_{1,x}(x)\, q_{1,xxx}(x) - 18\, c_1\, q_1(x)\, q_{1,x}(x)\, q_{1,xxx}(x) \\
&+ 24\, c_1\, q_{0,xx}(x)\, q_{1,xxx}(x) - 2\, c_1\, q_{1,xxx}(x)^2 + 36\, {d_1}^2\, q_{1,xxxx}(x) \\
&+ 24\, d_1\, q_0(x)\, q_{1,xxxx}(x) + 12\, d_1\, q_{1,x}(x)\, q_{1,xxxx}(x)\Big) \frac{1}{648},
\end{aligned}
$$

etc.

Concerning the dynamics of the zeros $\mu_j(x)$ and $\nu_\ell(x)$ of $D_{r-1}(z,x)$ and $N_r(z,x)$ one obtains the following analog of Dubrovin's equations in Lemma 3.2.

Lemma 6.4. *Assume (6.6) and let $x \in \mathbb{R}$. Then*

(i).

$$
\mu_{j,x}(x) = \frac{-\epsilon(r)\, F_r(\mu_j(x),x)\big[3\left(y(\hat{\mu}_j(x)) - k_r(\mu_j(x))\right)^2 + S_r(\mu_j(x))\big]}{\prod_{\substack{k=1\\k\neq j}}^{r-1}\big[\mu_j(x) - \mu_k(x)\big]},
\quad 1 \le j \le r-1. \tag{6.55}
$$

(ii).

$$\nu_{\ell,x}(x) = \frac{-\epsilon(r)\, J_r(\nu_\ell(x),x)\big[3\,\big(y(\hat\nu_\ell(x)) - k_r(\nu_\ell(x))\big)^2 + S_r(\nu_\ell(x))\big]}{\prod_{\substack{m=0\\ m\neq\ell}}^{r-1}\big[\nu_\ell(x)-\nu_m(x)\big]},$$

$$0 \le \ell \le r-1. \tag{6.56}$$

Proof. Combining (6.22), (6.23), (6.32) and (6.37) yields

$$\epsilon(r)D_{r-1,x}(\mu_j(x),x) = F_r(\mu_j(x),x)\big[3\,\big(y(\hat\mu_j(x)) - k_r(\mu_j(x))\big)^2 + S_r(\mu_j(x))\big] \tag{6.57}$$

which in turn implies (6.55) using (6.35). Similarly, combining (6.27), (6.28), (6.32) and (6.38) yields

$$\epsilon(r)\, N_{r,x}(\nu_\ell(x),x) = J_r(\nu_\ell(x),x)\big[3\,\big(y(\hat\nu_\ell(x)) - k_r(\nu_\ell(x))\big)^2 + S_r(\nu_\ell(x))\big] \tag{6.58}$$

implying (6.56) by means of (6.36). $\square$

We emphasize that $2\,(y-k_r)$ in (3.36) and (3.37) and $3\,(y-k_r)^2+S_r$ in (6.55) and (6.56) is precisely the y-derivative of the Burchnall-Chaundy polynomial, that is, $\frac{\partial}{\partial y}\,\mathcal{F}_{r-1}(z,y)$.

We conclude this section with a few trace formulas for the Bsq invariants in terms of $\mu_j(x)$ and $\nu_\ell(x)$ analogous to the KdV case in Lemma 3.3.

Lemma 6.5. *Assume (6.6) and let $x\in\mathbb{R}$. Then*

(i). For $r=2$:

$$\frac{1}{6}\,q_{1,x}(x) + q_0(x) + d_0\, q_1(x) + d_0^3 = \mu_1(x) \tag{6.59}$$

and for $r>2$:

$r\equiv 1 \pmod 3$:

$$\begin{aligned}
&\frac{1}{6}\,q_{1,x}(x) - q_0(x) - 3\,d_1 = \sum_{j_1=1}^{r-1}\mu_{j_1}(x),\\
&\frac{1}{18}\,q_{1,xxxx}(x) + \frac{1}{9}\,q_{0,xxx}(x) + \frac{5}{18}\,q_1(x)q_{1,xx}(x) + \frac{7}{36}\,q_{1,x}(x)^2\\
&\quad + \frac{1}{3}\,q_0(x)q_{1,x}(x) + \frac{1}{9}\,q_1(x)q_{0,x}(x) + \frac{2}{27}\,q_1(x)^3 - q_0(x)^2 + \frac{1}{2}\,d_1 q_{1,x}(x)\\
&\quad - 3\,d_1 q_0(x) + c_1^3 - 3\,d_1^2 - 3\,d_2 = -\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r-1}\mu_{j_1}(x)\,\mu_{j_2}(x),
\end{aligned} \tag{6.60}$$

etc.

$r\equiv 2 \pmod 3$:

$$\frac{1}{6}\,q_{1,x}(x) - q_0(x) + d_0^3 - 3c_1 = \sum_{j_1=1}^{r-1}\mu_{j_1}(x),$$

$$\frac{1}{18}\,q_{1,xxxx}(x) + \frac{1}{9}\,q_{0,xxx}(x) + \frac{5}{18}\,q_1(x)q_{1,xx}(x) + \frac{7}{36}\,q_{1,x}(x)^2 + \frac{1}{3}\,q_0(x)q_{1,x}(x)$$

$$+\frac{1}{9}\,q_1(x)q_{0,x}(x)+\frac{2}{27}\,q_1(x)^3-q_0(x)^2+\big(\frac{1}{2}\,c_1\ -\frac{1}{6}\,d_0^3\big)\,q_{1,x}(x)+(d_0^3-3\,c_1)q_0(x)$$
$$-3\,c_1^2-3\,c_2+3\,d_0^2d_1=-\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r-1}\mu_{j_1}(x)\,\mu_{j_2}(x), \tag{6.61}$$

etc.

(ii). For $r=2$:

$$\frac{1}{3}\,q_{1,x}(x)-2\,q_0(x)-d_0\,q_1(x)-d_0^3=-\nu_0(x)-\nu_1(x) \tag{6.62}$$

and for $r>2$:

$r\equiv 1 \pmod 3$:

$$\frac{1}{3}\,q_{1,x}(x)+3\,d_1=-\sum_{\ell_1=0}^{r-1}\nu_{\ell_1}(x),$$
$$\frac{2}{9}\,q_{0,xxx}(x)+\frac{1}{3}\,q_0(x)q_{1,x}(x)+\frac{1}{18}\,q_{1,x}(x)^2-\frac{1}{18}\,q_1(x)q_{1,xx}(x)+\frac{5}{9}\,q_{0,x}(x)q_1(x)$$
$$-\frac{2}{27}\,q_1(x)^3+d_1q_{1,x}(x)-c_1^3+3\,d_1^2+3\,d_2=\sum_{\substack{\ell_1,\ell_2=0\\ \ell_1<\ell_2}}^{r-1}\nu_{\ell_1}(x)\,\nu_{\ell_2}(x), \tag{6.63}$$

etc.

$r\equiv 2 \pmod 3$:

$$\frac{1}{3}\,q_{1,x}(x)+3\,c_1-d_0^3=-\sum_{\ell_1=0}^{r-1}\nu_{\ell_1}(x),$$
$$\frac{2}{9}\,q_{0,xxx}(x)+\frac{1}{3}\,q_0(x)q_{1,x}(x)+\frac{1}{18}\,q_{1,x}(x)^2-\frac{1}{18}\,q_1(x)q_{1,xx}(x)+\frac{5}{9}\,q_{0,x}(x)q_1(x)$$
$$-\frac{2}{27}\,q_1(x)^3-\big(\frac{1}{3}\,d_0^3-c_1\big)\,q_{1,x}(x)+3\,c_1^2+3\,c_2-3\,d_0^2\,d_1=\sum_{\substack{\ell_1,\ell_2=0\\ \ell_1<\ell_2}}^{r-1}\nu_{\ell_1}(x)\,\nu_{\ell_2}(x), \tag{6.64}$$

etc.

Here c_1,c_2,d_0,d_1,d_2 can be expressed in terms of zeros of $S_r(z)$ and $T_r(z)$ in analogy to (3.40).

Proof. It suffices to substitute (6.35) and (6.36) into (5.10) and (5.11) (taking into account (5.2)) and comparing powers of z. □

Explicit examples illustrating the formalism of this section are provided in Section 8.

7. The Time-Dependent Bsq Formalism

In our final Bsq section we extend the polynomial approach of Sections 5 and 6 to the time-dependent Bsq hierarchy.

We start with a stationary $(m-1)$-gap solution $(q_0^{(0)}(x),q_1^{(0)}(x))$ associated with $\mathcal{K}_{m-1}$ satisfying

$$\mathrm{Bsq}_m(q_0^{(0)},q_1^{(0)}) = \begin{cases} -3\,f_{n+1,x}=0 \\ \\ -3\,g_{n+1,x}=0 \end{cases}, \quad x\in\mathbb{R},\ m=3\,n+1 \text{ or } m=3\,n+2 \tag{7.1}$$

for some fixed $n\in\mathbb{N}_0$ and a given set of integration constants $\{c_\ell\}_{1\le\ell\le n}, \{d_\ell\}_{0\le\ell\le n}$. Our aim is to construct the r-th Bsq flow

$$\mathrm{Bsq}_r(q_0,q_1)=0, \quad (q_0(x,t_{0,r}),q_1(x,t_{0,r}))=(q_0^{(0)}(x),q_1^{(0)}(x)), \quad x\in\mathbb{R} \tag{7.2}$$

for some fixed $r\in 3\,\mathbb{N}_0+1$ or $r\in 3\,\mathbb{N}_0+2$ and $t_{0,r}\in\mathbb{R}$. In terms of Lax pairs this amounts to solving

$$\frac{d}{d\,t_r}L_3(t_r)-[\tilde P_r(t_r),L_3(t_r)] \;=\; 0, \quad t_r\in\mathbb{R}, \tag{7.3}$$

$$[P_m(t_{0,r}),L_3(t_{0,r})] \;=\; 0. \tag{7.4}$$

As a consequence one obtains

$$[P_m(t_r),L_3(t_r)]=0, \qquad t_r\in\mathbb{R}, \tag{7.5}$$

$$\big(P_m(t_r)-k_m(L_3(t_r))\big)^3+\big(P_m(t_r)-k_m(L_3(t_r))\big)S_m(L_3(t_r))-T_m(L_3(t_r))=0, \quad t_r\in\mathbb{R} \tag{7.6}$$

since the Bsq flows are isospectral deformations of $L_3(t_{0,r})$.

We follow our KdV convention of Section 4 and denote the integration constants in $\tilde P_r$ by $\{\tilde c_k\},\{\tilde d_k\}$ and use the notation $\tilde k_r,\tilde F_r,\tilde G_r,\tilde H_r$, etc., in order to distinguish them from k_m,F_m,G_m,H_m, etc., associated with P_m and the integration constants $\{c_\ell\},\{d_\ell\}$ in P_m.

Instead of working directly with (7.3) and (7.5) we find it preferable to take the following equations as our starting point,

$$\begin{aligned} q_{0,t_r} =& -\frac16\,\tilde F_{r,xxxxx}-\frac56\,q_1\,\tilde F_{r,xxx}-\frac54\,q_{1,x}\,\tilde F_{r,xx}-(\frac34\,q_{1,xx}+\frac23\,q_1^2)\,\tilde F_{r,x}\\ &-(\frac16\,q_{1,xxx}+\frac23\,q_1q_{1,x})\,\tilde F_r-3(z-q_0)\,\tilde G_{r,x}+q_{0,x}\,\tilde G_r,\\ q_{1,t_r} =&\ 2\,\tilde G_{r,xxx}+2\,q_1\,\tilde G_{r,x}+q_{1,x}\,\tilde G_r-3\,(z-q_0)\,\tilde F_{r,x}+2\,q_{0,x}\,\tilde F_r, \qquad (x,t_r)\in\mathbb{R}^2, \end{aligned} \tag{7.7}$$

$$\begin{aligned} &-\frac16\,F_{m,xxxx}F_m+\frac16\,F_{m,xxx}F_{m,x}-\frac1{12}\,F_{m,xx}^2-\frac56\,q_1F_{m,xx}F_m\\ &\quad-\frac5{12}\,q_{1,x}F_{m,x}F_m+\frac5{12}\,q_1F_{m,x}^2-\frac13\,\big(\frac12\,q_{1,xx}+q_1^2\big)F_m^2\\ &\quad+2\,G_{m,xx}G_m-G_{m,x}^2+q_1G_m^2-3\,(z-q_0)F_mG_m=S_m(z), \quad (x,t_r)\in\mathbb{R}^2, \end{aligned} \tag{7.8}$$

$$\begin{aligned} &\frac1{18}\,F_{m,xxxx}F_{m,xx}F_m-\frac1{24}\,F_{m,xxxx}F_{m,x}^2+\frac1{18}\,q_1F_{m,xxxx}F_m^2+\frac1{36}\,F_{m,xxx}F_{m,xx}F_{m,x}\\ &-\frac1{36}F_mF_{m,xxx}^2-\frac1{18}\,q_{1,x}F_{m,xxx}F_m^2-\frac19\,q_1F_{m,xxx}F_{m,x}F_m-\frac1{108}\,F_{m,xx}^3\\ &+\frac29\,q_{1,x}F_{m,xx}F_{m,x}F_m+\frac1{18}\,q_{1,xx}F_{m,xx}F_m^2-\frac7{72}\,q_1F_{m,xx}F_{m,x}^2+\frac5{18}\,q_1^2F_{m,xx}F_m^2\\ &+\frac7{36}\,q_1F_{m,xx}^2F_m-\frac1{24}\,q_{1,xx}F_{m,x}^2F_m-\frac7{48}\,q_{1,x}F_{m,x}^3-\frac16\,q_1^2F_{m,x}^2F_m+\frac1{12}\,q_{1,x}q_1F_{m,x}F_m^2 \end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{2}{27}\,q_1^3-\frac{1}{36}\,q_{1,x}^2+\frac{1}{18}\,q_{1,xx}q_1+(z-q_0)^2\Big)F_m^3+(z-q_0)G_m^3+\frac{1}{6}\,F_{m,xxxx}G_m^2\\
&-\frac{1}{3}\,F_{m,xxx}G_{m,x}G_m+F_mG_{m,xx}^2+\frac{1}{3}\,F_{m,xx}\big(G_{m,x}^2+G_{m,xx}G_m\big)-F_{m,x}G_{m,xx}G_{m,x}\\
&-q_1(z-q_0)F_m^2G_m+\frac{2}{3}\,q_1^2F_mG_m^2+\frac{5}{6}\,q_1F_{m,xx}G_m^2-\frac{4}{3}\,q_1F_{m,x}G_{m,x}G_m+\frac{1}{3}\,q_1F_mG_{m,x}^2\\
&+\frac{7}{12}\,q_{1,x}F_{m,x}G_m^2+\frac{4}{3}\,q_1F_mG_{m,xx}G_m+\frac{1}{6}\,q_{1,xx}F_mG_m^2-\frac{1}{3}\,q_{1,x}F_mG_{m,x}G_m\\
&+(z-q_0)F_{m,x}F_mG_{m,x}-\frac{1}{4}\,(z-q_0)F_{m,x}^2G_m-2\,(z-q_0)F_m^2G_{m,xx}=T_m(z), \qquad (7.9)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t_r)\in\mathbb{R}^2,
\end{aligned}
$$

where (cf. (5.10), (5.11))

$$
\begin{aligned}
F_m(z,x,t_r) &= \sum_{\ell=0}^{n} f_{n-\ell}(x,t_r)z^\ell, && (7.10)\\
F_m(z,x,t_{0,r}) &= \sum_{\ell=0}^{n} f_{n-\ell}^{(0)}(x)z^\ell, &&\\
G_m(z,x,t_r) &= \sum_{\ell=0}^{n} g_{n-\ell}(x,t_r)z^\ell, && (7.11)\\
G_m(z,x,t_{0,r}) &= \sum_{\ell=0}^{n} g_{n-\ell}^{(0)}(x)z^\ell &&
\end{aligned}
$$

for fixed $t_{0,r}\in\mathbb{R}$, $m\in 3\,\mathbb{N}_0+1$ or $m\in 3\,\mathbb{N}_0+2$, $r\in 3\,\mathbb{N}_0+1$ or $r\in 3\,\mathbb{N}_0+2$. Here $f_\ell(x,t_r), g_\ell(x,t_r)$ and $f_\ell^{(0)}(x), g_\ell^{(0)}(x)$ are defined as in (5.2) with $(q_0(x),q_1(x))$ replaced by $(q_0(x,t_r),q_1(x,t_r))$, and $(q_0^{(0)}(x),q_1^{(0)}(x))$, respectively.

In analogy to (6.35) and (6.36) we introduce (cf. (4.10), (4.11))

$$
\begin{aligned}
D_{m-1}(z,x,t_r) &= \prod_{j=1}^{m-1}[z-\mu_j(x,t_r)], && (7.12)\\
N_m(z,x,t_r) &= \prod_{\ell=0}^{m-1}[z-\nu_\ell(x,t_r)], && (7.13)
\end{aligned}
$$

where D_{m-1} and N_m are defined as in (6.16) and (6.17) (with F_r, G_r replaced by F_m, G_m). This implies in particular, that (cf. (6.25))

$$
\begin{aligned}
D_{m-1}(z,x,t_r)N_m(z,x,t_r) &= B_m(z,x,t_r)\,E_m(z,x,t_r)-T_m(z)[A_m(z,x,t_r)\\
&\quad\times\big(G_m(z,x,t_r)+\frac{1}{2}\,F_{m,x}(z,x,t_r)\big)-F_m(z,x,t_r)\,C_m(z,x,t_r)], \quad (7.14)
\end{aligned}
$$

where A_m, B_m, C_m and E_m are defined as in (6.11)–(6.14) (with F_r, G_r replaced by F_m, G_m). Hence Lemma 6.1 (with r replaced by m) holds in the present context. Similarly, Lemma 6.2 remains valid and one obtains

$$
\phi(P,x,t_r)=\frac{(G_m(z,x,t_r)+\frac{1}{2}\,F_{m,x}(z,x,t_r))\big(y(P)-k_m(z)\big)+C_m(z,x,t_r)}{F_m(z,x,t_r)\big(y(P)-k_m(z)\big)-A_m(z,x,t_r)} \qquad (7.15)
$$

$$= \frac{F_m(z,x,t_r)\big(y(P)-k_m(z)\big)^2 + A_m(z,x,t_r)\big(y(P)-k_m(z)\big) + B_m(z,x,t_r)}{\epsilon(m)D_{m-1}(z,x,t_r)} \tag{7.16}$$

$$\begin{aligned} &= -\,\epsilon(m)N_m(z,x,t_r)\Big((G_m(z,x,t_r) + \frac{1}{2}\,F_{m,x}(z,x,t_r))\big(y(P)-k_m(z)\big)^2 \\ &\quad - C_m(z,x,t_r)\big(y(P)-k_m(z)\big) - E_m(z,x,t_r)\Big)^{-1}, \\ &\qquad\qquad P=(z,y(P))\in\mathcal{K}_{m-1}. \end{aligned} \tag{7.17}$$

In analogy to (6.37) and (6.38) one then introduces Dirichlet and Neumann data by

$$\begin{aligned} \hat\mu_j(x,t_r) &= \big(\mu_j(x,t_r), y(\hat\mu_j(x,t_r))\big) \\ &= \big(\mu_j(x,t_r), k_m(\mu_j(x,t_r)) + \frac{A_m(\mu_j(x,t_r),x,t_r)}{F_m(\mu_j(x,t_r),x,t_r)}\big) \in \mathcal{K}_{m-1}, \\ &\qquad 1\le j\le m-1, \quad (x,t_r)\in\mathbb{R}^2, \end{aligned} \tag{7.18}$$

$$\begin{aligned} \hat\nu_\ell(x,t_r) &= \big(\nu_\ell(x,t_r), y(\hat\nu_\ell(x,t_r))\big) \\ &= \big(\nu_\ell(x,t_r), k_m(\nu_\ell(x,t_r)) - \frac{C_m(\nu_\ell(x,t_r),x,t_r)}{G_m(\nu_\ell(x,t_r),x,t_r) + \frac{1}{2}\,F_{m,x}(\nu_\ell(x,t_r),x,t_r)}\big) \in \mathcal{K}_{m-1}, \\ &\qquad 0\le \ell\le m-1, \quad (x,t_r)\in\mathbb{R}^2 \end{aligned} \tag{7.19}$$

and hence infers that the divisor $\big(\phi(P,x,t_r)\big)$ of $\phi(P,x,t_r)$ is given by

$$\big(\phi(P,x,t_r)\big) = \mathcal{D}_{\hat\nu_0(x,t_r),\dots,\hat\nu_{m-1}(x,t_r)}(P) - \mathcal{D}_{P_\infty,\hat\mu_1(x,t_r),\dots,\hat\mu_{m-1}(x,t_r)}(P). \tag{7.20}$$

Next we define the time-dependent BA-function $\psi(P,x,x_0,t_r,t_{0,r})$

$$\begin{aligned} \psi(P,x,x_0,t_r,t_{0,r}) &= \exp\Big\{\int_{x_0}^{x} d\,x'\phi(P,x',t_r) + \int_{t_{0,r}}^{t_r} d\,s\big[\tilde F_r(z,x_0,s)\big(\phi_x(P,x_0,s) \\ &\quad + \phi(P,x_0,s)^2\big) + (\tilde G_r(z,x_0,s) - \frac{1}{2}\,\tilde F_{r,x}(z,x_0,s))\phi(P,x_0,s) + \big(\frac{1}{6}\,\tilde F_{r,xx}(z,x_0,s) \\ &\quad + \frac{2}{3}\,q_1(x_0,s)\tilde F_r(z,x_0,s) - \tilde G_{r,x}(z,x_0,s) + \tilde k_r(z)\big)\big]\Big\}, \\ &\qquad P\in\mathcal{K}_{m-1}\setminus\{P_\infty\}, \quad (x,t_r)\in\mathbb{R}^2, \end{aligned} \tag{7.21}$$

with fixed $(x_0,t_{0,r})\in\mathbb{R}^2$. The following lemma records some properties of $\phi(P,x,t_r)$ and $\psi(P,x,x_0,t_r,t_{0,r})$.

Lemma 7.1. *Assume (7.7)–(7.11), $P=(z,y(P))\in\mathcal{K}_{m-1}\setminus\{P_\infty\}$ and let $(z,x,x_0,t_r,t_{0,r})\in\mathbb{C}\times\mathbb{R}^4$. Then*

(i). $\phi(P,x,t_r)$ satisfies

$$\begin{aligned} &\phi_{xx}(P,x,t_r) + 3\,\phi_x(P,x,t_r)\,\phi(P,x,t_r) + \phi(P,x,t_r)^3 + q_1(x,t_r)\,\phi(P,x,t_r) \\ &\qquad = z - q_0(x,t_r) - \frac{1}{2}\,q_{1,x}(x,t_r), \end{aligned} \tag{7.22}$$

$$\begin{aligned} \phi_{t_r}(P,x,t_r) &= \partial_x\big[\tilde F_r(z,x,t_r)(\phi(P,x,t_r)^2 + \phi_x(P,x,t_r)) \\ &\quad + (\tilde G_r(z,x,t_r) - \frac{1}{2}\,\tilde F_{r,x}(z,x,t_r))\phi(P,x,t_r) + \tilde H_r(z,x,t_r)\big]. \end{aligned} \tag{7.23}$$

(ii). $\psi(P,x,x_0,t_r,t_{0,r})$ satisfies

$$\psi_{xxx}(P,x,x_0,t_r,t_{0,r}) + q_1(x,t_r)\psi_x(P,x,x_0,t_r,t_{0,r})$$

$$+ (q_0(x,t_r) + \frac{1}{2}\, q_{1,x}(x,t_r) - z)\psi(P,x,x_0,t_r,t_{0,r}) = 0, \tag{7.24}$$

$$\psi_{t_r}(P,x,x_0,t_r,t_{0,r}) = \big(\tilde F_r(z,x,t_r)(\phi(P,x,t_r)^2 + \phi_x(P,x,t_r))$$
$$+ (\tilde G_r(z,x,t_r) - \frac{1}{2}\, \tilde F_{r,x}(z,x,t_r))\phi(P,x,t_r) + \tilde H_r(z,x,t_r)\big)\psi(P,x,x_0,t_r,t_{0,r}) \tag{7.25}$$

$(i.e.,\ (L_3 - z)\psi = 0,\ (P_m - y)\psi = 0,\ \psi_{t_r} = \tilde P_r \psi).$

$$(iii).\ \phi(P,x,t_r)\,\phi(P^*,x,t_r)\,\phi(P^{**},x,t_r) = \frac{N_m(z,x,t_r)}{D_{m-1}(z,x,t_r)}. \tag{7.26}$$

$$(iv).\ \phi(P,x,t_r) + \phi(P^*,x,t_r) + \phi(P^{**},x,t_r) = \frac{D_{m-1,x}(z,x,t_r)}{D_{m-1}(z,x,t_r)}. \tag{7.27}$$

$$(v).\ (y(P) - k_m(z))\,\phi(P,x,t_r) + (y(P^*) - k_m(z))\,\phi(P^*,x,t_r)$$
$$+ (y(P^{**}) - k_m(z))\,\phi(P^{**},x,t_r) = \frac{3\,T_m(z)\,F_m(z,x,t_r) - 2\,S_m(z)\,A_m(z,x,t_r)}{\epsilon(m)D_{m-1}(z,x,t_r)}. \tag{7.28}$$

Proof. (i). (7.22) follows from (7.8), (7.9) and (7.15). In order to prove (7.23) one first derives from (7.7)–(7.9) and (7.15) that

$$\big[\partial_x^2 + 3\phi\,\partial_x + 3(\phi^2 + \phi_x) + q_1\big]\Big(\phi_{t_r} - \partial_x\big(\tilde F_r(\phi^2 + \phi_x) + (\tilde G_r - \frac{1}{2}\, F_{r,x})\phi + \tilde H_r\big)\Big) = 0.$$

Thus

$$\phi_{t_r} - \partial_x\big(\tilde F_r(\phi^2 + \phi_x) + (\tilde G_r - \frac{1}{2}\, F_{r,x})\phi + \tilde H_r\big) = C_1 f_1 + C_2 f_2, \tag{7.29}$$

where $f_j,\ j = 1,2$ are two linearly independent solutions of

$$\big[\partial_x^2 + 3\phi\,\partial_x + 3(\phi^2 + \phi_x) + q_1\big] f = 0$$

and $C_j,\ j = 1,2$ are independent of x (but may depend on P and t_r). The high-energy behavior of $\phi(P,x,t_r) \underset{|z|\to\infty}{=} O(|z|^{1/3})$ (cf. (7.15)) then proves $C_1 = C_2 = 0$ since the left-hand side of (7.29) is meromorphic on $\mathcal{K}_{m-1}$ (and hence especially near P_∞).
(ii). (7.24) is clear from (7.21) ($\phi = \psi_x/\psi$) and (7.22). (7.25) follows from (7.21) and (7.23).
(iii)–(v) follow as in Lemma 6.3 (ii)–(iv). □

Lemma 7.2. *Assume (7.7)–(7.11) and let* $(z,x,t_r) \in \mathbb{C}\times\mathbb{R}^2$. *Then*

$$(i).\ D_{m-1,t_r}(z,x,t_r) = D_{m-1,x}(z,x,t_r)\Big(\tilde G_r(z,x,t_r) - \frac{1}{2}\, \tilde F_{r,x}(z,x,t_r)$$
$$- \frac{\tilde F_r(z,x,t_r)}{F_m(z,x,t_r)}\big(G_m(z,x,t_r) - \frac{1}{2}\, F_{m,x}(z,x,t_r)\big)\Big) + D_{m-1}(z,x,t_r)$$
$$\times 3\,\Big(\tilde H_r(z,x,t_r) - \tilde k_r(z) - \frac{\tilde F_r(z,x,t_r)}{F_m(z,x,t_r)}\,\big(H_m(z,x,t_r) - k_m(z)\big)\Big). \tag{7.30}$$

$$(ii).\ N_{m,t_r}(z,x,t_r) = N_{m,x}(z,x,t_r)\Big(\tilde G_r(z,x,t_r) + \frac{1}{2}\, \tilde F_{r,x}(z,x,t_r) - \frac{\tilde J_r(z,x,t_r)}{J_m(z,x,t_r)}$$
$$\times\big(G_m(z,x,t_r) + \frac{1}{2}\, F_{m,x}(z,x,t_r)\big)\Big) - N_m(z,x,t_r)\Big(q_1(x,t_r)\,\tilde F_r(z,x,t_r)$$

$$+ \tilde F_{r,xx}(z,x,t_r) - \frac{\tilde J_r(z,x,t_r)}{J_m(z,x,t_r)}\big(q_1(x,t_r)\,F_m(z,x,t_r) + F_{m,xx}(z,x,t_r)\big)\Big). \tag{7.31}$$

Proof. In order to prove (7.30) one combines

$$\begin{aligned}\partial_{t_r}\partial_x\big(\ln D_{m-1}(z,x,t_r)\big) &= \partial_x\partial_{t_r}\big(\ln D_{m-1}(z,x,t_r)\big)\\ &= \big(\phi(P,x,t_r) + \phi(P^*,x,t_r) + \phi(P^{**},x,t_r)\big)_{t_r},\end{aligned}$$

(7.23), (7.27), and

$$\begin{aligned}\phi(P)^2 + \phi(P^*)^2 + \phi(P^{**})^2 = &\; -\partial_x\big(\frac{D_{m-1,x}}{D_{m-1}}\big) - \frac{G_m - \frac{1}{2}F_{m,x}}{F_m}\frac{D_{m-1,x}}{D_{m-1}}\\ &- \frac{1}{F_m}\big(\frac{1}{2}F_{m,xx} + 2\,q_1 F_m - 3\,G_{m,x}\big).\end{aligned} \tag{7.32}$$

Similarly, in order to prove (7.31) one combines

$$\partial_{t_r}\Big(\frac{N_m(z,x,t_r)}{D_{m-1}(z,x,t_r)}\Big) = \big(\phi(P,x,t_r)\phi(P^*,x,t_r)\phi(P^{**},x,t_r)\big)_{t_r},$$

(6.28), (7.23), (7.32), and

$$\frac{1}{\phi(P)} + \frac{1}{\phi(P^*)} + \frac{1}{\phi(P^{**})} = \frac{2(G_m + \frac{1}{2}F_{m,x})S_m + 3\,E_m}{\epsilon(m)N_m}. \tag{7.33}$$

□

The remaining analogs of Lemma 6.3 (v)–(vii) then read

Lemma 7.3. *Assume (7.7)–(7.11), $P = (z, y(P)) \in \mathcal{K}_{m-1} \setminus \{P_\infty\}$ and let $(z, x, x_0, t_r, t_{0,r}) \in \mathbb{C} \times \mathbb{R}^4$. Then*

(*i*). $\psi(P,x,x_0,t_r,t_{0,r})\psi(P^*,x,x_0,t_r,t_{0,r})\psi(P^{**},x,x_0,t_r,t_{0,r})$

$$= e^{3\,k_r(z)(t_r - t_{0,r})}\,\frac{D_{m-1}(z,x,t_r)}{D_{m-1}(z,x_0,t_{0,r})}. \tag{7.34}$$

(*ii*). $\psi_x(P,x,x_0,t_r,t_{0,r})\psi_x(P^*,x,x_0,t_r,t_{0,r})\psi_x(P^{**},x,x_0,t_r,t_{0,r})$

$$= e^{3\,k_r(z)(t_r - t_{0,r})}\,\frac{N_m(z,x,t_r)}{D_{m-1}(z,x_0,t_{0,r})}. \tag{7.35}$$

$$\begin{aligned}(iii).\ \psi(P,x,x_0,t_r,t_{0,r}) = &\left[\frac{D_{m-1}(z,x,t_r)}{D_{m-1}(z,x_0,t_{0,r})}\right]^{1/3} \exp\Big\{\int_{x_0}^{x} d\,x'\epsilon(r)D_{m-1}(z,x',t_r)^{-1}\\ &\times\big[F_m(z,x',t_r)\,\big(y(P) - k_m(z)\big)^2 + A_m(z,x',t_r)\,\big(y(P) - k_m(z)\big)\\ &+ \frac{2}{3}\,F_m(z,x',t_r)\,S_m(z)\big] - \int_{t_{0,r}}^{t_r} d\,s\,\Big[\epsilon(r)D_{m-1}(z,x_0,s)^{-1}\big[F_m(z,x_0,s)\\ &\times\big(y(P) - k_m(z)\big)^2 + A_m(z,x_0,s)\,\big(y(P) - k_m(z)\big) + \frac{2}{3}\,F_m(z,x_0,s)\,S_m(z)\big]\\ &\times\big(\tilde G_r(z,x_0,s) - \frac{1}{2}\,\tilde F_{r,x}(z,x_0,s) - \big(G_m(z,x_0,s) - \frac{1}{2}\,F_{m,x}(z,x_0,s)\big)\,\frac{\tilde F_r(z,x_0,s)}{F_m(z,x_0,s)}\big)\\ &+ (y(P) - k_m(z))\,\frac{\tilde F_r(z,x_0,s)}{F_m(z,x_0,s)}\Big] + k_r(z)(t_r - t_{0,r})\Big\}.\end{aligned} \tag{7.36}$$

Proof. (7.34) follows from (7.21), (7.27), and (7.30). (7.35) follows from (7.26) and (7.34). (7.36) follows from (7.21), (7.16), (6.22), and (7.30). $\square$

The dynamics of the zeros $\mu_j(x,t_r)$ and $\nu_\ell(x,t_r)$ of $D_{m-1}(z,x,t_r)$ and $N_m(z,x,t_r)$, in analogy to Lemma 6.4, is then described in terms of Dubrovin-type equations as follows.

Lemma 7.4. *Assume (7.7)–(7.11) and let $(z,x,t_r)\in\mathbb{C}\times\mathbb{R}^2$. Then*

(i).
$$\mu_{j,x}(x,t_r) = -\epsilon(m)\, F_m(\mu_j(x,t_r),x,t_r) \times \frac{\big[3\,\big(y(\hat\mu_j(x,t_r)) - k_m(\mu_j(x,t_r))\big)^2 + S_m(\mu_j(x,t_r))\big]}{\prod_{\substack{k=1\\k\neq j}}^{m-1}\big[\mu_j(x,t_r)-\mu_k(x,t_r)\big]}, \quad 1\le j\le m-1, \tag{7.37}$$

$$\begin{aligned}\mu_{j,t_r}(x,t_r) = \Big(&F_m(\mu_j(x,t_r),x,t_r)\big(\tilde G_r(\mu_j(x,t_r),x,t_r) - \frac{1}{2}\,\tilde F_{r,x}(\mu_j(x,t_r),x,t_r)\big)\\ &- \tilde F_r(\mu_j(x,t_r),x,t_r)\big(G_m(\mu_j(x,t_r),x,t_r) - \frac{1}{2}\,F_{m,x}(\mu_j(x,t_r),x,t_r)\big)\Big)\\ &\times \frac{-\epsilon(m)\big[3\,\big(y(\hat\mu_j(x,t_r)) - k_m(\mu_j(x,t_r))\big)^2 + S_m(\mu_j(x,t_r))\big]}{\prod_{\substack{k=1\\k\neq j}}^{m-1}\big[\mu_j(x,t_r)-\mu_k(x,t_r)\big]}, \quad 1\le j\le m-1.\end{aligned} \tag{7.38}$$

(ii).
$$\nu_{\ell,x}(x,t_r) = -\epsilon(m)\, J_m(\nu_\ell(x),x,t_r) \times \frac{\big[3\,\big(y(\hat\nu_\ell(x,t_r)) - k_m(\nu_\ell(x,t_r))\big)^2 + S_m(\nu_\ell(x,t_r))\big]}{\prod_{\substack{k=0\\k\neq \ell}}^{m-1}\big[\nu_\ell(x,t_r)-\nu_k(x,t_r)\big]}, \quad 0\le \ell\le m-1, \tag{7.39}$$

$$\begin{aligned}\nu_{\ell,t_r}(x,t_r) = \Big(&J_m(\nu_\ell(x,t_r),x,t_r)\big(\tilde G_r(\nu_\ell(x,t_r),x,t_r) + \frac{1}{2}\,\tilde F_{r,x}(\nu_\ell(x,t_r),x,t_r)\big)\\ &- \tilde J_r(\nu_\ell(x,t_r),x,t_r)\big(G_m(\nu_\ell(x,t_r),x,t_r) + \frac{1}{2}\,F_{m,x}(\nu_\ell(x,t_r),x,t_r)\big)\Big)\\ &\times \frac{-\epsilon(m)\,\big[3\,\big(y(\hat\nu_\ell(x,t_r)) - k_m(\nu_\ell(x,t_r))\big)^2 + S_m(\nu_\ell(x,t_r))\big]}{\prod_{\substack{k=0\\k\neq \ell}}^{m-1}\big[\nu_\ell(x,t_r)-\nu_k(x,t_r)\big]}, \quad 0\le \ell\le m-1.\end{aligned} \tag{7.40}$$

Proof. (7.37) and (7.39) are analogous to (6.55) and (6.56). (7.38) follows from (7.30) and (7.40) follows from (7.31). $\square$

The initial condition

$$(q_0(x,t_{0,r}),q_1(x,t_{0,r})) = (q_0^{(0)}(x),q_1^{(0)}(x)), \quad x\in\mathbb{R} \tag{7.41}$$

effects

$$\hat\mu_j(x,t_{0,r}) = \hat\mu_j^{(0)}(x), \quad 1\le j\le m-1, \quad x\in\mathbb{R}, \tag{7.42}$$

$$\hat{\nu}_\ell(x,t_{0,r}) = \hat{\nu}_\ell^{(0)}(x), \quad 0 \le \ell \le m-1, \quad x \in \mathbb{R} \tag{7.43}$$

(cf. (7.10)–(7.13)).

Finally, the trace relations in Lemma 6.5 extend in a one-to-one manner to the present time-dependent setting by substituting,

$$\begin{aligned} &(q_0(x), q_1(x)) \to (q_0(x,t_r), q_1(x,t_r)), \\ &\qquad \mu_j(x) \to \mu_j(x,t_r), \ \ 1 \le j \le m-1, \qquad \nu_\ell(x) \to \nu_\ell(x,t_r), \ \ 0 \le \ell \le m-1, \end{aligned} \tag{7.44}$$

keeping $\{c_\ell\}_{1\le\ell\le n}, \{d_\ell\}_{1\le\ell\le n}$ as in Lemma 6.5 since $\mathcal{K}_{m-1}$ is t_r-independent.

8. Illustrations

Our final section illustrates the stationary formalism of Sections 2, 3, 5, and 6 and provides specific KdV and Bsq examples.

We start with the KdV case.

Example 8.1. Rational KdV potentials.

We abbreviate $y_j = \omega_2^j y, \ j=1,2, \ \omega_2=-1$.

(i). $r=3$ (genus $g=1$):

$$L_2 = \frac{d^2}{dx^2} - \frac{2}{x^2}, \qquad P_3 = \frac{d^3}{dx^3} - \frac{3}{x^2}\frac{d}{dx} + \frac{3}{x^3}, \tag{8.1}$$

$$\mathcal{F}_1(z,y) = y^2 - z^3 = 0, \qquad E_m = 0, \ \ 0 \le m \le 2, \tag{8.2}$$

$$F_3(z,x) = z - \frac{1}{x^2}, \tag{8.3}$$

$$D_1(z,x) = z - \frac{1}{x^2}, \qquad N_2(z,x) = z^2 + \frac{1}{x^2}z + \frac{1}{x^4}, \tag{8.4}$$

$$\phi_j(z,x) = \frac{y_j + \frac{1}{x^3}}{z - \frac{1}{x^2}} \tag{8.5}$$

$$= \frac{z^2 + \frac{1}{x^2}z + \frac{1}{x^4}}{y_j - \frac{1}{x^3}}, \qquad j=1,2. \tag{8.6}$$

(ii). $r=5$ (genus $g=2$):

$$L_2 = \frac{d^2}{dx^2} - \frac{6}{x^2}, \qquad P_5 = \frac{d^5}{dx^5} - \frac{15}{x^2}\frac{d^3}{dx^3} + \frac{45}{x^3}\frac{d^2}{dx^2} - \frac{45}{x^4}\frac{d}{dx}, \tag{8.7}$$

$$\mathcal{F}_2(z,y) = y^2 - z^5 = 0, \qquad E_m = 0, \ \ 0 \le m \le 4, \tag{8.8}$$

$$F_5(z,x) = z^2 - \frac{3}{x^2}z + \frac{9}{x^4}, \tag{8.9}$$

$$D_2(z,x) = z^2 - \frac{3}{x^2}z + \frac{9}{x^4}, \qquad N_3(z,x) = z^3 + \frac{3}{x^2}z^2 - \frac{36}{x^6}, \tag{8.10}$$

$$\phi_j(z,x) = \frac{y_j + \frac{3}{x^3}z - \frac{18}{x^5}}{z^2 - \frac{3}{x^2}z + \frac{9}{x^4}} \tag{8.11}$$

$$= \frac{z^3 + \frac{3}{x^2}z^2 - \frac{36}{x^6}}{y_j - \frac{3}{x^3}z + \frac{18}{x^5}}, \qquad j=1,2. \tag{8.12}$$

(iii). $r = 7$ (genus $g = 3$):

$$L_2 = \frac{d^2}{d\,x^2} - \frac{12}{x^2}, \qquad P_7 = \frac{d^7}{d\,x^7} - \frac{42}{x^2}\frac{d^5}{d\,x^5} + \frac{210}{x^3}\frac{d^4}{d\,x^4} - \frac{315}{x^4}\frac{d^3}{d\,x^3} - \frac{630}{x^5}\frac{d^2}{d\,x^2} + \frac{2835}{x^6}\frac{d}{d\,x} - \frac{2835}{x^7}, \tag{8.13}$$

$$\mathcal{F}_3(z,y) = y^2 - z^7 = 0, \qquad E_m = 0, \quad 0 \le m \le 6, \tag{8.14}$$

$$F_7(z,x) = z^3 - \frac{6}{x^2}\,z^2 + \frac{45}{x^4}\,z - \frac{225}{x^6}, \tag{8.15}$$

$$D_3(z,x) = z^3 - \frac{6}{x^2}\,z^2 + \frac{45}{x^4}\,z - \frac{225}{x^6},$$

$$N_4(z,x) = z^4 + \frac{6}{x^2}\,z^3 - \frac{9}{x^4}\,z^2 - \frac{135}{x^6}\,z + \frac{2025}{x^8}, \tag{8.16}$$

$$\phi_j(z,x) = \frac{y_j + \frac{6}{x^3}\,z^2 - \frac{90}{x^5}\,z + \frac{675}{x^7}}{z^3 - \frac{6}{x^2}\,z^2 + \frac{45}{x^4}\,z - \frac{225}{x^6}} \tag{8.17}$$

$$= \frac{z^4 + \frac{6}{x^2}\,z^3 - \frac{9}{x^4}\,z^2 - \frac{135}{x^6}\,z + \frac{2025}{x^8}}{y_j - \frac{6}{x^3}\,z^2 + \frac{90}{x^5}\,z - \frac{675}{x^7}}, \qquad j = 1,2. \tag{8.18}$$

Example 8.2. Elliptic KdV potentials.

Here $\wp(x) := \wp(x,\omega_1,\omega_2)$ denotes the Weierstrass $\wp-$function with periods $2\omega_j$, $j = 1,3$, $\mathrm{Im}(\omega_3/\omega_1) \neq 0$, and invariants g_2 and g_3, see [1], Ch. 18.

(i). $r = 3$ (genus $g = 1$):

$$L_2 = \frac{d^2}{d\,x^2} - 2\,\wp(x), \qquad P_3 = \frac{d^3}{d\,x^3} - 3\,\wp(x)\,\frac{d}{d\,x} - \frac{3}{2}\,\wp'(x), \tag{8.19}$$

$$\mathcal{F}_1(z,y) = y^2 - \big(z^3 - \frac{g_2}{4}\,z - \frac{g_3}{4}\big) = 0,$$

$$E_0 = \wp(\omega_1), \quad E_1 = \wp(\omega_2), \quad E_2 = \wp(\omega_3), \tag{8.20}$$

$$F_3(z,x) = z - \wp(x), \tag{8.21}$$

$$D_1(z,x) = z - \wp(x), \qquad N_2(z,x) = z^2 + \wp(x)\,z + \wp(x)^2 - \frac{g_2}{4}, \tag{8.22}$$

$$\phi_j(z,x) = \frac{y_j - \frac{1}{2}\,\wp'(x)}{z - \wp(x)} \tag{8.23}$$

$$= \frac{z^2 + \wp(x)\,z + \wp(x)^2 - \frac{g_2}{4}}{y_j + \frac{1}{2}\,\wp'(x)},$$

$$y_j = (-1)^j\Big(z^3 - \frac{g_2}{4}\,z - \frac{g_3}{4}\Big)^{1/2}, \qquad j = 1,2. \tag{8.24}$$

(ii). $r = 5$ (genus $g = 2$):

$$L_2 = \frac{d^2}{d\,x^2} - 6\,\wp(x),$$

$$P_5 = \frac{d^5}{d\,x^5} - 15\,\wp(x)\,\frac{d^3}{d\,x^3} - \frac{45}{2}\,\wp'(x)\,\frac{d^2}{d\,x^2} + \big(\frac{27}{4}\,g_2 - 45\,\wp(x)^2\big)\,\frac{d}{d\,x}, \tag{8.25}$$

$$\mathcal{F}_2(z,y) = y^2 - \Big(z^5 - \frac{21}{4}\,g_2 z^3 + \frac{27}{4}\,g_3 z^2 + \frac{27}{4}\,g_2^2 z - \frac{81}{4} g_2 g_3\Big)$$

$$= y^2 - \Big((z^2 - 3g_2)(z^3 - \frac{9}{4}\,g_2 z + \frac{27}{4}\,g_3)\Big) = 0, \tag{8.26}$$

$$E_0 = (3g_2)^{1/2},\ E_1 = -3\,\wp(\omega_3),\ E_2 = -3\,\wp(\omega_2),\ E_3 = -3\,\wp(\omega_1),$$
$$E_4 = -(3g_2)^{1/2},$$
$$F_5(z,x) = z^2 - 3\,\wp(x)z + 9\,\wp(x)^2 - \frac{9}{4}\,g_2, \tag{8.27}$$
$$D_2(z,x) = z^2 - 3\,\wp(x)z + 9\,\wp(x)^2 - \frac{9}{4}\,g_2,$$
$$N_3(z,x) = z^3 + 3\,\wp(x)\,z^2 - 3\,g_2 z + 9g_3 - 36\,\wp(x)^3, \tag{8.28}$$
$$\phi_j(z,x) = \frac{y_j - \frac{3}{2}\,\wp'(x)z + 9\,\wp(x)\wp'(x)}{z^2 - 3\,\wp(x)z + 9\,\wp(x)^2 - \frac{9}{4}\,g_2} \tag{8.29}$$
$$= \frac{z^3 + 3\,\wp(x)\,z^2 - 3\,g_2 z + 9g_3 - 36\,\wp(x)^3}{y_j + \frac{3}{2}\,\wp'(x)z - 9\,\wp(x)\wp'(x)}, \tag{8.30}$$
$$y_j = (-1)^j\Big(z^5 - \frac{21}{4}\,g_2z^3 + \frac{27}{4}\,g_3z^2 + \frac{27}{4}\,g_2^2z - \frac{81}{4}g_2g_3\Big)^{1/2}, \qquad j = 1,2.$$

Finally, we turn to the Bsq case.

Example 8.3. Rational Bsq potentials.

We abbreviate $y_j = \omega_3^j y,\ 1 \le j \le 3,\ \omega_3 = \exp(2\pi i/3)$.

(i). $r = 2$ (genus $g = 1$):

$$L_3 = \frac{d^3}{d\,x^3} - \frac{3}{x^2}\,\frac{d}{d\,x} + \frac{3}{x^3}, \qquad P_2 = \frac{d^2}{d\,x^2} - \frac{2}{x^2}, \tag{8.31}$$
$$\mathcal{F}_1(z,y) = y^3 - z^2 = 0, \tag{8.32}$$
$$F_2(z,x) = 1, \qquad G_2(z,x) = 0, \tag{8.33}$$
$$D_1(z,x) = z - \frac{1}{x^3}, \qquad N_2(z,x) = z^2 + \frac{2}{x^3}\,z + \frac{1}{x^6}, \tag{8.34}$$
$$\phi_j(z,x) = \frac{(z + \frac{1}{x^3})}{y_j - \frac{1}{x^2}} \tag{8.35}$$
$$= \frac{y_j^2 + y_j\frac{1}{x^2} + \frac{1}{x^4}}{z - \frac{1}{x^3}} \tag{8.36}$$
$$= \frac{(z + \frac{1}{x^3})^2}{(z + \frac{1}{x^3})y_j - \frac{1}{x^2}(z + \frac{1}{x^3})}, \quad 1 \le j \le 3. \tag{8.37}$$

(ii). $r = 4$ (genus $g = 3$):

$$L_3 = \frac{d^3}{d\,x^3} - \frac{15}{x^2}\,\frac{d}{d\,x} + \frac{15}{x^3}, \quad P_4 = \frac{d^4}{d\,x^4} - \frac{20}{x^2}\,\frac{d^2}{d\,x^2} + \frac{40}{x^3}\,\frac{d}{d\,x}, \tag{8.38}$$
$$\mathcal{F}_3(z,y) = y^3 - z^4 = 0, \tag{8.39}$$
$$F_4(z,x) = -\frac{5}{x^2}, \qquad G_4(z,x) = z, \tag{8.40}$$
$$D_3(z,x) = z^3 - \frac{5}{x^3}\,z^2 - \frac{200}{x^6}\,z - \frac{1000}{x^9},$$
$$N_4(z,x) = z^4 + \frac{10}{x^3}\,z^3 + \frac{225}{x^6}\,z^2 + \frac{27000}{x^{12}}, \tag{8.41}$$
$$\phi_j(z,x) = \frac{(z + \frac{5}{x^3})\,y_j + (\frac{300}{x^7} - \frac{20}{x^4}\,z)}{-\frac{5}{x^2}\,y_j - (\frac{100}{x^6} - z^2)} \tag{8.42}$$

$$= \frac{\frac{5}{x^2}\, y_j^2 + (z^2 - \frac{100}{x^6})\, y_j + (\frac{5}{x^4}\, z^2 + \frac{400}{x^7}\, z + \frac{3000}{x^{10}})}{z^3 - \frac{5}{x^3}\, z^2 - \frac{200}{x^6}\, z - \frac{1000}{x^9}} \tag{8.43}$$

$$= \frac{z^4 + \frac{10}{x^3}\, z^3 + \frac{225}{x^6}\, z^2 + \frac{27000}{x^{12}}}{(z + \frac{5}{x^3})\, y_j^2 + (\frac{20}{x^4}\, z - \frac{300}{x^7})\, y_j - (\frac{5}{x^2}\, z^3 + \frac{25}{x^5}\, z^2 + \frac{600}{x^8}\, z + \frac{9000}{x^{11}})}, \quad 1 \le j \le 3. \tag{8.44}$$

(iii). $r = 5$ (genus $g = 4$):

$$L_3 = \frac{d^3}{d\,x^3} - \frac{24}{x^2}\frac{d}{d\,x} + \frac{24}{x^3},$$

$$P_5 = \frac{d^5}{dx^5} - \frac{40}{x^2}\frac{d^3}{dx^3} + \frac{120}{x^3}\frac{d^2}{dx^2} + \frac{40}{x^4}\frac{d}{dx} - \frac{320}{x^5}, \tag{8.45}$$

$$\mathcal{F}_4(z, y) = y^3 - z^5 = 0, \tag{8.46}$$

$$F_5(z, x) = z, \qquad G_5(z, x) = -\frac{56}{x^4}, \tag{8.47}$$

$$D_4(z, x) = z^4 - \frac{8}{x^3}\, z^3 - \frac{224}{x^6}\, z^2 + \frac{12544}{x^9}\, z + \frac{175616}{x^{12}},$$

$$N_5(z, x) = z^5 + \frac{16}{x^3}\, z^4 + \frac{960}{x^6}\, z^3 - \frac{17920}{x^9}\, z^2 - \frac{200704}{x^{12}}\, z - \frac{11239424}{x^{15}}, \tag{8.48}$$

$$\phi_j(z, x) = \frac{-\frac{56}{x^4}\, y_j + (z^3 + \frac{8}{x^3}\, z^2 + \frac{224}{x^6}\, z - \frac{12544}{x^9})}{z\, y_j - (\frac{8}{x^2}\, z^2 - \frac{3136}{x^8})} \tag{8.49}$$

$$= \frac{z\, y_j^2 + \left(\frac{8}{x^2} z^2 - \frac{3136}{x^8}\right) y_j + \frac{8}{x^4}\, z^3 + \frac{448}{x^7}\, z^2 - \frac{37632}{x^{10}}\, z - \frac{702464}{x^{13}}}{z^4 - \frac{8}{x^3}\, z^3 - \frac{224}{x^6}\, z^2 + \frac{12544}{x^9}\, z + \frac{175616}{x^{12}}} \tag{8.50}$$

$$= \frac{z^5 + \frac{16}{x^3}\, z^4 + \frac{960}{x^6}\, z^3 - \frac{17920}{x^9}\, z^2 - \frac{200704}{x^{12}}\, z - \frac{11239424}{x^{15}}}{\frac{56}{x^4}\, y_j^2 + (z^3 + \frac{8}{x^3}\, z^2 + \frac{224}{x^6}\, z - \frac{12544}{x^9})\, y_j + (\frac{-8}{x^2}\, z^4 - \frac{64}{x^5}\, z^3 + \frac{896}{x^8}\, z^2 + \frac{100352}{x^{11}}\, z + \frac{2809856}{x^{14}})}, \quad 1 \le j \le 3. \tag{8.51}$$

Example 8.4. Elliptic Bsq potentials.

(i). $r = 2$ (genus $g = 1$):

$$L_3 = \frac{d^3}{d\,x^3} - 3\,\wp(x)\,\frac{d}{d\,x} - \frac{3}{2}\,\wp'(x), \qquad P_2 = \frac{d^2}{d\,x^2} - 2\,\wp(x), \tag{8.52}$$

$$\mathcal{F}_1(z, y) = y^3 - \frac{g_2}{4}\, y - z^2 - \frac{g_3}{4} = 0, \tag{8.53}$$

$$F_2(z, x) = 1, \qquad G_2(z, x) = 0, \tag{8.54}$$

$$D_1(z, x) = z + \frac{1}{2}\,\wp'(x), \qquad N_2(z, x) = \big(z - \frac{1}{2}\,\wp'(x)\big)^2, \tag{8.55}$$

$$\phi_j(z, x) = \frac{z - \frac{1}{2}\,\wp'(x)}{y_j - \wp(x)} \tag{8.56}$$

$$= \frac{y_j^2 + y_j\,\wp(x) + \wp(x)^2 - \frac{g_2}{4}}{z + \frac{1}{2}\,\wp'(x)} \tag{8.57}$$

$$= \frac{(z - \frac{1}{2}\,\wp'(x))^2}{(z - \frac{1}{2}\,\wp'(x))y_j - \wp(x)(z - \frac{1}{2}\,\wp'(x))}, \qquad 1 \le j \le 3, \tag{8.58}$$

where $y_j,\ 1 \le j \le 3$ denote the roots of (8.53).

(ii). $r = 4$ (genus $g = 3$):

$$L_3 = \frac{d^3}{d\,x^3} + \big(2\,\sqrt{3g_2} - 15\,\wp(x)\big)\,\frac{d}{d\,x} - \frac{15}{2}\,\wp'(x), \tag{8.59}$$
$$P_4 = \frac{d^4}{d\,x^4} + \big(\frac{\sqrt{3g_2}}{3} - 20\,\wp(x)\big)\,\frac{d^2}{d\,x^2} - 20\,\wp'(x)\frac{d}{d\,x} + \big(10\,\sqrt{3g_2}\,\wp(x) - \frac{5}{2}\,g_2\big),$$

$$\begin{aligned}\mathcal{F}_3(z,y) \;=\; & y^3 + y\Big(- \frac{375}{16}\,{g_2}^2 - \frac{225}{4}\,\sqrt{3g_2}\,g_3 + 7\,\sqrt{3g_2}\,z^2\Big) \\ & + \frac{1375}{32}\,{g_2}^3 + \frac{2625}{16}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,g_3 + \frac{3375}{16}\,g_3^2 \\ & + \frac{1505}{36}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,z^2 + \frac{55}{2}\,g_3\,z^2 - z^4 = 0,\end{aligned} \tag{8.60}$$

$$F_4(z,x) = \big(- \tfrac{5}{3}\,\sqrt{3g_2} - 5\,\wp(x)\big), \qquad G_4(z,x) = z, \tag{8.61}$$

$$\begin{aligned}D_3(z,x) = {} & z^3 + \frac{5}{2}\,\wp'(x)\,z^2 + z\,\Big(\frac{1025}{36}\,\sqrt{3}\,g_2^{\frac{3}{2}} + \frac{25}{4}\,g_3 + 100g_2\,\wp(x) \\ & - 50\,\sqrt{3g_2}\,\wp(x)^2 - 200\,\wp(x)^3\Big) + \frac{125}{8}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,\wp'(x) + \frac{125}{8}\,g_3\,\wp'(x) \\ & + 250g_2\,\wp(x)\wp'(x) + 375\,\sqrt{3g_2}\,\wp(x)^2\wp'(x) + 500\,\wp(x)^3\wp'(x), \\ N_4(z,x) = {} & z^4 - 5\,\wp'(x)z^3 + z^2\,\Big(\frac{1145}{36}\,\sqrt{3}\,g_2^{\frac{3}{2}} - 40g_3 + \frac{135}{4}\,g_2\,\wp(x) \\ & - 90\,\sqrt{3g_2}\,\wp(x)^2 + 225\,\wp(x)^3\Big) + z\,\Big(\frac{75}{4}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,\wp'(x) + \frac{675}{4}\,g_3\,\wp'(x) \\ & - 900g_2\wp(x)\wp'(x) - 450\,\sqrt{3g_2}\,\wp(x)^2\wp'(x)\Big) + \frac{375}{16}\,\sqrt{3}\,g_2^{\frac{3}{2}}g_3 + \frac{3375}{16}\,g_3^2 \\ & + \frac{375}{16}\,\sqrt{3}\,g_2^{\frac{5}{2}}\,\wp(x) + \frac{3375}{16}\,g_2g_3\wp(x) - \frac{3375}{2}\,\sqrt{3g_2}\,g_3\,\wp(x)^2 - 6750\,g_2\,\wp(x)^4 \\ & - (\frac{7125}{4}\,\sqrt{3}\,g_2^{\frac{3}{2}} + \frac{30375}{4}\,g_3)\,\wp(x)^3 + 6750\,\sqrt{3g_2}\,\wp(x)^5 + 27000\,\wp(x)^6,\end{aligned} \tag{8.62}$$

$$\begin{aligned}\phi_j(z,x) = {} & \Big(y_j\,\big(z - \frac{5}{2}\,\wp'(x)\big) + \frac{175}{12}\,g_2\,z + \frac{20}{3}\,\sqrt{3g_2}z\,\wp(x) - 20z\wp(x)^2 - \frac{25}{8}\,g_2\,\wp'(x) \\ & - 50\,\sqrt{3g_2}\,\wp(x)\wp'(x) - 150\,\wp(x)^2\wp'(x)\Big)\Big(y_j\,\big(- \frac{5}{3}\,\sqrt{3g_2} - 5\wp(x)\big) \\ & + z^2 + \frac{25}{6}\,\sqrt{3}g_2^{\frac{3}{2}} + \frac{25}{4}\,g_3 - \frac{25}{4}\,g_2\,\wp(x) - 50\sqrt{3g_2}\,\wp(x)^2 - 100\wp(x)^3\Big)^{-1}\end{aligned} \tag{8.63}$$
$$\begin{aligned}= {} & \frac{1}{D_3(z,x)}\Big(y_j^2\,\big(\frac{5}{3}\,\sqrt{3g_2} + 5\,\wp(x)\big) + y_j\,\Big(\frac{25}{6}\,\sqrt{3}\,g_2^{\frac{3}{2}} + \frac{25}{4}\,g_3 + z^2 - \frac{25}{4}\,g_2\,\wp(x) \\ & - 50\,\sqrt{3g_2}\,\wp(x)^2 - 100\,\wp(x)^3\Big) + \frac{275}{12}\,g_2\,z^2 - \frac{875}{2}\,\sqrt{3g_2}\,g_3\,\wp(x) + 5\,z^2\,\wp(x)^2 \\ & - \frac{1625}{8}\,g_2^2\,\wp(x) + \frac{70}{3}\,\sqrt{3g_2}\,z^2\,\wp(x) - \frac{1125}{4}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,\wp(x)^2 - \frac{1875}{4}\,g_3\,\wp(x)^2 \\ & + 250\,g_2\,\wp(x)^3 + 1750\,\sqrt{3g_2}\,\wp(x)^4 + \frac{100}{3}\,g_2\,z\,\wp'(x) - \frac{100}{3}\,\sqrt{3g_2}\,z\,\wp(x)\,\wp'(x) \\ & - 200\,z\,\wp(x)^2\,\wp'(x) - \frac{1375}{48}\,\sqrt{3}\,g_2^{\frac{5}{2}} - \frac{4375}{16}\,g_2\,g_3 + 3000\,\wp(x)^5\Big)\end{aligned} \tag{8.64}$$

$$
\begin{aligned}
&= N_4(z,x)\bigg(y_j^2\left(z-\frac{5}{2}\,\wp'(x)\right)+y_j\,\bigg(-\frac{175}{12}\,g_2\,z-\frac{20}{3}\,\sqrt{3g_2}\,z\,\wp(x)+20\,z\,\wp(x)^2\\
&\quad+\frac{25}{8}\,g_2\,\wp'(x)+50\,\sqrt{3g_2}\,\wp(x)\,\wp'(x)+150\,\wp(x)^2\,\wp'(x)\bigg)+(\frac{16}{3}\,\sqrt{3g_2}-5\,\wp(x))z^3\\
&\quad+\frac{725}{24}\,g_2^2\,z-\frac{125}{4}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,z\,\wp(x)+\frac{675}{4}\,g_3\,z\,\wp(x)-200\,g_2\,z\,\wp(x)^2-600\,z\,\wp(x)^4\\
&\quad+\frac{125}{16}\,g_2^2\,\wp'(x)-\frac{40}{3}\,\sqrt{3g_2}\,z^2\,\wp'(x)-\frac{1375}{8}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,\wp(x)\,\wp'(x)+\frac{25}{2}\,z^2\,\wp(x)\,\wp'(x)\\
&\quad-\frac{3375}{8}\,g_3\,\wp(x)\,\wp'(x)+1875\,\sqrt{3g_2}\,\wp(x)^3\,\wp'(x)-150\,\sqrt{3g_2}\,z\,\wp(x)^3\\
&\quad+4500\,\wp(x)^4\,\wp'(x)\bigg)^{-1}, \qquad 1\le j\le 3, \qquad (8.65)
\end{aligned}
$$

where $y_j,\ 1\le j\le 3$ denote the roots of (8.60).

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