On Gelfand–Dickey and Drinfeld–Sokolov Systems

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Abstract.

We study the connections between Gelfand–Dickey (GD) systems and their modified counterparts, the Drinfeld–Sokolov (DS) systems in the case of general matrix–valued coefficients with entries in a commutative algebra over an arbitrary field. Our main results describe auto–Bäcklund transformations for the GD hierarchy based on Miura–type transformations associated with factorizations of n-th order linear differential expressions.

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1 Introduction

The main topic of this paper is to detail the connections between Gelfand–Dickey (GD) systems and their modified counterparts, the Drinfeld–Sokolov (DS) systems. Similar to the well known case of the Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) hierarchy [28], these connections are effected by Miura–type transformations associated with factorizations of n–th order linear differential expressions. A characteristic feature of the underlying formalism is an explicit description of auto–Bäcklund transformations of the GD equations.

In order to explain some of these points in more detail we consider matrix–valued formal pseudo–differential expressions of the type

$$L_n = \sum_{j=0}^{n} q_j(x)\partial_x^j, \tag{1.1}$$

$$P_r = \sum_{j=-\infty}^{r} p_j(x)\partial_x^j, \quad x \in \mathbb{R}, \quad n, r \in \mathbb{N}, \tag{1.2}$$

commuting with each other, i.e.,

$$[P_r, L_n] = 0. \tag{1.3}$$

Here we assume $q_j(x)$, $0 \leq j \leq n$ (and hence $p_j(x)$, $-\infty \leq j \leq r$) to be $m \times m$ matrices, $m \in \mathbb{N}$ with smooth entries in x and, in order to avoid technicalities in the introduction, that the highest coefficient q_n is a (nonzero) constant (i.e. independent of $x \in \mathbb{R}$) multiple of the identity $m \times m$ matrix

$$q_n = \mathrm{diag}(c, \dots, c), \quad c \neq 0. \tag{1.4}$$

(q_n an invertible diagonal $m \times m$ matrix, or even more generally, q_n invertible, would be sufficient.) Moreover, we assume that

$$q_{n-1,\mu,\nu} = 0, \quad 1 \leq \mu, \nu \leq m \tag{1.5}$$

and that $P_{0,r} = p_r\partial_x^r$ is $L_{0,n} = q_n\partial_x^n$ admissible (see Section 2). Under the present simplifying assumption (1.4) this is equivalent to the fact that p_r is also a (nonzero) constant multiple of the $m \times m$ identity matrix

$$p_r = \mathrm{diag}(d, \dots, d), \quad d \neq 0. \tag{1.6}$$

(Actually, we deal later on with general matrix algebras with entries in a commutative algebra over an arbitrary field, but for the sake of simplicity we only consider the algebra of smooth complex–valued functions in the introduction.) The Gelfand–Dickey equations associated with the Lax pair $((P_r)_+, L_n)$ are then defined by

$$\frac{d}{dt_r}L_n = [(P_r)_+, L_n], \quad r \in \mathbb{N}, \tag{1.7}$$

where $(P_r)_+ = \sum_{j=0}^{r} p_j \partial_x^j$ represents the formal differential operator part of P_r. In terms of the coefficients q_j of L_n, (1.7) yields the GD system

$$\partial_{t_r} q_{j,\mu,\nu} = f_{j,r,\mu,\nu}(q_0, \dots, q_{n-2}), \quad 0 \le j \le n-2, \quad 1 \le \mu, \nu \le m, \tag{1.8}$$

where

$$[(P_r)_+, L_n]_{\mu,\nu} = \sum_{j=0}^{n-2} f_{j,r,\mu,\nu} \partial_x^j, \quad 1 \le \mu, \nu \le m. \tag{1.9}$$

The special scalar case $m = 1$, $n = 2$, $r \in \mathbb{N}$, $r \ne 0 (\mod 2)$ then represents the KdV hierarchy whereas $n = 3$, $r \in \mathbb{N}$, $r \ne 0 (\mod 3)$ describes the Boussinesq hierarchy of equations. The modified Gelfand–Dickey or Drinfeld–Sokolov system associated with (1.7) is defined as follows. Consider $m \times m$ matrices $\phi_k(x,t)$, $1 \le k \le n$ with smooth entries in (x,t) such that

$$\sum_{k=1}^{n} \phi_k = 0. \tag{1.10}$$

Introduce the matrix differential expressions

$$\mathcal{M}_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & q_n^{1/n}(\partial_x + \phi_n) \\ q_n^{1/n}(\partial_x + \phi_1) & 0 & \cdots & 0 & 0 \\ 0 & q_n^{1/n}(\partial_x + \phi_2) & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & q_n^{1/n}(\partial_x + \phi_{n-1}) & 0 \end{pmatrix}, \tag{1.11}$$

$$\mathcal{M}_{0,n} = \begin{pmatrix} 0 & 0 & \cdots & 0 & q_n^{1/n}\partial_x \\ q_n^{1/n}\partial_x & 0 & & 0 & 0 \\ 0 & q_n^{1/n}\partial_x & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & q_n^{1/n}\partial_x & 0 \end{pmatrix}, \tag{1.12}$$

$$q_n^{1/n} = \operatorname{diag}(c^{1/n}, \dots, c^{1/n}). \tag{1.13}$$

Then

$$\mathcal{M}_n^n = \operatorname{diag}(L_{n,1}, \dots, L_{n,n}), \tag{1.14}$$

where

$$L_{n,k} = A_{k+n-1} \cdots A_{k+1} A_k = \sum_{j=0}^{n} q_{j,k} \partial_x^j, \quad q_{n,k} = q_n, \tag{1.15}$$

$$A_k = q_n^{1/n}(\partial_x + \phi_k), \quad A_{k+n} = A_k, \quad 1 \leq k \leq n. \tag{1.16}$$

Because of condition (1.9), $q_{n-1,k}$ satisfy (1.5), i.e.,

$$q_{n-1,k,\mu,\nu} = 0, \quad 1 \leq k \leq n, \quad 1 \leq \mu, \nu \leq m \tag{1.17}$$

and hence all $L_{n,k}$ are of the type (1.1). Moreover, due to (1.17), one can conjugate $L_{n,k}$ into $L_{0,n} = q_n \partial_x^n$,

$$L_{n,k} = K_{n,k} L_{0,n} K_{n,k}^{-1}, \quad 1 \leq k \leq n, \tag{1.18}$$

where $K_{n,k}$ denotes a certain formal pseudo–differential expression (the Zakharov–Shabat dressing operator)

$$K_{n,k} = 1 + \sum_{j=-\infty}^{-1} \chi_{n,k,j} \partial_x^j, \quad 1 \leq k \leq n. \tag{1.19}$$

Next, suppose that $P_{0,r} = p_r \partial_x^r$, $r \in \mathbb{N}$ is $L_{0,n}$ admissible in the sense of (1.6) and define

$$\begin{aligned} Q_{0,r} &= \operatorname{diag}(P_{0,r}, \ldots, P_{0,r}), && (1.20)\\ \mathcal{Q}_r &= \mathcal{K}_n \mathcal{Q}_{0,r} \mathcal{K}_n^{-1} = \operatorname{diag}(P_{r,1}, \ldots, P_{r,n}), &&\\ P_{r,k} &= K_{n,k} P_{0,r} K_{n,k}^{-1}, \quad 1 \leq k \leq n, && (1.21) \end{aligned}$$

where

$$\mathcal{K}_n = \operatorname{diag}(K_{n,1}, \ldots, K_{n,n}). \tag{1.22}$$

Then the Drinfeld–Sokolov (DS) equations associated with the Lax pair $((\mathcal{Q}_r)_+, \mathcal{M}_n)$ are defined by

$$\frac{d}{dt_r} \mathcal{M}_n = [(\mathcal{Q}_r)_+, \mathcal{M}_n], \quad r \in \mathbb{N}. \tag{1.23}$$

Equation (1.23), rewritten in terms of the coefficients ϕ_k of $\mathcal{M}_n$, then yields the DS system

$$\partial_{t_r} \phi_{k,\mu,\nu} = g_{k,r,\mu,\nu}(\phi_1, \ldots, \phi_n), \quad 1 \leq k \leq n, \quad 1 \leq \mu, \nu \leq m. \tag{1.24}$$

At this point we are ready to describe the close connections between GD and DS systems. In fact, suppose $(\phi_1, \ldots, \phi_n)$ satisfy the DS system (1.23) resp. (1.24). Define A_k, $L_{n,k}$, and $q_{j,k}$, $1 \leq k \leq n$ as in (1.15). Then the identity (1.14) together with the diagonal structure of $\mathcal{Q}_r$

in (1.21) readily proves that $(q_{0,k}, \dots, q_{n-2,k})$, $1 \leq k \leq n$ are n solutions of the GD system (1.7) resp. (1.8). In short, one solution $(\phi_1, \dots, \phi_n)$ of the DS equations implies n solutions $(q_{0,k}, \dots, q_{n-2,k})$, $1 \leq k \leq n$ of the GD equations. In the scalar case $m = 1$ this observation goes back to Sokolov and Shabat [33] (see also [2], [4], [8], [23], [30], [35], [36]). Our main purpose in this paper is to prove a converse of this statement in the following sense: Given a solution $(q_{0,1}, \dots, q_{n-2,1})$ of the GD equations (corresponding to the Lax pair $((P_{r,1})_+, L_{n,1})$), construct a solution $(\phi_1, \dots, \phi_n)$ of the corresponding DS equations and hence $n-1$ further solutions $(q_{0,k}, \dots, q_{n-2,k})$, $2 \leq k \leq n$ of the GD equations related to each other by (1.11), (1.15), and (1.16) and hence by generalized Miura–type transformations. This result will be proven in a general algebraic setting in Section 4 and supplemented by a detailed analytic treatment in the scalar case $m = 1$ in Section 5. As a by–product one obtains the auto–Bäcklund transformations for the GD equations in terms of the factorization of $L_{n,1}$ in (1.15).

Returning to the special case $m = 1$, it should perhaps be mentioned at this point that besides the $n \times n$ matrix Lax pair $((\mathcal{Q}_r)_+, \mathcal{M}_n)$ for the DS equations (1.23), originally due to Sokolov and Shabat [33] and further developed in [2], [4], [8], [23], [30], [35], [36], scalar Lax pairs have also been developed for (1.24), see, e.g., [20]–[22], [27]. Since the Bäcklund transformations, however, are most naturally described in terms of factorizations (1.15) of $L_n = L_{n,1}$, we have consistently chosen the approach effected by (1.11), (1.21).

In Section 2 we recall the general algebraic framework in connection with GD systems as developed by Wilson [34] (see also [8], [23], [25], [35], [36]). Although we provide a fairly complete collection of the results of [34] in order to set–up the basic notation needed in Sections 3–5, we assume a certain familiarity of the reader with this material (and hence provide no proofs).

In Section 3 we develop the corresponding algebraic set–up for the DS equations. We derive the analogs of all the GD results of Section 2 for the general matrix–valued DS systems ($m \in \mathbb{N}$) following the lines of Kuperschmidt and Wilson [23] and Wilson [36] (see also [30]) in the scalar case $m = 1$. Although our generalization to arbitrary $m \in \mathbb{N}$ appears to be novel, we only sketch some of the proofs since the main techniques involving circulant matrices are familiar from the work [23].

In Section 4 we prove our main results concerning the connections between GD and DS systems within the general algebraic approach developed in Sections 2 and 3. In Theorem 4.1 we recall the well known fact that a solution of the DS equations yields n solutions of the corresponding GD equations. In Theorem 4.2 we prove a first form of a converse to Theorem 4.1 by assuming the existence of a factorization of $L_n = L_{n,1}$ of the type (1.15), (1.16). If in addition the coefficients $(q_{0,k}, \dots, q_{n-2,k})$ associated with $L_{n,k}$, $1 \leq k \leq n$ (see (1.15)) all satisfy the corresponding GD equations (1.7) respectively (1.8), then $(\phi_1, \dots, \phi_n)$ in (1.15), (1.16) satisfies the DS equations (1.23) respectively (1.24). Under the additional hypothesis of the existence of formal eigenvectors of $L_n = L_{n,1}$ we prove our main result in Theorem 4.9: given a solution $(q_{0,1}, \dots, q_{n-2,1})$ of the GD equations, we find necessary and sufficient conditions in terms of a basis of the formal eigenspace of L_n such that $(\phi_1, \dots, \phi_n)$, constructed with the help of this basis, satisfies the DS equations. Corollary 4.10 then describes the auto–Bäcklund transformations of the GD equations associated with the factorization of $L_n = L_{n,1}$.

Our general approach includes recent generalized (m)KdV equations discussed e.g., in [3], [24] and the references therein.

While Sections 2–4 are purely algebraic in nature, we finally give a detailed analytical treatment of the particular scalar case $m = 1$ in our final Section 5 by specializing to sufficiently differentiable complex–valued coefficients q_j in the n–th order (scalar) differential expression L_n. Under minimal differentiability requirements on the coefficients q_j we derive the zero–curvature representation of the GD equations as a by–product of Theorem 5.8. We then continue along these lines and reprove all major results of Section 4 in the analytical context. In particular, we provide a detailed study of factorizations of L_n and its relation to solutions $(\phi_1, \dots, \phi_n)$. Our hypotheses in Section 5 are sufficiently general to include singular solutions (such as rational ones). This point is significant since blow–up of solutions in finite time even for initial data in the Schwartz space is known to occur for GD and DS equations under appropriate conditions (see, e.g., [6] and [19]). Due to the very general conditions on the coefficients q_j the results in Section 5 are new. In particular, in contrast to other possible approaches based on bi–Hamiltonian structures or inverse scattering techniques [4], [7], [31] we do not require (almost) periodicity or decay conditions on the coefficients q_j as $|x| \to \infty$. The special cases $n = 2, r = 3$ and $n = 3, r = 2$ representing the (modified) KdV and Boussinesq equation were separately studied in [16] and [15] respectively.

Due to the very general framework developed in Section 4, our methods can be applied to the case where L_n is a formal pseudo–differential expression and hence to the Kadomtsev–Petviashvili (KP) hierarchy. These results will appear elsewhere [17].

2 Gelfand–Dickey Systems

In this section we give a short summary of the algebraic set–up of Wilson [34] (see also [8], [23], [25], [35], [36]) in the context of Gelfand–Dickey (GD) systems [13], [14] (see also [1], [8], [32]). Let A be a commutative algebra over the field $(F, +, \cdot)$ (as usual, we denote the unit element for addition by 0, and the unit element for multiplication by e), ∂ a derivation on A and

$$A[\xi] = \left\{ \sum_{j=0}^{N} a_j \xi^j \mid a_j \in A,\ 0 \le j \le N,\ N \in \mathbb{N}_0 \right\} \tag{2.1}$$

the polynomial algebra generated by $A \cup \{\xi\}$, where

$$\begin{aligned} &\xi^0 a = a,\ \xi^j a = \sum_{l=0}^{j} \binom{j}{l} a^{(l)} \xi^{j-l},\ j \in \mathbb{N}, \\ &\quad a^{(0)} = a,\ a^{(l)} = (\partial^l a),\ l \in \mathbb{N},\ a \in A. \end{aligned} \tag{2.2}$$

We also need

$$A((\xi^{-1})) = \left\{ \sum_{j=-\infty}^{M} a_j \xi^j \mid a_j \in A,\ j \le M,\ M \in \mathbb{Z} \right\}, \tag{2.3}$$

where

$$\xi^{-j}a = \sum_{l=0}^{\infty}(-1)^l \binom{j+l-1}{l} a^{(l)}\xi^{-j-l}, \ j \in \mathbb{N}, \ a \in A. \tag{2.4}$$

Denoting by $M_m(A)$, $m \in \mathbb{N}$ the algebra of $m \times m$–matrices over F with entries in A, we introduce

$$L_n = \sum_{j=0}^{n} q_j \xi^j, \ q_j = [q_{j,\mu,\nu}]_{\mu,\nu=1}^m \in M_m(A), \ 1 \leq j \leq n, \ n \in \mathbb{N} \tag{2.5}$$

assuming the following basic hypothesis on q_n and q_{n-1} for the rest of this section.

Hypothesis 2.1. *(i). Suppose q_n is a diagonal matrix of the type*

$$q_n = \mathrm{diag}(c_1, \ldots, c_m), \ c_\mu \in F\backslash\{0\}, \quad 1 \leq \mu \leq m. \tag{2.6}$$

(ii).

$$\textit{If } \ c_\mu = c_\nu \ \textit{ then } q_{n-1,\mu,\nu} = 0, \ 1 \leq \mu, \nu \leq m. \tag{2.7}$$

At this point it suffices to restrict A to

$$B = F\left[\left\{q_{j,\mu,\nu}^{(l)}\right\}_{\substack{l\in\mathbb{N}_0 \\ 0\leq j\leq n-1}} \setminus \left\{q_{n-1,\mu',\nu'}^{(l)} \text{ if } c_{\mu'} = c_{\nu'}\right\}_{l\in\mathbb{N}_0}\right] \tag{2.8}$$

and denote again by ∂ the corresponding restriction of ∂ to B. ∂ naturally extends to the algebra $M_m(B)$ by $(\partial q)_{\mu,\nu} = \partial(q_{\mu,\nu})$. We also introduce

$$M_m(B)[\xi] = \left\{\sum_{j=0}^{N} r_j \xi^j \mid r_j \in M_m(B), \ 0 \leq j \leq N, \ N \in \mathbb{N}_0\right\}, \tag{2.9}$$

whose elements are called formal differential operators, and

$$M_m(B)((\xi^{-1})) = \left\{\sum_{j=-\infty}^{M} s_j \xi^j \mid s_j \in M_m(B), \ j \leq M, \ M \in \mathbb{Z}\right\}, \tag{2.10}$$

whose elements are called formal pseudo–differential operators. For $S = \sum_{j=-\infty}^{M} s_j \xi^j$ $\in M_m(B)((\xi^{-1}))$ one writes

$$S_+ = \sum_{j=0}^{M} s_j \xi^j, \quad S_- = \sum_{j=-\infty}^{-1} s_j \xi^j \tag{2.11}$$

and calls the differential expression S_+ the (formal) differential operator part of S. The order of S is defined by

$$\text{order}\,(S) = \max\{j \in \mathbb{Z} \,|\, s_j \neq 0\}. \tag{2.12}$$

Associating the degree (weight)

$$\deg(q^{(l)}_{j,\mu,\nu}) = n + l - j \tag{2.13}$$

with $q^{(l)}_{j,\mu,\nu}$, B becomes a $\mathbb{Z}$–graded algebra and ∂ is then homogeneous of degree 1. This grading naturally extends to $M_m(B)$ and, defining

$$\deg(\xi) = 1, \tag{2.14}$$

extends to $M_m(B)[\xi]$ and $M_m(B)((\xi^{-1}))$. $M_m(B)[\xi]$ is then a $\mathbb{Z}$–graded algebra and L_n is homogeneous of degree n. The product of two homogeneous elements of $M_m(B)((\xi^{-1}))$ of degree r and s respectively is then homogeneous of degree $r+s$. The derivation ∂ on B has kernel equal to F,

$$\text{Ker}\,(\partial) = F, \tag{2.15}$$

but in general ∂ will not be surjective, i.e., $\text{Ran}\,(\partial) \subset B$. As shown in [34], it is possible to extend B to an algebra $\bar{B}$ and ∂ to a derivation $\bar{\partial}$ on $\bar{B}$ such that

$$\text{Ker}\,(\bar{\partial}) = F, \ \text{Ran}\,(\bar{\partial}) = \bar{B} \tag{2.16}$$

and also the grading extends to $\bar{B}$ with $\bar{\partial}$ being homogeneous of degree 1. The following key result (which requires $\bar{B}$ instead of B) describes the Zakharov–Shabat dressing operation in an algebraic setting:

Theorem 2.2. *There exists an element $K_n \in M_m(\bar{B})((\xi^{-1}))$ of the type*

$$K_n = e_m + \sum_{j=-\infty}^{-1} \chi_{n,j}\xi^j, \quad \chi_{n,j} \in M_m(\bar{B}), \quad j \leq -1, \ e_m = \text{diag}(e, \dots, e) \tag{2.17}$$

such that

$$L_n = K_n L_{0,n} K_n^{-1}, \tag{2.18}$$

where

$$L_{0,n} = q_n \xi^n. \tag{2.19}$$

The normalization $K_n = e_m +$ [lower order terms] together with the assumption that K_n be homogeneous of degree zero uniquely determines K_n.

In the following we shall, without further notice, always work with the unique degree zero part of K_n. Without repeating the details of the proof of Theorem 2.2 in [34], it should be mentioned that it is Hypothesis 2.1 which allows one to determine the coefficients $\chi_{n,j}$ of K_n. For a given element $L \in M_m(B)((\xi^{-1}))$ we denote by

$$C_B(\{L\}) = \{P \in M_m(B)((\xi^{-1})) \mid [P, L] = 0\} \tag{2.20}$$

($[P, Q] = PQ - QP$ the commutator of P and Q) the centralizer of $\{L\}$ in $M_m(B)((\xi^{-1}))$ and its center by

$$Z(C_B(\{L\})) = \{Q \in C_B(\{L\}) \mid [Q, P] = 0 \text{ for all } P \in C_B(\{L\})\}. \tag{2.21}$$

Similarly $C_{\bar{B}}(\{L\})$, $Z(C_{\bar{B}}(\{L\}))$ denote the corresponding subalgebras with B replaced by its extension $\bar{B}$. We also denote by $C_B(\{q\})$ the centralizer of $\{q\}$ in $M_m(B)$ and by $Z(C_B(\{q\}))$ its center.

Theorem 2.3. *(i).*

$$C_{\bar{B}}(\{L_{0,n}\}) = C_B(\{L_{0,n}\}). \tag{2.22}$$

(ii). $S = \sum_{j=-\infty}^{M} s_j \xi^j \in M_m(\bar{B})((\xi^{-1}))$ *commutes with* $L_{0,n} = q_n \xi^n$ *iff each* s_j, $j \leq M$ *is a constant matrix commuting with* q_n. $C_B(\{L_{0,n}\})$ *is commutative iff* $c_\mu \neq c_\nu$ *for all* $\mu \neq \nu$. *(*$C_B(\{L_{0,n}\})$ *is of course commutative if* $m = 1$.*)*

(iii).

$$Z(C_{\bar{B}}(\{L_n\})) = C_B(\{L_n\}) = Z(C_B(\{L_n\})). \tag{2.23}$$

In particular, $C_B(\{L_n\})$ is commutative. If $c_\mu \neq c_\nu$ for all $\mu \neq \nu$ (or if $m = 1$) then $C_{\bar{B}}(\{L_n\})$ is commutative too and hence coincides with $C_B(\{L_n\})$. Next, following [34], we call $P_{0,r} = p_r \xi^r \in M_m(B)[\xi]$, $r \in \mathbb{N}_0$, $L_{0,n}$–admissible iff

$$p_r = \operatorname{diag}(d_1, \dots, d_m), \ d_\mu \in F \tag{2.24}$$

and $d_\mu = d_\nu$ whenever $c_\mu = c_\nu$, $1 \leq \mu, \nu \leq m$.

One then proves

Lemma 2.4. $P_{0,r} = p_r \xi^r \in M_m(B)[\xi]$ *is* $L_{0,n}$*–admissible iff* $P_{0,r} \in Z(C_B(\{L_{0,n}\}))$ $(= Z(C_{\bar{B}}(\{L_{0,n}\})))$. *Or equivalently,* $P_{0,r} = p_r \xi^r$ *is* $L_{0,n}$*–admissible iff* $p_r \in Z(C_B(\{q_n\}))$.

Theorem 2.5. *Assume that* $P_{0,r} = p_r\xi^r \in M_m(B)[\xi]$, $r \in \mathbb{N}$ *commutes with* $L_{0,n}$, *i.e.,* $[P_{0,r}, L_{0,n}] = 0$. *Let* $K_n \in M_m(\bar{B})((\xi^{-1}))$ *be given by (2.17), i.e.,* $L_n = K_n L_{0,n} K_n^{-1}$, *and define*

$$P_r = K_n P_{0,r} K_n^{-1} \in M_m(\bar{B})((\xi^{-1})) \tag{2.25}$$

so that P_r *commutes with* L_n, *i.e.,* $[P_r, L_n] = 0$. *Then actually*

$$P_r \in M_m(B)((\xi^{-1})) \tag{2.26}$$

iff $P_{0,r}$ *is* $L_{0,n}$*–admissible, i.e., iff* $P_{0,r} \in Z(C_B(\{L_{0,n}\}))$.

Theorem 2.6. *Let* $P_{0,r} = p_r\xi^r \in M_m(B)[\xi]$ *be* $L_{0,n}$*–admissible. Then*
(i). $C_B(\{L_n\})$ *contains a unique element of the form*

$$P_r = P_{0,r} + [\textit{ lower order terms }] \tag{2.27}$$

which is homogeneous of degree r.
(ii). $C_B(\{L_n\})$ *consists of the (in general infinite) sums of elements in* (i). *In particular, the leading coefficient in any element of* $C_B(\{L_n\})$ *is a constant matrix commuting with* q_n.

If d is a derivation on B, then d is called an evolutionary derivation on B iff d and ∂ commute i.e., iff

$$d(\partial b) = \partial(db), \ b \in B. \tag{2.28}$$

d (like ∂) naturally extends to $M_m(B)$ by $(dq)_{\mu,\nu} = d(q_{\mu,\nu})$ and also extends to $M_m(B)[\xi]$, $M_m(B)((\xi^{-1}))$ coefficientwise, i.e.,

$$d\left(\sum_{j=-\infty}^{M} s_j\xi^j\right) = \sum_{j=-\infty}^{M} (ds_j)\xi^j. \tag{2.29}$$

(An evolutionary derivation is uniquely defined by its values on all $q_{j,\mu,\nu}\backslash\{q_{n-1,\mu',\nu'}$ if $c_{\mu'} = c_{\nu'}\}$, $0 \leq j \leq n-1$, above. These values can be arbitrarily prescribed.) For $P \in C_B(\{L_n\})$, order $(P) \geq 0$, the evolutionary derivation ∂_P associated with P is defined by

$$\begin{aligned} \partial_P q_{n,\mu,\nu} &= 0, \\ \partial_P q_{j,\mu,\nu} &= \{\text{ coefficient of } \xi^j \text{ in } [P_+, L_n]_{\mu,\nu}\}, \quad 0 \leq j \leq n-1, \quad 1 \leq \mu,\nu \leq m. \end{aligned} \tag{2.30}$$

Associating a "time" parameter $t_P \in \mathbb{R}$ with P,

$$\partial_{t_P} q_{j,\mu,\nu} = \partial_P q_{j,\mu,\nu}, \quad 0 \leq j \leq n-1, \quad 1 \leq \mu,\nu \leq m \tag{2.31}$$

represents the system of nonlinear evolution equations corresponding to the Lax pair (P_+, L_n). If $P = P_r$ is homogeneous of degree $r \geq 1$ then the right–hand–side of equation (2.31) are differential polynomials in q_j homogeneous of degree $n-j+r$. Since $[P, L_n] = 0$ by hypothesis ($P \in C_B(\{L_n\})$),

$$[P_+, L_n] = [-P_-, L_n]. \tag{2.32}$$

Thus (2.32) represents a formal differential operator of order at most $n-1$,

$$[P_+, L_n] = [-P_-, L_n] \in M_m(B)[\xi]. \tag{2.33}$$

Since

$$[-P_-, L_n] = (c_\mu - c_\nu) p_{-1,\mu,\nu} \xi^{n-1} + [\text{ lower order terms }], \tag{2.34}$$

the coefficient of ξ^{n-1} of $[P_+, L_n]_{\mu,\nu} = [-P_-, L_n]_{\mu,\nu}$ vanishes for $c_\mu = c_\nu$ as in the case of L_n (see (H.2.1)). (2.31) can be rewritten as

$$\frac{d}{dt_P} L_n = \partial_P L_n = [P_+, L_n] \quad (= [-P_-, L_n]). \tag{2.35}$$

In the special case where $m = 1$, the sequence of nonlinear evolution equations

$$\frac{d}{dt_r} L_n = [(P_r)_+, L_n], \quad r \in \mathbb{N}, \tag{2.36}$$

where $q_n = e$, $q_{n-1} = 0$, $t_r = t_{P_r}$, and

$$P_r = e\xi^r + [\text{ lower order terms }] \in C_B(\{L_n\}), \ r \in \mathbb{N} \tag{2.37}$$

are homogeneous of degree r, represents the Gelfand–Dickey hierarchy [13]. Whenever r is a multiple of n, i.e., $r = hn$ for some $h \in \mathbb{N}$ and hence $P_r = L^h$, the evolution equations (2.36) are trivial. More generally, one calls (2.31), (2.35) a Gelfand–Dickey system.

Example 2.7. $m = 1, q_n = e, q_{n-1} = 0$.

For $n > 1, r \geq 1$ we have

$$P_r = (L_n)^{\frac{r}{n}} = e\xi^r + \sum_{j=-\infty}^{r-2} p_j \xi^j, \tag{2.38}$$

i.e., for $r = 1, 2, 3$,

$$\begin{aligned} (P_1)_+ &= e\xi, && (2.39)\\ (P_2)_+ &= e\xi^2 + \frac{2}{n} q_{n-2}, && (2.40)\\ (P_3)_+ &= e\xi^3 + \frac{3}{n} q_{n-2}\xi + \frac{3}{n}\left(q_{n-3} + \frac{3-n}{2}\partial q_{n-2}\right). && (2.41) \end{aligned}$$

$n = 2, r = 1, 3, 5, 7:$

$$
\begin{aligned}
L_2 &= e\xi^2 + q_0, && (2.42)\\
(P_1)_+ &= e\xi, && (2.43)\\
(P_3)_+ &= e\xi^3 + \frac{3}{2}q_0\xi + \frac{3}{4}\partial q_0, && (2.44)\\
(P_5)_+ &= e\xi^5 + \frac{5}{2}q_0\xi^3 + \frac{15}{4}\partial q_0\xi^2 + \frac{5}{8}(3q_0^2 + 5\partial^2 q_0)\xi + \frac{5}{16}(3\partial^3 q_0 + 6q_0\partial q_0), && (2.45)\\
(P_7)_+ &= e\xi^7 + \frac{7}{2}q_0\xi^5 + \frac{35}{4}\partial q_0\xi^4 + \frac{1}{8}(35q_0^2 + 105\partial^2 q_0)\xi^3 + \frac{1}{16}(175\partial^3 q_0 + 210q_0\partial q_0)\xi^2 \\
&\quad + \frac{1}{32}(70q_0^3 + 245(\partial q_0)^2 + 350q_0\partial^2 q_0 + 161\partial^4 q_0)\xi \\
&\quad + \frac{1}{64}(210q_0^2\partial q_0 + 420\partial q_0\partial^2 q_0 + 210q_0\partial^3 q_0 + 63\partial^5 q_0). && (2.46)
\end{aligned}
$$

$$
\begin{aligned}
q_{0,t_1} &= \partial q_0, && (2.47)\\
q_{0,t_3} &= \frac{1}{4}\partial^3 q_0 + \frac{3}{2}q_0\partial q_0, && (2.48)\\
q_{0,t_5} &= \frac{1}{16}\partial^5 q_0 + \frac{5}{8}q_0\partial^3 q_0 + \frac{5}{4}\partial q_0\partial^2 q_0 + \frac{15}{8}q_0^2\partial q_0, && (2.49)\\
q_{0,t_7} &= \frac{1}{64}\Big(\partial^7 q_0 + 14q_0\partial^5 q_0 + 42\partial q_0\partial^4 q_0 + 70\partial^2 q_0\partial^3 q_0 + 70q_0^2\partial^3 q_0 \\
&\quad + 280q_0\partial q_0\partial^2 q_0 + 70(\partial q_0)^3 + 140q_0^3\partial q_0\Big). && (2.50)
\end{aligned}
$$

These are the first four equations of the KdV hierarchy.

$n = 3, r = 1, 2, 4:$

$$
\begin{aligned}
L_3 &= e\xi^3 + q_1\xi + q_0, && (2.51)\\
(P_1)_+ &= e\xi, && (2.52)\\
(P_2)_+ &= e\xi^2 + \frac{2}{3}q_1, && (2.53)\\
(P_4)_+ &= e\xi^4 + \frac{4}{3}q_1\xi^2 + (\frac{4}{3}q_0 + \frac{2}{3}\partial q_1)\xi^1 + \frac{2}{9}(\partial^2 q_1 + 3\partial q_0 + q_1^2). && (2.54)
\end{aligned}
$$

$$\begin{aligned} q_{1,t_1} &= \partial q_1, \\ q_{0,t_1} &= \partial q_0, \end{aligned} \tag{2.55}$$

$$\begin{aligned} q_{1,t_2} &= -\partial^2 q_1 + 2\partial q_0, \\ q_{0,t_2} &= \partial^2 q_0 - \frac{2}{3}\partial^3 q_1 - \frac{2}{3} q_1 \partial q_1, \end{aligned} \tag{2.56}$$

$$\begin{aligned} q_{1,t_4} &= -\frac{1}{3}\partial^4 q_1 + \frac{2}{3}\partial^3 q_0 - \frac{2}{3} q_1 \partial^2 q_1 - \frac{2}{3}(\partial q_1)^2 + \frac{4}{3} q_1 \partial q_0 + \frac{4}{3} q_0 \partial q_1, \\ q_{0,t_4} &= -\frac{2}{9}\partial^5 q_1 + \frac{1}{3}\partial^4 q_0 - \frac{2}{3} q_1 \partial^3 q_1 - \frac{4}{3}\partial q_1 \partial^2 q_1 + \frac{2}{3}\partial q_1 \partial q_0 + \frac{2}{3} q_1 \partial^2 q_0 \\ &\quad -\frac{4}{9} q_1^2 \partial q_1 + \frac{4}{3} q_0 \partial q_0. \end{aligned} \tag{2.57}$$

These are the first three members of the Boussinesq hierarchy.

Theorem 2.8. *Let* $P, Q \in C_B(\{L_n\})$.
(i). If $\operatorname{order}(P) \geq 0$ *then*

$$\partial_P Q = [P_+, Q] = [-P_-, Q]. \tag{2.58}$$

(ii). If $\operatorname{order}(P) \geq 0$, $\operatorname{order}(Q) \geq 0$ *then*

$$\partial_P Q_+ = [-P_-, Q_+]_+ \tag{2.59}$$

and the corresponding evolutionary derivations ∂_P *and* ∂_Q *commute,*

$$[\partial_P, \partial_Q] = 0, \tag{2.60}$$

i.e., the flows defined in accordance with (2.31) commute.

For $P = \sum_{j=-\infty}^{M} p_j \xi^j \in M_m(B)((\xi^{-1}))$, the coefficient p_{-1} is called the residue of P and one writes $\operatorname{Res}(P) = p_{-1}$. If d is an evolutionary derivation on B, then $T \in B$ is called a (polynomially) conserved density for d iff $dT \in \operatorname{Ran}(\partial)$ (i.e., iff $dT = \partial X$ for some $X \in B$). One then proves

Theorem 2.9. *For any* $Q \in C_B(\{L_n\})$, $\operatorname{Tr}[\operatorname{Res}(Q)]$ *is a conserved density for all evolutionary derivations* ∂_P *associated with* $P \in C_B(\{L_n\})$ ($\operatorname{order}(P) \geq 0$, *see (2.30)). If* Q *is homogeneous of degree* s, *then* $\operatorname{Tr}[\operatorname{Res}(Q)]$ *is homogeneous of degree* $s+1$. *(Here* $\operatorname{Tr}(.)$ *denotes the trace of* $m \times m$ *matrices.)*

3 Drinfeld–Sokolov Systems

In this section we provide a short algebraic treatment of Drinfeld–Sokolov (DS) (or modified Gelfand–Dickey) systems [9] following and extending the treatment in Kuperschmidt and Wilson [23] and Wilson [36] (see also [30]). We freely use the notation employed in Section 2 and assume the following hypothesis for the rest of this section.

Hypothesis 3.1. *(i). $q_n = \mathrm{diag}(c_1, \dots, c_m)$ has an n–th root (i.e., c_μ, $1 \leq \mu \leq m$ have an n–th root)*

$$q_n^{1/n} = \mathrm{diag}(c_1^{1/n}, \dots, c_m^{1/n}). \tag{3.1}$$

(In order to avoid problems with the nonuniqueness of the n–th roots of c_μ, we shall fix the choice of $c_\mu^{1/n}$, $1 \leq \mu \leq m$ in Sections 3 and 4.)
(ii). $\phi_k \in M_m(A)$, $1 \leq k \leq n$ and

$$\text{if } c_\mu = c_\nu \text{ then } \sum_{k=1}^{n} \phi_{k,\mu,\nu} = 0, \quad 1 \leq \mu, \nu \leq m. \tag{3.2}$$

Given Hypothesis 3.1, we restrict the algebra A to

$$D = F\left[\left\{\phi_{k,\mu,\nu}^{(l)}\right\}_{\substack{l\in\mathbb{N}_0 \\ 1\leq k\leq n}} \setminus \left\{\phi_{n,\mu',\nu'}^{(l)} \text{ if } c_{\mu'} = c_{\nu'}\right\}_{l\in\mathbb{N}_0}\right] \tag{3.3}$$

and denote the restriction of ∂ to D again by ∂. ∂ naturally extends to $M_m(D)$. On D one defines a $\mathbb{Z}$–grading by associating the degree

$$\deg(\phi_{k,\mu,\nu}^{(l)}) = l + 1 \tag{3.4}$$

with $\phi_{k,\mu,\nu}^{(l)}$. ∂ is then homogeneous of degree 1. Since we are now interested in $n \times n$ formal matrix (pseudo–) differential operators with entries in $M_m(D)[\xi]$, $(M_m(D)((\xi^{-1})))$ we introduce

$$(M_m(D)[\xi])^n = \left\{\sum_{j=0}^{N} \mathcal{R}_j \xi^j \mid \mathcal{R}_j \in M_{mn}(D),\ 0 \leq j \leq N,\ N \in \mathbb{N}_0\right\} \tag{3.5}$$

and

$$(M_m(D)((\xi^{-1})))^n = \left\{\sum_{j=-\infty}^{M} \mathcal{S}_j \xi^j \,\middle|\, \mathcal{S}_j \in M_{mn}(D),\ j \leq M,\ M \in \mathbb{Z}\right\}. \tag{3.6}$$

The role of the basic operator L_n of Section 2 is now played by

$$\mathcal{M}_n = \begin{pmatrix} 0 & 0 & \cdots & q_n^{1/n}(e_m\xi+\phi_n) & \\ q_n^{1/n}(e_m\xi+\phi_1) & 0 & \ddots & & 0 \\ 0 & q_n^{1/n}(e_m\xi+\phi_2) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & q_n^{1/n}(e_m\xi+\phi_{n-1}) & 0 \end{pmatrix} \in (M_m(D)[\xi])^n,$$
$$e_m = \mathrm{diag}(e,\dots,e) \tag{3.7}$$

and on $(M_m(D))^n$, the algebra of $n\times n$ matrices over F with entries in $M_m(D)$, one can introduce a $(\mod n)$–grading as usual. More precisely, on any algebra of $n\times n$ matrices (with entries in any associative algebra) the natural $(\mod n)$–grading is defined as follows: $\mathcal{R}=[R_{k,l}]_{k,l=1}^n$ is called homogeneous of degree r iff $R_{k,l}=0$ except when $k-l=r(\mod n)$. Thus $\mathcal{M}_n$ is homogeneous of degree $1(\mod n)$ and $\mathcal{M}_n^n$ is homogeneous of degree $0(\mod n)$, i.e., $\mathcal{M}_n^n$ is a diagonal matrix differential expression.

Defining

$$\deg(\xi)=1 \tag{3.8}$$

enables one to extend our $\mathbb{Z}$–grading to $(M_m(D)[\xi])^n$ and $(M_m(D)((\xi^{-1})))^n$. In fact, one has

Lemma 3.2.

$$\mathcal{M}_n^n = \mathrm{diag}(L_{n,1},\dots,L_{n,n}), \tag{3.9}$$

where

$$L_{n,k} = A_{k+n+1}\cdots A_{k+1}A_k, \tag{3.10}$$
$$A_k = q_n^{1/n}(e_m\xi+\phi_k),\ A_{k+n}=A_k,\ 1\le k\le n. \tag{3.11}$$

Thus each operator $L_{n,k}$ is of the type of the operator L_n considered in Section 2.

In order to derive the analog of Theorem 2.2 in connection with $\mathcal{M}_n$ one again needs to extend the algebra D and the derivation ∂ on D to an algebra $\bar{D}$ with derivation $\bar{\partial}$ on $\bar{D}$ such that

$$\ker(\bar{\partial})=F,\ \mathrm{Ran}\,(\bar{\partial})=\bar{D}. \tag{3.12}$$

Then also the $\mathbb{Z}$–grading on D extends to one on $\bar{D}$ with $\bar{\partial}$ being homogeneous of degree 1.

Theorem 3.3. *There exists an element $\mathcal{K}_n \in (M_m(\bar{D})((\xi^{-1})))^n$ of the type*

$$\mathcal{K}_n = \mathrm{diag}(K_{n,1},\dots,K_{n,n}), \tag{3.13}$$

$$
\begin{aligned}
K_{n,k} &= e_m + \sum_{j=-\infty}^{-1} \chi_{n,k,j}\xi^j \in M_m(\bar{D})((\xi^{-1})), \\
K_{n,k+n} &= K_{n,k},\ 1 \le k \le n
\end{aligned}
\tag{3.14}
$$

such that

$$
\mathcal{M}_n = \mathcal{K}_n \mathcal{M}_{0,n} \mathcal{K}_n^{-1},
\tag{3.15}
$$

where

$$
\mathcal{M}_{0,n} = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 & q_n^{1/n}\xi \\
q_n^{1/n}\xi & 0 & \ddots & & & 0 \\
0 & q_n^{1/n}\xi & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & q_n^{1/n}\xi & 0
\end{pmatrix}
\tag{3.16}
$$

is the (constant coefficient) leading order term of $\mathcal{M}_n$. The normalization $K_{n,k} = e_m +$ [lower order terms] together with the assumption that each $K_{n,k}$, $1 \le k \le n$ be homogeneous of degree zero uniquely determines $\mathcal{K}_n$. In particular,

$$
L_{n,k} = K_{n,k} L_{0,n} K_{n,k}^{-1}, \quad 1 \le k \le n.
\tag{3.17}
$$

The proof requires a straightforward generalization of the arguments in [34], [35] and relies on Hypothesis 3.1. We shall always use the unique degree zero part of $\mathcal{K}_n$ in the following. For a given element $\mathcal{M} \in (M_m(D)((\xi^{-1})))^n$ we denote by

$$
C_D(\{\mathcal{M}\}) = \{\mathcal{P} \in (M_m(D)((\xi^{-1})))^n \mid [\mathcal{P}, \mathcal{M}] = 0\}
\tag{3.18}
$$

the centralizer of $\{\mathcal{M}\}$ in $(M_m(D)((\xi^{-1})))^n$ and its center by

$$
Z(C_D(\{\mathcal{M}\})) = \{Q \in C_D(\{\mathcal{M}\}) \mid [Q, \mathcal{P}] = 0 \text{ for all } \mathcal{P} \in C_D(\{\mathcal{M}\})\}.
\tag{3.19}
$$

Similarly $C_{\bar{D}}(\{\mathcal{M}\})$, $Z(C_{\bar{D}}(\{\mathcal{M}\}))$ denote the corresponding subalgebras with D replaced by its extension $\bar{D}$.

For the uniqueness of the degree zero part of $\mathcal{K}_n$ in Theorem 3.3 one uses part (i) of

Theorem 3.4. *Given $\mathcal{M}_{0,n}$ as in (3.16) let $\mathcal{R}$ be defined by*

$$
\mathcal{R} = [\mathcal{R}_{k,l}]_{k,l=1}^n = \begin{pmatrix}
R_{1,1} & \cdots & R_{1,n} \\
\vdots & \ddots & \vdots \\
R_{n,1} & \cdots & R_{n,n}
\end{pmatrix}, \qquad \mathcal{R} \in \left(M_m(\bar{D})((\xi^{-1}))\right)^n,
\tag{3.20}
$$

where

$$R_{k,l} = \sum_{j=-\infty}^{s} R_{k,l,j}\xi^j, \quad R_{k,l,j} \in M_m(\bar{D}), \tag{3.21}$$

i.e.,

$$R_{k,l,j} = [R_{k,l,j}]_{\mu,\nu=1}^{m} = \begin{pmatrix} R_{k,l,j,1,1} & \cdots & R_{k,l,j,1,m} \\ \vdots & \ddots & \vdots \\ R_{k,l,j,m,1} & \cdots & R_{k,l,j,m,m} \end{pmatrix}, \quad 1 \le k,l \le n, \ -\infty < \ j \le s. \tag{3.22}$$

Then
(i). $\mathcal{R}$ *commutes with* $\mathcal{M}_{0,n}$ *iff each* $R_{k,l,j}$ *is a constant matrix which commutes with* $q_n^{1/n}$ *and* $R_{k+1,l+1} = R_{k,l}$ *(i.e.,* $\mathcal{R}$ *is circulant as an* $n \times n$ *matrix).*
(ii).

$$C_D(\{\mathcal{M}_{0,n}\}) = C_{\bar{D}}(\{\mathcal{M}_{0,n}\}). \tag{3.23}$$

(iii).

$$Z(C_{\bar{D}}(\{\mathcal{M}_n\})) = Z(C_D(\{\mathcal{M}_n)) = C_D(\{\mathcal{M}_n\}). \tag{3.24}$$

In particular $C_D(\{\mathcal{M}_n\})$ *is commutative.*

Sketch of Proof. (i).

$$\begin{aligned} &[\mathcal{R}, \mathcal{M}_{0,n}] = 0 \quad \text{implies} \quad [\mathcal{R}, (\mathcal{M}_{0,n})^n] = 0, \quad \text{or equivalently,} \\ &[\mathcal{R}, \operatorname{diag}(L_{0,n}, \dots, L_{0,n})] = 0, \quad \text{i.e.,} \quad [R_{k,l}, L_{0,n}] = 0, \qquad 1 \le k,l \le n. \end{aligned} \tag{3.25}$$

Theorem 2.3 (ii) now implies that all $R_{k,l,j}$ are constant matrices and $[R_{k,l,j}, q_n] = 0$ which is equivalent to $[R_{k,l,j}, q_n^{1/n}] = 0$, i.e., $R_{k,l,j,\mu,\nu}(c_\mu^{1/n} - c_\nu^{1/n}) = 0$, or equivalently, $R_{k,l,j,\mu,\nu}(c_\mu - c_\nu) = 0$. This implies

$$\mathcal{R}\mathcal{M}_{0,n} - \mathcal{M}_{0,n}\mathcal{R} = 0, \qquad \text{i.e.,} \qquad R_{k+1,l+1}q_n^{1/n} = q_n^{1/n}R_{k,l} = R_{k,l}q_n^{1/n}. \tag{3.26}$$

Hence $\mathcal{R}$ must be a circulant. Parts (ii) and (iii) then follow as in [34], [23] in connection with Theorem 2.3.

Remark 3.5. Circulant matrices commute with each other. Let

$$C = \operatorname{circ}(c_1, \dots, c_n) = \begin{pmatrix} c_1 & c_2 & \cdots & \cdots & c_{n-1} & c_n \\ c_n & \ddots & \ddots & & & c_{n-1} \\ c_{n-1} & \ddots & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & c_2 \\ c_2 & \cdots & \cdots & c_{n-1} & c_n & c_1 \end{pmatrix}, \tag{3.27}$$

then writing $C_{1,j} = c_j$ we get for circulant matrices

$$C_{j,k} = c_{k-j+1} \qquad \text{(subscripts are taken } (\mod n)). \tag{3.28}$$

Using the same notation for $B = \mathrm{circ}(b_1, \dots, b_n)$ one infers that

$$\begin{aligned} [B, C] &= \sum_{m=1}^{n} (B_{j,m} B_{m,k} - B_{j,m} B_{m,k}) = \sum_{m=1}^{n} (b_{m-j+1} c_{k-m+1} - c_{m-j+1} b_{k-m+1}) \\ &= \sum_{m=1}^{n} b_{m-j+1} c_{k-m+1} - \sum_{\tilde{m}=1}^{n} b_{\tilde{m}-j+1} c_{k-\tilde{m}+1} = 0. \end{aligned} \tag{3.29}$$

In analogy to (2.24) we call

$$\mathcal{Q}_{0,r} = \mathrm{circ}(P_{0,r_1}, \dots, P_{0,r_n}) \in \Big(M_m(D)[\xi]\Big)^n, \qquad P_{0,r_k} = p_{r_k} \xi^{r_k}, \quad p_{r_k} \in M_m(D), \\ r_k \in \mathbb{N}, \quad 1 \le k \le n \tag{3.30}$$

$\mathcal{M}_{0,n}$–admissible iff $p_{r_k}, 1 \le k \le n$ are constant matrices lying in the center of the centralizer of $q_n^{1/n}$, i.e.,

$$p_{r_k} \in Z\Big(C_D(\{q_n^{1/n}\})\Big) = Z\Big(C_D(\{q_n\})\Big), \qquad 1 \le k \le n. \tag{3.31}$$

Remark 3.6. Hence $\mathcal{Q}_{0,r}$ is $\mathcal{M}_{0,n}$-admissible iff P_{0,r_k} is $L_{0,n}$-admissible for all $1 \le k \le n$.

The analog of Theorem 2.5 is then given by

Theorem 3.7. *Given $\mathcal{M}_n \in \Big(M_m(D)[\xi]\Big)^n$, let $\mathcal{K}_n = \mathrm{diag}(K_{n,1}, \dots, K_{n,n})$ be the corresponding formal dressing operator (3.13) satisfying $\mathcal{M}_{0,n} = \mathcal{K}_n^{-1} \mathcal{M}_n \mathcal{K}_n$. Assume that $\mathcal{Q}_{0,r} = \mathrm{circ}(P_{0,r_1}, \dots, P_{0,r_n})$, $P_{0,r_k} = p_{r_k} \xi^{r_k} \in M_m(D)[\xi]$ commutes with $\mathcal{M}_{0,n}$, i.e. $[\mathcal{Q}_{0,r}, \mathcal{M}_{0,n}] = 0$. Define*

$$\mathcal{Q}_r = \mathcal{K}_n \mathcal{Q}_{0,r} \mathcal{K}_n^{-1} \in \Big(M_m(\bar{D})((\xi^{-1}))\Big)^n \tag{3.32}$$

(which implies $[\mathcal{Q}_r, \mathcal{M}_n] = 0$). Then actually

$$\mathcal{Q}_r \in \Big(M_m(D)((\xi^{-1}))\Big)^n \tag{3.33}$$

iff $\mathcal{Q}_{0,r}$ is $\mathcal{M}_{0,n}$-admissible.

The elements of $C_D(\{\mathcal{M}_n\})$ are characterized by

Theorem 3.8. *Let $P_{0,r} = p_r \xi^r$ and $\mathcal{Q}_{0,k,r} = \mathrm{circ}(0, \dots, 0, \underbrace{P_{0,r}}_{k}, 0, \dots, 0)$ be $\mathcal{M}_{0,n}$–admissible. (Note that $\mathcal{Q}_{0,k,r}$ is homogeneous of degree r with respect to the $\mathbb{Z}$–grading and homogeneous of degree $-k+1$ with respect to the $(\mod n)$–grading.) Then*

(i). $C_D(\{\mathcal{M}_n\})$ contains a unique element of the form

$$\mathcal{Q}_r = \mathcal{Q}_{0,k,r} + [\text{ lower order terms }] \tag{3.34}$$

which is homogeneous of degree r and $-k+1$ with respect to the $\mathbb{Z}$ and $(\mod n)$–grading.

(ii). $C_D(\{\mathcal{M}_n\})$ consists precisely of the (in general infinite) sums of elements in (i). In particular, the leading term in any element of $C_D(\{\mathcal{M}_n\})$ is a circulant constant $n \times n$ matrix with $m \times m$ matrix entries commuting with q_n.

In analogy to (2.30) an evolutionary derivation $\partial_{\mathcal{Q}}$ associated with $\mathcal{Q} \in C_D(\{\mathcal{M}_n\})$, order $(\mathcal{Q}) \geq 0$, degree$(\mathcal{Q}) = 0 (\mod n)$, is defined as a derivation commuting with ∂ that satisfies

$$\partial_{\mathcal{Q}}(q_n^{1/n}\phi_k)_{\mu,\nu} = \left([\mathcal{Q}_+, \mathcal{M}_n]_{k+1,k}\right)_{\mu,\nu}, \quad 1 \leq k \leq n, \quad 1 \leq \mu, \nu \leq m. \tag{3.35}$$

Associating a time–parameter $t_{\mathcal{Q}} \in \mathbb{R}$ with $\mathcal{Q}$,

$$\partial_{t_{\mathcal{Q}}}\phi_{k,\mu,\nu} = \partial_{\mathcal{Q}}\phi_{k,\mu,\nu}, \quad 1 \leq k \leq n, \quad 1 \leq \mu, \nu \leq m \tag{3.36}$$

then represents the system of nonlinear evolution equations corresponding to the Lax pair $(\mathcal{Q}_+, \mathcal{M}_n)$. If $\mathcal{Q} = \mathcal{Q}_r$ is homogeneous of degree $r \geq 1$ (with respect to the $\mathbb{Z}$–grading) then the right–hand–side of equation (3.35) are differential polynomials in the ϕ_k homogeneous of degree $r+1$. (3.36) can be rewritten as

$$\frac{d}{dt_{\mathcal{Q}}}\mathcal{M}_n = \partial_{\mathcal{Q}}\mathcal{M}_n = [\mathcal{Q}_+, \mathcal{M}_n] = [-\mathcal{Q}_-, \mathcal{M}_n]. \tag{3.37}$$

By (3.11) and $\mathcal{Q} = \mathrm{diag}(P_1, \dots, P_n)$ this is equivalent to

$$\begin{aligned} \frac{d}{dt_{\mathcal{Q}}}A_k &= \partial_{\mathcal{Q}}A_k = (P_{k+1})_+A_k - A_k(P_k)_+ \\ &= -(P_{k+1})_-A_k + A_k(P_k)_- \\ &= \mathrm{Res}\,(P_k - P_{k+1}), \quad 1 \leq k \leq n. \end{aligned} \tag{3.38}$$

Note that A_k is of order 1 and $(P_k)_-$ is of order -1, so that the right hand side of (3.35) is a multiplication operator, i.e., a differential expression of order zero. (3.38) is equivalent to

$$\partial_{t_{\mathcal{Q}}}\phi_k = p_{-1,k} - (q_n^{1/n})^{-1}p_{-1,k+1}q_n^{1/n}, \quad 1 \leq k \leq n, \tag{3.39}$$

i.e., to

$$\partial_{t_{\mathcal{Q}}}\phi_{k,\mu,\nu} = p_{-1,k,\mu,\nu} - (c_\mu^{1/n})^{-1}c_\nu^{1/n}p_{-1,k+1,\mu,\nu}, \quad 1 \leq k \leq n,\ 1 \leq \mu, \nu \leq m. \tag{3.40}$$

Remark 3.9. If $c_{\mu_0} = c_{\nu_0}$, (3.40) simplifies to

$$\partial_{t_{\mathcal{Q}}} \phi_{k,\mu_0,\nu_0} = p_{-1,k,\mu_0,\nu_0} - p_{-1,k+1,\mu_0,\nu_0}, \quad 1 \leq k \leq n. \tag{3.41}$$

Summing from $k = 1$ to $k = n-1$ then gives

$$\partial_{t_{\mathcal{Q}}} \phi_{n,\mu_0,\nu_0} = -\sum_{k=1}^{n-1} \partial_{t_{\mathcal{Q}}} \phi_{k,\mu_0,\nu_0} = p_{-1,n,\mu_0,\nu_0} - p_{-1,1,\mu_0,\nu_0} \tag{3.42}$$

which shows that the evolutionary derivation $\partial_{\mathcal{Q}}$ is compatible with the specialization (restriction) of the algebra in (3.2), (3.3). In particular, if $q_n = \mathrm{diag}(c, \dots, c)$, i.e., $c_\mu = c$, $1 \leq \mu \leq m$, then the first $n-1$ DS equations for $\phi_1, \dots, \phi_{n-1}$ imply the last one for ϕ_n.

In the special case where $m = 1$ the sequence of nonlinear evolution equations

$$\frac{d}{dt_r} \mathcal{M}_n = [(\mathcal{Q}_r)_+, \mathcal{M}_n], \quad r \in \mathbb{N}, \tag{3.43}$$

where $q_n = q_n^{1/n} = e$, $t_r = t_{\mathcal{Q}_r}$, and

$$\mathcal{Q}_r = \mathrm{diag}(P_{r,1}, \dots, P_{r,n}) \in C_D(\{\mathcal{M}_n\}) \tag{3.44}$$

with

$$P_{r,k} = e\xi^r + [\text{ lower order terms }], \quad 1 \leq k \leq n \tag{3.45}$$

being homogeneous of degree r, represents the Drinfeld–Sokolov hierarchy [9]. Whenever r is a multiple of n, i.e., $r = hn$ for some $h \in \mathbb{N}$, the equations (3.43) are trivial since then $\mathcal{Q}_r = \mathcal{M}_n^h$. More generally, (3.36), (3.37) are called Drinfeld–Sokolov systems.

Example 3.10. $m = 1, q_n^{1/n} = e$.

For $n > 1$ we have

$$\begin{aligned} q_{n,k} &= e, \quad q_{n-1,k} = 0, \\ q_{n-2,k} &= \sum_{l=1}^{n} \sum_{m=1}^{l-1} \phi_l \phi_m + \partial \left(\sum_{l=1}^{n-1} \sum_{m=1}^{l} \phi_{k+m-1} \right) \\ &= \partial\Big((n-1)\phi_k + (n-2)\phi_{k+1} + \dots + \phi_{k+n-2}\Big) + (\phi_1\phi_2 + \ldots + \phi_{n-1}\phi_n), \\ & \qquad 1 \leq k \leq n. \end{aligned} \tag{3.46}$$

$n = 2, r = 1, 3, 5, 7:$

$$A_k = e\xi + \phi_k, \qquad k = 1, 2, \qquad \phi_2 = -\phi_1, \tag{3.47}$$

$$\mathcal{M}_2 = \begin{pmatrix} 0 & A_2 \\ A_1 & 0 \end{pmatrix}, \qquad \mathcal{Q}_r = \begin{pmatrix} P_{r,1} & 0 \\ 0 & P_{r,2} \end{pmatrix}, \tag{3.48}$$

$$(P_{1,k})_+ = e\xi, \tag{3.49}$$

$$(P_{3,k})_+ = e\xi^3 + \frac{3}{2}q_{0,k}\xi + \frac{3}{4}\partial q_{0,k}, \tag{3.50}$$

$$\begin{aligned}(P_{5,k})_+ &= e\xi^5 + \frac{5}{2}q_{0,k}\xi^3 + \frac{15}{4}\partial q_{0,k}\xi^2 + \frac{5}{8}(3q_{0,k}^2 + 5\partial^2 q_{0,k})\xi \\ &\quad + \frac{5}{16}(3\partial^3 q_{0,k} + 6q_{0,k}\partial q_{0,k}),\end{aligned} \tag{3.51}$$

$$\begin{aligned}(P_{7,k})_+ &= e\xi^7 + \frac{7}{2}q_{0,k}\xi^5 + \frac{35}{4}\partial q_{0,k}\xi^4 + \frac{1}{8}(35q_{0,k}^2 + 105\partial^2 q_{0,k})\xi^3 \\ &\quad + \frac{1}{16}(175\partial^3 q_{0,k} + 210q_{0,k}\partial q_{0,k})\xi^2 \\ &\quad + \frac{1}{32}(70q_{0,k}^3 + 245(\partial q_{0,k})^2 + 350q_{0,k}\partial^2 q_{0,k} + 161\partial^4 q_{0,k})\xi \\ &\quad + \frac{1}{64}(210q_{0,k}^2\partial q_{0,k} + 420\partial q_{0,k}\partial^2 q_{0,k} + 210q_{0,k}\partial^3 q_{0,k} + 63\partial^5 q_{0,k}),\end{aligned} \tag{3.52}$$

where

$$q_{0,k} = \phi_1\phi_2 + \partial\phi_k, \qquad k = 1, 2. \tag{3.53}$$

This yields the first four equations of the modified Korteweg-de Vries (mKdV) hierarchy,

$$\phi_{1,t_1} = \partial\phi_1, \tag{3.54}$$

$$\phi_{1,t_3} = \frac{1}{4}\partial^3\phi_1 - \frac{3}{2}\phi_1^2\partial\phi_1, \tag{3.55}$$

$$\phi_{1,t_5} = \frac{1}{16}\partial^5\phi_1 - \frac{5}{8}\phi_1^2\partial^3\phi_1 - \frac{5}{2}\phi_1\partial\phi_1\partial^2\phi_1 - \frac{5}{8}(\partial\phi_1)^3 + \frac{15}{8}\phi_1^4\partial\phi_1, \tag{3.56}$$

$$\begin{aligned}\phi_{1,t_7} &= \frac{1}{64}\Big(\partial^7\phi_1 - 14\phi_1^2\partial^5\phi_1 - 84\phi_1\partial\phi_1\partial^4\phi_1 - 140\phi_1\partial^2\phi_1\partial^3\phi_1 - 126(\partial\phi_1)^2\partial^3\phi_1 \\ &\quad + 70\phi_1^4\partial^3\phi_1 - 182\partial\phi_1(\partial^2\phi_1)^2 + 560\phi_1^3\partial\phi_1\partial^2\phi_1 + 420\phi_1^2(\partial\phi_1)^3 - 140\phi_1^6\partial\phi_1\Big).\end{aligned} \tag{3.57}$$

$n = 3, r = 1, 2:$

$$A_k = e\xi + \phi_k, \quad k = 1, 2, 3, \qquad \phi_3 = -(\phi_1 + \phi_2), \tag{3.58}$$

$$\mathcal{M}_3 = \begin{pmatrix} 0 & 0 & A_3 \\ A_1 & 0 & 0 \\ 0 & A_2 & 0 \end{pmatrix}, \qquad \mathcal{Q}_r = \begin{pmatrix} P_{r,1} & 0 & 0 \\ 0 & P_{r,2} & 0 \\ 0 & 0 & P_{r,3} \end{pmatrix}, \tag{3.59}$$

$$(P_{1,k})_+ = e\xi, \tag{3.60}$$

$$(P_{2,k})_+ = e\xi^2 + \frac{2}{3}q_{1,k}, \tag{3.61}$$

where

$$q_{1,k} = \partial\Big(2\phi_k + \phi_{k+1}\Big) + (\phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3) \tag{3.62}$$

$$q_{0,k} = \partial^2\phi_k + \phi_k\Big(\partial\phi_{k+1} - \partial\phi_k\Big) + \phi_1\phi_2\phi_3, \qquad \phi_k = \phi_{k+3},, \quad k = 1,2,3. \tag{3.63}$$

This yields the first two equations of the modified Boussinesq hierarchy,

$$\phi_{k,t_1} = \partial\phi_k, \tag{3.64}$$

$$\phi_{k,t_2} = -\frac{1}{3}\partial^2\phi_k - \frac{2}{3}\partial^2\phi_{k+1} - 2\phi_k\partial\phi_k - \frac{2}{3}\partial(\phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3), \tag{3.65}$$

$$\phi_k = \phi_{k+3}, \qquad k = 1,2,3.$$

Finally, the analogs of Theorems 2.8 and 2.9 read

Theorem 3.11. *Let* $\mathcal{P}$, $\mathcal{Q} \in C_D(\{\mathcal{M}_n\})$
(i). If $\operatorname{order}(\mathcal{P}) \geq 0$, $\operatorname{degree}(\mathcal{P}) = 0 \pmod{n}$ *then*

$$\partial_{\mathcal{P}}\mathcal{Q} = [\mathcal{P}_+, \mathcal{Q}] = [-\mathcal{P}_-, \mathcal{Q}]. \tag{3.66}$$

(ii). If $\operatorname{order}(\mathcal{P}) \geq 0$, $\operatorname{order}(\mathcal{Q}) \geq 0$, $\operatorname{degree}(\mathcal{P}) = 0 \pmod{n}$, $\operatorname{degree}(\mathcal{Q}) = 0 \pmod{n}$ *then*

$$\partial_{\mathcal{P}}(\mathcal{Q}_+) = [-\mathcal{P}_-, \mathcal{Q}_+]_+ \tag{3.67}$$

and the corresponding evolutionary derivations $\partial_{\mathcal{P}}$ *and* $\partial_{\mathcal{Q}}$ *commute,*

$$[\partial_{\mathcal{P}}, \partial_{\mathcal{Q}}] = 0, \tag{3.68}$$

i.e., the flows defined in accordance with (3.36) commute.

Theorem 3.12. *For any* $\mathcal{P} \in C_D(\{\mathcal{M}_n\})$, $\operatorname{Tr}[\operatorname{tr}(\operatorname{Res}(\mathcal{P}))]$ *is a conserved density for all evolutionary derivations* $\partial_{\mathcal{Q}}$ *associated with* $\mathcal{Q} \in C_D(\{\mathcal{M}_n\})(\operatorname{order}(\mathcal{Q}) \geq 0$, $\operatorname{degree}(\mathcal{Q}) = 0 \pmod{n}$ *see (3.35)). If* $\mathcal{P}$ *is homogeneous of degree* s*, then* $\operatorname{Tr}[\operatorname{tr}(\operatorname{Res}(\mathcal{P}))]$ *is homogeneous of degree* $s+1$*. (Here* $\operatorname{tr}(.)$ *denotes the trace of* $n \times n$ *matrices.)*

4 Connections between GD and DS Systems

In this section we exhibit connections between GD systems and their modified versions, the DS systems, in the spirit of Miura's transformation connecting the Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) equation. As in Sections 2 and 3 we shall rely on algebraic methods deferring an analytical treatment to Section 5.

Assuming Hypothesis 3.1, let

$$L_{n,k} = A_{k+n-1}\cdots A_{k+1}A_k = \sum_{j=0}^{n} q_{j,k}\xi^j, \quad 1 \leq k \leq n \tag{4.1}$$

be the operators defined in (3.10), (3.11). Then the $q_{j,k}$ are matrix–valued differential polynomials in the ϕ_k of degree $n-j$ (since $\deg(\phi_k^{(l)})=l+1$). Especially,

$$q_{n,k}=q_n, \tag{4.2}$$

$$q_{n-1,k}=\sum_{l=0}^{n-1}(q_n^{1/n})^{n-l}\phi_{l+k}(q_n^{1/n})^l, \quad 1\le k\le n. \tag{4.3}$$

Due to the well known special case $n=2$, $m=1$,

$$q_{2,k}=q_{2,k}^{1/2}=e,\ q_{1,k}=0, \quad q_{0,k}=\phi_1\phi_2+\partial\phi_k, \quad \phi_1+\phi_2=0,\ k=1,2 \tag{4.4}$$

which represents Miura's transformation [28] linking solutions of the KdV and mKdV hierarchy, the expression of $q_{j,k}$, $0\le j\le n-1$, $1\le k\le n$ in terms of the $(\phi_1,\dots,\phi_n)$ in (4.1) corresponds to the generalized Miura transformations between GD and DS systems. As will become clear during the course of this section, these generalized Miura transformations, together with the DS equations, provide auto–Bäcklund transformations for the GD system. As a simple consequence of Lemma 3.2 we shall next prove that any solution $(\phi_1,\dots,\phi_n)$ of the DS system (3.36) yields n solutions of the associated GD system (2.31) in terms of the $q_{j,k}$ in (4.1). More precisely, we have

Theorem 4.1. *Assume Hypothesis 3.1. Suppose* $\mathcal{Q}=\mathrm{diag}(P_1,\dots,P_n)\in C_D(\{\mathcal{M}_n\})$ *and that* $(\phi_1,\dots,\phi_n)$ *satisfies the DS system (3.36),*

$$\partial_{t_{\mathcal{Q}}}\phi_{k,\mu,\nu}=\partial_{\mathcal{Q}}\phi_{k,\mu,\nu}, \quad 1\le k\le n,\ 1\le\mu,\nu\le m. \tag{4.5}$$

Define $q_{j,k}$ *by (4.1). Then* $(q_{0,k},\dots,q_{n-1,k})$, $1\le k\le n$ *satisfy the GD system (2.31) with* $P=P_k$, $1\le k\le n$,

$$\partial_{t_{P_k}}q_{j,k,\mu,\nu}=\partial_{P_k}q_{j,k,\mu,\nu}, \quad 0\le j\le n-1,\ 1\le k\le n,\ 1\le\mu,\nu\le m. \tag{4.6}$$

Proof. Since (4.5) is equivalent to

$$\frac{d}{dt_{\mathcal{Q}}}\mathcal{M}_n=[\mathcal{Q}_+,\mathcal{M}_n], \tag{4.7}$$

one infers

$$\frac{d}{dt_{\mathcal{Q}}}\mathcal{M}_n^n=[\mathcal{Q}_+,\mathcal{M}_n^n]. \tag{4.8}$$

Taking into account the diagonal structure of $\mathcal{M}_n^n$ by Lemma 3.2,

$$\mathcal{M}_n^n=\mathrm{diag}(L_{n,1},\dots,L_{n,n}), \tag{4.9}$$

(4.8) is equivalent to

$$\frac{d}{dt_{P_k}} L_{n,k} = [(P_k)_+, L_{n,k}], \quad 1 \leq k \leq n \tag{4.10}$$

which in turn is equivalent to (4.6). $\square$

Since ∂_Q in (4.5) is defined as a derivation over D but ∂_{P_k} in (4.6) is a derivation over B, our notation in Theorem 4.1 and its proof implicitly imply that the coefficients in P_k, originally expressed in terms of the (matrix) differential polynomials in the ϕ_k, are rewritten as differential polynomials in the $q_{j,k}$. This notational convention will be assumed in the following.

A first form of a converse of Theorem 4.1 is provided by

Theorem 4.2 . *Assume Hypothesis 2.1 and suppose that* $L_n = \sum_{j=0}^{n} q_j \xi^j$, $P_{0,r} = p_r \xi^r$, $p_r \in Z(C_B(\{q_n\}))$, $P_r = K_n P_{0,r} K_n^{-1}$ *for some* $r \in \mathbb{N}$ *are homogeneous of degree* n *and* r *respectively. Moreover, suppose* $(q_0, \dots, q_{n-1})$ *satisfies the GD system*

$$\frac{d}{dt_r} L_n = \partial_{P_r} L_n = [(P_r)_+, L_n]. \tag{4.11}$$

Assume there exists a factorization of $L_n = L_{n,1}$ *of the form*

$$L_{n,1} = A_n \cdots A_1, \tag{4.12}$$

where

$$A_k = q_n^{1/n}(\xi + \phi_k), \qquad \phi_k \in M_m(A), \qquad 1 \leq k \leq n. \tag{4.13}$$

This determines $(K_n = K_{n,1}, P_r = P_{r,1})$

$$\mathcal{M}_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & A_n \\ A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & A_{n-1} & 0 \end{pmatrix},$$

$$(\mathcal{M}_n)^n = \operatorname{diag}(L_{n,1}, \dots, L_{n,n}), \qquad L_{n,k} = \sum_{j=0}^{n} q_{j,k} \xi^j, \quad 1 \leq k \leq n, \tag{4.14}$$

and

$$\mathcal{Q}_r = \operatorname{diag}(P_{r,1}, \dots, P_{r,n}), \qquad P_{r,k} = K_{n,k} P_{0,r} K_{n,k}^{-1}, \qquad 1 \leq k \leq n. \tag{4.15}$$

If $(q_{0,k},\dots,q_{n-1,k}), 1\le k\le n$ *satisfy the GD system*

$$\frac{d}{dt_r}L_{n,k}=[(P_{r,k})_+,L_{n,k}],\qquad 1\le k\le n, \tag{4.16}$$

then $(\phi_1,\dots,\phi_n)$ *fulfill the DS system*

$$\frac{d}{dt_r}\mathcal{M}_n=\partial_{\mathcal{Q}_r}\mathcal{M}_n=[(\mathcal{Q}_r)_+,\mathcal{M}_n]. \tag{4.17}$$

Proof.

$$A_k=K_{n,k+1}q_n^{1/n}\xi K_{n,k}^{-1} \tag{4.18}$$

implies

$$\frac{d}{dt_r}A_k=(\frac{d}{dt_r}K_{n,k+1})K_{n,k+1}^{-1}A_k-A_k(\frac{d}{dt_r}K_{n,k})K_{n,k}^{-1}, \tag{4.19}$$

and

$$L_{n,k}=K_{n,k}q_n\xi^nK_{n,k}^{-1} \tag{4.20}$$

yields

$$\frac{d}{dt_r}L_{n,k}=\left[(\frac{d}{dt_r}K_{n,k})K_{n,k}^{-1},L_{n,k}\right]. \tag{4.21}$$

Moreover, (4.16) implies

$$(\frac{d}{dt_r}K_{n,k})K_{n,k}^{-1}=-(P_{r,k})_- \tag{4.22}$$

which together with (4.19) yields

$$\frac{d}{dt_r}A_k \;=\; -(P_{r,k+1})_-A_k+A_k(P_{r,k})_-=(P_{r,k+1})_+A_k-A_k(P_{r,k})_+. \qquad \Box \tag{4.23}$$

For the rest of Section 4 we assume the simplifying hypothesis

Hypothesis 4.3. *Let* $q_n^{1/n}=q_n=p_r=e_m=\mathrm{diag}(e,\dots,e)$.

By Hypothesis 2.1 (ii), Hypothesis 4.3 implies that $q_{n-1,\mu,\nu} = 0, 1 \leq \mu, \nu \leq m$. (While Hypothesis 4.3 is not essential for the rest of Section 4, it considerably simplifies the amount of notations involved.)

In the following we need to postulate the existence of a formal eigenvector associated with $e_m\xi$. To this effect we introduce hypothesis

Hypothesis 4.4. *For a fixed $\lambda \in F \setminus \{0\}$ assume there exist n different roots κ_k, i.e. $(\kappa_k)^n = \lambda$, $\kappa_k \neq \kappa_{k'}, k \neq k', 1 \leq k, k' \leq n$. Suppose there exist $\Psi_0(\kappa_k) \in A$ such that $\partial\Psi_0(\kappa_k) = \kappa_k\Psi_0(\kappa_k), 1 \leq k \leq n$.*

Then

$$\partial\psi_0(\kappa_k) = \kappa_k\psi_0(\kappa_k), \quad 1 \leq k \leq n, \tag{4.24}$$

where $\psi_0(\kappa_k)$ denotes the $m \times m$ matrix

$$\psi_0(\kappa_k) = e_m\Psi_0(\kappa_k) = \operatorname{diag}\left(e\Psi_0(\kappa_k), \ldots, e\Psi_0(\kappa_k)\right), \quad 1 \leq k \leq n. \tag{4.25}$$

We then define the action of $e_m\xi$ on $\psi_0(\kappa_k)$ by

$$e_m\xi^j\psi_0(\kappa_k) = \partial^j\psi_0(\kappa_k), \qquad j \in \mathbb{N}, \ \ 1 \leq k \leq n \tag{4.26}$$

and consequently,

$$e_m\xi^j\psi_0(\kappa_k) = \kappa_k^j\psi_0(\kappa_k), \qquad j \in \mathbb{Z}, \ \ 1 \leq k \leq n. \tag{4.27}$$

Next we introduce the $m \times m$ matrices $\psi_{n,k}$

$$\psi_{n,k} = K_n\psi_0(\kappa_k) = \psi_0(\kappa_k) + \sum_{j=-\infty}^{-1} \chi_{n,j}\kappa_k^j\psi_0(\kappa_k), \qquad 1 \leq k \leq n. \tag{4.28}$$

Using Hypothesis 4.3, L_n becomes

$$L_n = e_m\xi^n + q_{n-2}\xi^{n-2} + \cdots + q_0, \qquad q_j \in M_m(A), \quad 0 \leq j \leq n-2. \tag{4.29}$$

According to Theorem 2.2 this determines K_n given $L_0 = e_m\xi^n$. The n $m \times m$ matrices $\psi_{n,k}$ now solve

$$(L_n - \lambda)\psi_{n,k} = 0, \qquad \kappa_k^n = \lambda, \ 1 \leq k \leq n. \tag{4.30}$$

Furthermore, let the matrices $\psi_{n,k}$ satisfy the following conditions:

Hypothesis 4.5. *(i). Assume that* $\{\psi_{n,k}\}_{1\leq k\leq n}$ *is a basis of the (algebraic) nullspace* $\ker(L_n-\lambda)$.

(ii). Define the $m\times m$ *matrices* $\Gamma_{j,k,l}$ *by*

$$\begin{aligned}
\Gamma_{j,k,0} &= \partial^{(j)}\psi_{n,k}, \quad 1\leq k\leq n-1,\\
\Gamma_{j,k,l} &= \Gamma_{j,k,l-1}\Gamma^{-1}_{l-1,k,l-1}-\Gamma_{j,k+1,l-1}\Gamma^{-1}_{l-1,k+1,l-1},\\
&\quad\ 1\leq k\leq n-1-l, \quad 1\leq l\leq j, \quad 1\leq j\leq n-1
\end{aligned}\tag{4.31}$$

and suppose that the matrices $\Gamma_{j,k,j},\ 1\leq j\leq n-2$ *are invertible.*

Explicitly, one computes

$$\begin{aligned}
\Gamma_{0,k,0} &= \psi_{n,k}, \quad 1\leq k\leq n-1,\\
\Gamma_{1,k,1} &= \left((\partial\psi_{n,k})\psi^{-1}_{n,k}-(\partial\psi_{n,k+1})\psi^{-1}_{n,k+1}\right), \quad 1\leq k\leq n-2,\\
\Gamma_{2,k,2} &= \Big(\left((\partial^2\psi_{n,k})\psi^{-1}_{n,k}-(\partial^2\psi_{n,k+1})\psi^{-1}_{n,k+1}\right)\left((\partial\psi_{n,k})\psi^{-1}_{n,k}-(\partial\psi_{n,k+1})\psi^{-1}_{n,k+1}\right)^{-1}\\
&\quad-\left((\partial^2\psi_{n,k+1})\psi^{-1}_{n,k+1}-(\partial^2\psi_{n,k+2})\psi^{-1}_{n,k+2}\right)\left((\partial\psi_{n,k+1})\psi^{-1}_{n,k+1}-(\partial\psi_{n,k+2})\psi^{-1}_{n,k+2}\right)^{-1}\Big),\\
&\qquad 1\leq k\leq n-3.
\end{aligned}\tag{4.32}$$

Condition (ii) guarantees that one can recover $q_j, 0\leq j\leq n-2$ from the $\psi_{n,k}$, $1\leq k\leq n-1$ by successive elimination from the system $L_n\psi_{n,k}=0, 1\leq k\leq n-1$. It also enables one to factorize $(L_n-\lambda)$ into n first-order differential expressions (see (4.43) below). In the special scalar case $m=1$, these conditions reduce to the nonvanishing of Wronskians $W(\psi_1,\dots,\psi_k)\neq 0, 1\leq k\leq n$ as in Theorem 5.11.

Given Hypothesis 4.3, we shall abbreviate the corresponding GD system (2.31) (2.31) as

$$\mathrm{GD}_{n,r,j}(q_0,\dots,q_{n-2})=0, \qquad 0\leq j\leq n-2,\tag{4.33}$$

where

$$\mathrm{GD}_{n,r,j,\mu,\nu}(q_0,\dots,q_{n-2})=\partial_{t_r}q_{j,\mu,\nu}-\partial_{P_r}q_{j,\mu,\nu}=0, \quad 0\leq j\leq n-2, \quad 1\leq\mu,\nu\leq m.\tag{4.34}$$

Theorem 4.6. *Assume Hypotheses 4.3, 4.4, and 4.5, let* L_n *be defined by (4.29),* $P_{0,r}=e_m\xi^r$, *and* $P_r=K_nP_{0,r}K_n^{-1}$. *Then the GD system (2.31) (2.31) is fulfilled*

$$\mathrm{GD}_{n,r,j}(q_0,\dots,q_{n-2})=0, \qquad 0\leq j\leq n-2\tag{4.35}$$

iff

$$\left(\partial_{t_r}-(P_r)_+\right)\psi_{n,k}=\sum_{l=1}^{n}\psi_{n,l}\alpha_{n,k,l}, \qquad 1\leq k\leq n-1,\tag{4.36}$$

where $\alpha_{n,k,l}$ *are constant* $m\times m$ *matrices.*

Proof. We have

$$\left(\mathrm{GD}_{n,r,n-2}(q_0,\dots,q_{n-2})\xi^{n-2}+\ldots+\mathrm{GD}_{n,r,0}(q_0,\dots,q_{n-2})\right)\psi_{n,k} \tag{4.37}$$
$$=\left[\left(\frac{d}{dt_r}-(P_r)_+\right),L_n\right]\psi_{n,k}=\left[\left(\frac{d}{dt_r}-(P_r)_+\right),(L_n-\lambda)\right]\psi_{n,k}$$
$$=-(L_n-\lambda)(\partial_{t_r}-(P_r)_+)\psi_{n,k},\qquad 1\le k\le n-1. \tag{4.38}$$

Condition (i) of Hypothesis 4.5 then gives the equivalence of

$$(\partial_{t_r}-(P_r)_+)\,\psi_{n,k}=\sum_{l=1}^{n}\psi_{n,l}\alpha_{n,k,l} \tag{4.39}$$

and

$$(L_n-\lambda)(\partial_{t_r}-(P_r)_+)\psi_{n,k}=0,\qquad 1\le k\le n-1. \tag{4.40}$$

Hence (4.35) implies (4.36).

Conversely, Hypothesis 4.5 (ii) is precisely the condition which allows one to eliminate $\mathrm{GD}_{n,r,j}(.)$ in the system

$$\left(\mathrm{GD}_{n,r,n-2}(q_0,\dots,q_{n-2})\xi^{n-2}+\ldots+\mathrm{GD}_{n,r,0}(q_0,\dots,q_{n-2})\right)\psi_{n,k}=0,\quad 1\le k\le n-1.$$

Thus (4.36) implies (4.35). □

In order to handle the case $\lambda=0$ excluded in Hypothesis 4.4, one assumes the existence of an element $x\in A$ such that $\partial x=e$. Then

$$\psi_{0,k}(0)=e_m x^{k-1},\qquad 1\le k\le n, \tag{4.41}$$
$$\psi_{n,k}(0)=K_n\psi_{0,k}(0)=\psi_{0,k}(0)+\sum_{j=-\infty}^{-1}\chi_{n,j}\frac{1}{k-j-1}x^{k-j-1} \tag{4.42}$$

and $\psi_{n,k}(0)$ then replaces $\psi_{n,k}$ in the treatment above.

Next set $\tilde L_{n,1}=L_{n,1}-\lambda=L_n-\lambda$. Condition (ii) of Hypothesis 4.5 allows one to calculate a factorization of $\tilde L_{n,1}$ as described below.

$$\tilde L_{n,1}=\tilde A_n\cdots\tilde A_2\tilde A_1,\qquad \tilde A_k=(e_m\xi+\tilde\phi_k),\qquad 1\le k\le n, \tag{4.43}$$

where $\tilde{\phi}_k$ are defined by

$$\tilde{\phi}_1 = -\left(\partial\psi_{n,1}\right)\psi_{n,1}^{-1}, \qquad \tilde{A}_1\psi_{n,1} = 0, \tag{4.44}$$

$$\begin{aligned}\tilde{\phi}_2 &= \left((\partial^2\psi_{n,1})\psi_{n,1}^{-1} - (\partial^2\psi_{n,2})\psi_{n,2}^{-1}\right)\left((\partial\psi_{n,1})\psi_{n,1}^{-1} - (\partial\psi_{n,2})\psi_{n,2}^{-1}\right)^{-1} + \left(\partial\psi_{n,1}\right)\psi_{n,1}^{-1}, \\ &= -\left(\partial(\tilde{A}_1\psi_{n,2})\right)\left(\tilde{A}_1\psi_{n,2}\right)^{-1}, \qquad \tilde{A}_2\tilde{A}_1\psi_{n,2} = 0,\end{aligned} \tag{4.45}$$

$$\vdots$$

$$\begin{aligned}\tilde{\phi}_{n-1} &= -\left(\partial(\tilde{A}_{n-2}\cdots\tilde{A}_1\psi_{n,n-1})\right)\left(\tilde{A}_{n-2}\cdots\tilde{A}_1\psi_{n,n-1}\right)^{-1}, \\ &\qquad \tilde{A}_{n-1}\cdots\tilde{A}_1\psi_{n,n-1} = 0,\end{aligned} \tag{4.46}$$

$$\tilde{\phi}_n = -\sum_{k=1}^{n-1}\tilde{\phi}_k. \tag{4.47}$$

Note that $\sum_{k=1}^{j}\tilde{\phi}_k = \Gamma_{j,1,j-1}\Gamma_{j-1,1,j-1}^{-1}, 1 \le j \le n-1$.
This determines $\tilde{\mathcal{M}}_n$ and $\tilde{L}_{n,k}$ by

$$\tilde{\mathcal{M}}_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & \tilde{A}_n \\ \tilde{A}_1 & 0 & \cdots & 0 & 0 \\ 0 & \tilde{A}_2 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \tilde{A}_{n-1} & 0 \end{pmatrix}, \qquad (\tilde{\mathcal{M}}_n)^n = \mathrm{diag}\left(\tilde{L}_{n,1}, \ldots, \tilde{L}_{n,n}\right), \tag{4.48}$$

and $L_{n,k}$ by $L_{n,k} = \tilde{L}_{n,k} + \lambda$. $\tilde{K}_{n,k}$ and $K_{n,k}$ are then determined by $L_{n,k} = K_{n,k}L_{0,n}K_{n,k}^{-1}$, $\tilde{L}_{n,k} = \tilde{K}_{n,k}L_{0,n}\tilde{K}_{n,k}^{-1}$. Note that $K_{n,k}$ and $\tilde{K}_{n,k}$ have the same structure. Only the terms which contain $q_{0,k}$, $\tilde{q}_{0,k}$ differ since $q_{0,k} = \tilde{q}_{0,k} + \lambda$.
Introducing

$$P_{r,k} = K_{n,k}P_{0,r}K_{n,k}^{-1}, \qquad 1 \le k \le n, \qquad P_r = P_{r,1}, \tag{4.49}$$

and

$$\mathcal{Q}_r = \mathrm{diag}\left(P_{r,1}\ldots P_{r,n}\right), \tag{4.50}$$

we can define the corresponding DS system by

$$\frac{d}{dt_r}\tilde{\mathcal{M}}_n - [(\mathcal{Q}_r)_+, \tilde{\mathcal{M}}_n] = 0. \tag{4.51}$$

We shall abbreviate (4.51) by

$$\widetilde{\mathrm{DS}}_{n,r,k}(\tilde{\phi}_1, \ldots, \tilde{\phi}_n) = 0, \qquad 1 \le k \le n, \tag{4.52}$$

where

$$\widetilde{\mathrm{DS}}_{n,r,k,\mu,\nu}(\tilde{\phi}_1,\dots,\tilde{\phi}_n) = \partial_{t_r}\tilde{\phi}_{k,\mu,\nu} - \partial_{\mathcal{Q}_r}\tilde{\phi}_{k,\mu,\nu} = 0, \quad 1 \le k \le n, \quad 1 \le \mu,\nu \le m. \tag{4.53}$$

Since for $1 \le k \le n$

$$\tilde{K}_{n,k}^{-1} L_{n,k} \tilde{K}_{n,k} = e_m(\xi^n + \lambda), \qquad P_{0,r} = e_m \xi^r, \tag{4.54}$$

$$P_{r,k} = K_{n,k} P_{0,r} K_{n,k}^{-1} = \tilde{K}_{n,k} \tilde{P}_{0,r} \tilde{K}_{n,k}^{-1}, \qquad \tilde{P}_{0,r} = e_m(\xi^n + \lambda)^{r/n}, \tag{4.55}$$

we see that $\mathcal{Q}_r$ and $\tilde{\mathcal{M}}_n$ commute and therefore the evolution equations (4.51) are well defined.

Example 4.7. $n = 2, m = 1, q_2^{1/2} = e, \tilde{\phi}_2 = -\tilde{\phi}_1$. In this case we have $\widetilde{\mathrm{DS}}_{2,r,1}(\tilde{\phi}_1,\tilde{\phi}_2) = \widetilde{\mathrm{mKdV}}_r(\tilde{\phi}_1)$.
$r = 1,3,5,7$:

$$\widetilde{\mathrm{mKdV}}_1(\tilde{\phi}_1) = \tilde{\phi}_{1,t_1} - \partial\tilde{\phi}_1 = 0, \tag{4.56}$$

$$\widetilde{\mathrm{mKdV}}_3(\tilde{\phi}_1) = \tilde{\phi}_{1,t_3} - (\frac{1}{4}\partial^3\tilde{\phi}_1 - \frac{3}{2}\tilde{\phi}_1^2\partial\tilde{\phi}_1 + \frac{3}{2}\lambda\partial\tilde{\phi}_1) = 0, \tag{4.57}$$

$$\begin{aligned}\widetilde{\mathrm{mKdV}}_5(\tilde{\phi}_1) &= \tilde{\phi}_{1,t_5} - \frac{1}{16}\Big(30\lambda^2\partial\tilde{\phi}_1 + \lambda(10\partial^3\tilde{\phi}_1 - 60\tilde{\phi}_1^2\partial\tilde{\phi}_1) + 30\tilde{\phi}_1^4\partial\tilde{\phi}_1 \\ &\quad -10(\partial\tilde{\phi}_1)^3 - 40\tilde{\phi}_1\partial\tilde{\phi}_1\partial^2\tilde{\phi}_1 - 10\tilde{\phi}_1^2\partial^3\tilde{\phi}_1 + \partial^5\tilde{\phi}_1\Big) = 0,\end{aligned} \tag{4.58}$$

$$\begin{aligned}\widetilde{\mathrm{mKdV}}_7(\tilde{\phi}_1) &= \tilde{\phi}_{1,t_7} - \frac{1}{64}\Big[140\lambda^3\partial\tilde{\phi}_1 + \lambda^2\left(-420\tilde{\phi}_1^2\partial\tilde{\phi}_1 + 70\partial^3\tilde{\phi}_1\right) \\ &\quad +\lambda\left(420\tilde{\phi}_1^4\partial\tilde{\phi}_1 - 140(\partial\tilde{\phi}_1)^3 - 560\tilde{\phi}_1\partial\tilde{\phi}_1\partial^2\tilde{\phi}_1 - 140\tilde{\phi}_1^2\partial^3\tilde{\phi}_1 + 14\partial^5\tilde{\phi}_1\right) \\ &\quad -140\tilde{\phi}_1^6\partial\tilde{\phi}_1 + 420\tilde{\phi}_1^2(\partial\tilde{\phi}_1)^3 + 560\tilde{\phi}_1^3\partial\tilde{\phi}_1\partial^2\tilde{\phi}_1 182\tilde{\partial}\phi_1(\partial^2\tilde{\phi}_1)^2 \\ &\quad +70\tilde{\phi}_1^4\partial^3\tilde{\phi}_1 - 126(\partial\tilde{\phi}_1)^2\partial^3\tilde{\phi}_1 - 140\tilde{\phi}_1\partial^2\tilde{\phi}_1\partial^3\tilde{\phi}_1 \\ &\quad -84\tilde{\phi}_1\partial\tilde{\phi}_1\partial^4\tilde{\phi}_1 - 14\tilde{\phi}_1^2\partial^5\tilde{\phi}_1 + \partial^7\tilde{\phi}_1\Big] = 0.\end{aligned} \tag{4.59}$$

In order to display the λ-dependence of these modified equations we note that q_0 in $P_{r,k}$ is expressed by $q_0 = \lambda + F(\tilde{\phi}_1, \tilde{\phi}_2, ...)$. This λ-dependence comes exclusively from the term $\tilde{P}_{0,r}$ in (4.56) since the explicit λ–dependence cancels in $\tilde{K}_{n,k}$. Thus we get

$$P_{r,k} = \tilde{K}_{n,k}\, e_m(\xi^n + \lambda)^{r/n} \tilde{K}_{n,k}^{-1} = \sum_{l=0}^{\infty} \binom{r/n}{l} \tilde{K}_{n,k}\, e_m \lambda^l \xi^{r-n\,l} \tilde{K}_{n,k}^{-1}. \tag{4.60}$$

Writing $\tilde{P}_{r-nl,k} = \tilde{K}_{n,k}\, e_m \xi^{r-nl} \tilde{K}_{n,k}^{-1}$, allows us to simplify the differential expression

$$(P_{r,k})_+ \;=\; \sum_{l=0}^{nl\le r} \binom{r/n}{l} \lambda^l (\tilde{P}_{r-nl,k})_+. \tag{4.61}$$

Example 4.8. $n = 2, m = 1, r = 1, 3, 5, 7:$

$$(P_{1,k})_+ = (\tilde{P}_{1,k})_+, \tag{4.62}$$
$$(P_{3,k})_+ = (\tilde{P}_{3,k})_+ + \frac{3}{2}\lambda(\tilde{P}_{1,k})_+, \tag{4.63}$$
$$(P_{5,k})_+ = (\tilde{P}_{5,k})_+ + \frac{5}{2}\lambda(\tilde{P}_{3,k})_+ + \frac{15}{8}\lambda^2(\tilde{P}_{1,k})_+, \tag{4.64}$$
$$(P_{7,k})_+ = (\tilde{P}_{7,k})_+ + \frac{7}{2}\lambda(\tilde{P}_{5,k})_+ + \frac{35}{8}\lambda^2(\tilde{P}_{3,k})_+ + \frac{35}{16}\lambda^3(\tilde{P}_{1,k})_+. \tag{4.65}$$

Now we are in position to formulate our main new result concerning the converse of Theorem 4.1.

Theorem 4.9. *Assume Hypotheses 4.3, 4.4, and 4.5, let L_n be defined as in (4.29), $P_{0,r} = e_m\xi^r$, and $P_r = K_n P_{0,r} K_n^{-1}$. Suppose that $(q_0, \dots, q_{n-2})$ satisfies the GD system (4.33)*

$$\mathrm{GD}_{n,r,j}(q_0, \dots, q_{n-2}) = 0, \qquad 0 \le j \le n-2, \tag{4.66}$$

or equivalently, that

$$(\partial_{t_r} - (P_r)_+)\, \psi_{n,k} = \sum_{l=1}^{n} \psi_{n,l}\alpha_{n,k,l}, \qquad 1 \le k \le n-1. \tag{4.67}$$

Define $(\tilde{\phi}_1, \dots, \tilde{\phi}_n)$ by (4.44)–(4.47). Then $(\tilde{\phi}_1, \dots, \tilde{\phi}_n)$ satisfy the DS system (4.52)

$$\widetilde{\mathrm{DS}}_{n,r,k}(\tilde{\phi}_1, \dots, \tilde{\phi}_n) = 0, \qquad 1 \le k \le n \tag{4.68}$$

iff

$$\alpha_{n,h,l} = 0 \quad \text{for} \quad h+1 \le l \le n, \quad 1 \le h \le n-1. \tag{4.69}$$

Proof. We have $(P_{r,1} = P_r)$

$$[(\frac{d}{dt_r} - \mathcal{Q}_r)_+, \tilde{\mathcal{M}}_n] \begin{pmatrix} \psi_{n,1} \\ \tilde{A}_1\psi_{n,2} \\ \vdots \\ \tilde{A}_{n-1}\cdots\tilde{A}_1\psi_{n,n} \end{pmatrix}$$

$$= \Big[\widetilde{\mathrm{DS}}_{n,r,k}(\tilde{\phi}_1, \dots, \tilde{\phi}_n)\delta_{k,(l+1)(\mod n)}(\prod_{m=1}^{k-1}\tilde{A}_m)\psi_{n,k}\Big]_{k,l=1}^n, \tag{4.70}$$

$$\begin{aligned} \widetilde{\mathrm{DS}}_{n,r,1}(\tilde{\phi}_1, \dots, \tilde{\phi}_n)\psi_{n,1} &= \Big((\partial_{t_r} - (P_{r,2})_+)\tilde{A}_1 - \tilde{A}_1(\partial_{t_r} - (P_{r,1})_+)\Big)\psi_{n,1} \\ &= -\tilde{A}_1(\partial_{t_r} - (P_{r,1})_+)\psi_{n,1}, \\ &= -\tilde{A}_1\sum_{l=1}^n \psi_{n,l}\alpha_{n,1,l} = -\sum_{l=2}^n \tilde{A}_1\psi_{n,l}\alpha_{n,1,l}. \end{aligned} \tag{4.71}$$

Thus $\widetilde{\mathrm{DS}}_{n,r,1}(\tilde{\phi}_1, \dots, \tilde{\phi}_n) = 0$ iff $\alpha_{n,1,l} = 0, l \geq 2$.

$$\begin{aligned} \widetilde{\mathrm{DS}}_{n,r,2}(\tilde{\phi}_1, \dots, \tilde{\phi}_n)\tilde{A}_1\psi_{n,2} &= \Big((\partial_{t_r} - (P_{r,3})_+)\tilde{A}_2 - \tilde{A}_2(\partial_{t_r} - (P_{r,2})_+)\Big)\tilde{A}_1\psi_{n,2} \\ &= -\Big(\tilde{A}_2(\partial_{t_r} - (P_{r,2})_+)\Big)\tilde{A}_1\psi_{n,2}, \\ &= -\tilde{A}_2\tilde{A}_1(\partial_{t_r} - (P_{r,1})_+)\psi_{n,2}, \\ &= -\tilde{A}_2\tilde{A}_1\sum_{l=1}^n \psi_{n,l}\alpha_{n,2,l} = -\sum_{l=3}^n \tilde{A}_2\tilde{A}_1\psi_{n,l}\alpha_{n,2,l}, \end{aligned} \tag{4.72}$$

where we used (4.71). Therefore $\widetilde{\mathrm{DS}}_{n,r,2}(\tilde{\phi}_1, \dots, \tilde{\phi}_n) = 0$ iff $\alpha_{n,2,l} = 0, l \geq 3$.
Iterating this process we finally get

$$\begin{aligned} &\widetilde{\mathrm{DS}}_{n,r,n}(\tilde{\phi}_1, \dots, \tilde{\phi}_n)\tilde{A}_{n-1}\cdots\tilde{A}_1\psi_{n,n} \\ &\quad = \Big((\partial_{t_r} - (P_{r,1})_+)\tilde{A}_n - \tilde{A}_n(\partial_{t_r} - (P_{r,n})_+)\Big)\tilde{A}_{n-1}\cdots\tilde{A}_1\psi_{n,n} \\ &\quad = -\tilde{A}_n\dots\tilde{A}_1(\partial_{t_r} - (P_{r,1})_+)\psi_{n,n} = -\tilde{L}_n\sum_{l=1}^n \psi_{n,l}\alpha_{n,k,l} = 0 \end{aligned} \tag{4.73}$$

and hence (4.68) holds iff (4.69) is valid. □

Clearly the case $\lambda = 0$ formally recovers the homogeneous GD and DS systems of Sections 2 and 3.

The auto-Bäcklund transformations of the GD equations are then described in

Corollary 4.10. *In addition to the hypotheses in Theorem 4.9 assume that*

$$\partial_{t_r}\psi_{n,k} = (P_r)_+\psi_{n,k}, \qquad 1 \le k \le n-1 \tag{4.74}$$

instead of (4.36). Then by (4.1), the solution $(\tilde{\phi}_1, \dots \tilde{\phi}_n)$ *constructed in Theorem 4.9 of the* $\widetilde{\mathrm{DS}}_{n,r,k}$ *equations (4.52) yields* $(n-1)$ *further solutions* $(q_{0,k}, \dots, q_{n-2,k})$, $2 \le k \le n$ *of the* $GD_{n,r,j}$ *equations (4.34), i.e.,* $q_{j,k}$ *satisfy*

$$\mathrm{GD}_{n,r,j}(q_{0,k}, \dots, q_{n-2,k}) = 0, \qquad 0 \le j \le n-2, \qquad 2 \le k \le n. \tag{4.75}$$

We might recall at the end that due to Hypothesis 4.3, the first $n-1$ $\widetilde{\mathrm{DS}}_{n,r,k}$ equations, $1 \le k \le n-1$ imply the last $\widetilde{\mathrm{DS}}_{n,r,n}$ equation by the argument in Remark 3.9.

5 Scalar GD and DS Hierarchies

In our final section we provide a detailed analytical treatment of the scalar GD and DS hierarchies. Factorizations of L_n will be used to describe the generalized Miura transformations linking the GD hierarchy and its modified analog, the DS hierarchy. The associated auto–Bäcklund transformations for the GD hierarchy are also studied in detail.

In order to fix the notation we now choose $F = \mathbb{C}$, $m = 1$ and A to be the algebra of C^∞–functions or $\mathbb{R}$ with $\partial = \partial_x$ the corresponding derivation on A.

Hypothesis 2.1, without loss of generality, is then replaced by

Hypothesis 5.1.

$$q_n = 1, \quad q_{n-1} = 0 \tag{5.1}$$

and hence the differential expression L_n is now of the type

$$L_n = \partial_x^n + q_{n-2}(x)\partial_x^{n-2} + \cdots + q_1(x)\partial_x + q_0(x). \tag{5.2}$$

The algebra $B = \mathbb{C}\left[\left\{q_j^{(l)}\right\}_{\substack{l\in\mathbb{N}_0\\ 0\le j\le n-2}}\right]$, the grading (2.13), (2.14), identifying $\xi = \partial_x$, are then defined as in Section 2. Choosing

$$P_r = K_n\partial_x^r K_n^{-1} = L_n^{r/n} = \partial_x^r + [\text{ lower order terms }], \quad r \le \mathbb{N} \tag{5.3}$$

the sequence of nonlinear evolution equations

$$\partial_{P_r} L_n = [(P_r)_+, L_n], \quad r \in \mathbb{N}, \tag{5.4}$$

or equivalently, viewing $q_j = q_j(x, t_r)$ as a function of $(x, t_r) \in \mathbb{R}^2$,

$$\partial_{t_r} q_j = \partial_{P_r} q_j = f_{n,r,j}(q_0, \dots, q_{n-2}), \quad 0 \le j \le n-2, \ r \in \mathbb{N} \tag{5.5}$$

then represents the GD hierarchy [13]. Since P_r is homogeneous of degree r and $\partial_x^l q_j = q_j^{(l)}$ has degree $n + l - j$, the $f_{j,r}$ in (5.5) are differential polynomials in the $q_{j'}$, $0 \le j' \le n-2$ homogeneous of degree $n + r - j$.

We shall abbreviate the system (5.5) also by

$$\mathrm{GD}_{n,r,j}(q_0, \dots, q_{n-2}) = 0, \quad 0 \le j \le n-2, \tag{5.6}$$

where

$$\mathrm{GD}_{n,r,j}(q_0, \dots, q_{n-2}) = \partial_{t_r} q_j - f_{n,r,j}(q_0, \dots, q_{n-2}), \ 0 \le j \le n-2. \tag{5.7}$$

Turning briefly to the DS hierarchy, we define the matrix differential expression

$$\mathcal{M}_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & \partial_x + \phi_n(x), \\ \partial_x + \phi_1(x) & 0 & \ddots & & 0 \\ 0 & \partial_x + \phi_2(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \partial_x + \phi_{n-1}(x) & 0 \end{pmatrix}, \tag{5.8}$$

where, according to Hypothesis 3.1, we now assume

Hypothesis 5.2.

$$q_n = q_n^{1/n} = 1. \tag{5.9}$$

$$\phi_k \in A, \quad 1 \le k \le n \quad \text{and} \quad \sum_{k=1}^{n} \phi_k = 0. \tag{5.10}$$

The algebra $D = \mathbb{C}\left[\left\{\phi_k^{(l)}\right\}_{\substack{l \in \mathbb{N}_0 \\ 1 \le k \le n}}\right]$, the grading (3.4), (3.8), identifying $\xi = \partial_x$, are then defined as in Section 3. Then

$$\mathcal{M}_n^n = \operatorname{diag}(L_{n,1}, \dots, L_{n,n}), \tag{5.11}$$

$$L_{n,k} = A_{k+n-1} \cdots A_{k+1} A_k = \partial_x^n + q_{n-2,k} \partial_x^{n-2} + \cdots + q_{1,k} \partial_x + q_{0,k}, \tag{5.12}$$

$$A_k = \partial_x + \phi_k, \ A_{k+n} = A_k, \quad 1 \le k \le n. \tag{5.13}$$

Since $\deg(\phi_k^{(l)}) = l+1$, the $q_{j,k}$ in (5.12) are differential polynomials in the $\phi_1, \dots, \phi_n$ homogeneous of degree $n-j$. Introducing

$$\mathcal{Q}_r = \operatorname{diag}(L_{n,1}^{r/n}, \dots, L_{n,n}^{r/n}) = \operatorname{diag}(P_{r,1}, \dots, P_{r,n}), \quad r \in \mathbb{N}, \tag{5.14}$$

the sequence of nonlinear evolution equations

$$\frac{d}{dt_r}\mathcal{M}_n = [(\mathcal{Q}_r)_+, \mathcal{M}_n], \quad r \in \mathbb{N}, \tag{5.15}$$

or equivalently, viewing $\phi_k = \phi_k(x, t_r)$ as a function of $(x, t_r) \in \mathbb{R}^2$

$$\partial_{t_r}\phi_k = \partial_{\mathcal{Q}_r}\phi_k = g_{n,k,r}(\phi_1, \dots, \phi_n), \quad 1 \le k \le n, \ r \in \mathbb{N} \tag{5.16}$$

then represents the DS hierarchy [9]. Since $\mathcal{Q}_r$ is homogeneous of degree zero (mod n), $L_{n,k}^{r/n}$ are homogeneous of degree r, and $\phi_k^{(l)}$ are homogeneous of degree $l+1$, the $g_{k,r}$ in (5.16) are differential polynomials in the $\phi_{k'}$, $1 \le k' \le n$ homogeneous of degree $r+1$.
We shall abbreviate the system (5.16) also by

$$\mathrm{DS}_{n,r,k}(\phi_1, \dots, \phi_n) = 0, \quad 1 \le k \le n, \tag{5.17}$$

where

$$\mathrm{DS}_{n,r,k}(\phi_1, \dots, \phi_n) = \partial_{t_r}\phi_k - g_{n,r,k}(\phi_1, \dots, \phi_n), \quad 1 \le k \le n. \tag{5.18}$$

At this point we turn from an algebraic description of the GD and DS hierarchy, which we used to define (5.6) and (5.17), to an analytic one and replace Hypotheses 5.1 and 5.2 as follows: Throughout this section we denote by Ω a simply connected open subset of $\mathbb{R}^2$ unless specified otherwise and consider arbitrary but fixed integers $n \ge 2$ and $r \not\equiv 0$ (mod n). In order to simplify our notation we shall write t instead of t_r in the following. We then assume

Hypothesis 5.3. *Let $q_j, 0 \le j \le n-2$ be complex–valued functions on Ω such that*

$$q_j^{(l)} = \frac{\partial^l q_j}{\partial x^l} \in C^0(\Omega), \quad 0 \le l \le r+j. \tag{5.19}$$

Hypothesis 5.4. *Let ϕ_k, $1 \le k \le n$ be complex–valued functions on Ω such that*

$$\phi_k^{(l)} = \frac{\partial^l \phi_k}{\partial x^l} \in C^0(\Omega), \quad 0 \le l \le n+r-1 \tag{5.20}$$

and

$$\sum_{k=1}^{n} \phi_k(x,t) = 0 \quad \text{on} \quad \Omega. \tag{5.21}$$

Our choice of Ω instead of $\mathbb{R}^2$ in particular, enables us to consider singular solutions (e.g., rational ones) by restricting the attention to those connected components in the (x,t)–plane where the solutions are finite. It also takes into account that even for initial data in the Schwartz space one cannot expect global solutions for the GD and DS equations in general. (For the Boussinesq equation, the case $n = 3, r = 2$, blow-up of the solutions in finite time is described, e.g., in [6] and [19]).

We start our analytical treatment by a careful investigation of differentiability properties of both solutions q_j, ϕ_k of the $\mathrm{GD}_{n,r}$ and $\mathrm{DS}_{n,r}$ equations as well as solutions ψ of $L_n\psi = 0$. By solutions of linear or nonlinear (partial) differential equations we mean classical solutions throughout this section. If ψ is a solution of $L_n\psi = 0$, we define the degree of $\psi^{(l)} = \dfrac{\partial^l\psi}{\partial x^l}$ as

$$\deg(\psi^{(l)}) = l, \quad l \in \mathbb{N}_0. \tag{5.22}$$

Lemma 5.5. *(i). The highest x–derivative of $q_{j'}$ occurring in $f_{n,r,j}$ in (5.5) is of the order $r + j' - j$.*

(ii). The highest x–derivative of $\phi_{k'}$ occurring in $g_{n,r,k}$ in (5.16) is of the order r.

(iii). Expressing $q_{j,k}$, $0 \le j \le n-2$ as a differential polynomial in terms of $\phi_1, \dots, \phi_n$, the highest x–derivative of $\phi_{k'}$ in $q_{j,k}$ is of the order $n - j - 1$.

Proof. The highest x–derivative of $q_{j'}$ in $f_{n,r,j}$, say $q_{j'}^{(l)}$ occurs if $q_{j'}^{(l)}$ is an isolated summand in $f_{n,r,j}$ (possibly multiplied by a constant). Thus

$$\deg(q_{j'}^{(l)}) = n + l - j' = \deg(f_{n,r,j}) = n + r - j \tag{5.23}$$

and hence $l = r + j' - j$. This proves (i). (ii) and (iii) are proved analogously. □

Lemma 5.6. *Let $q_j, 0 \le j \le n-2$ satisfy Hypothesis 5.3 and let ψ_k, $1 \le k \le n$ be a system of solutions of $L_n\psi = 0$. Introduce the Wronskians $W_k = W(\psi_1, \dots, \psi_k)$, $1 \le k \le n$. Then*

$$W_k^{(l)} = \frac{\partial^l W_k}{\partial x^l} \in C^0(\Omega), \quad 0 \le l \le n + r. \tag{5.24}$$

Proof. Since $\psi_{k'}^{(l)}$ has degree l, the Wronskian W_k is homogeneous of degree $k(k-1)/2$ and hence $W_k^{(l)}$ has degree $[k(k-1)/2] + l$, $0 \le l \le n + r$. Any determinant with rows of the type $(\psi_1^{(l)}, \dots, \psi_k^{(l)})$ with degree less than $k(k-1)/2$ vanishes since then at least two rows are equal. Using $L_n\psi_k = 0$, $1 \le k \le n$, one can reduce $W_k^{(l)}$ to a sum of terms consisting of products of differential polynomials of the q_j and $k \times k$ determinants involving only derivatives of the $\psi_{k'}$ up to order $n-1$. Suppose $q_k^{(r+k+m)}$ is among these terms for some $m \ge 1$. Then $[k(k-1)/2] + l = n - j + (r + j + m) + c + d$, where d is the degree of the $k \times k$ determinant and $c \ge 0$ accounts for the degree of other factors which might occur in the product. Thus, if $l \le n+r$, then $d < k(k-1)/2$ (since $m + c \ge 1$) and hence this term vanishes by the previous argument. Hence $W_k^{(l)}$, $0 \le l \le n+r$ is a sum of products of the q_j and their x–derivatives up to the order $r+j$ and the $\psi_{k'}$ and their x–derivatives up to order $n-1$ all of which are continuous on Ω. $\square$

The next two theorems show that under Hypothesis 5.3, the system $L_n\psi = 0$, $\psi_t = (P_r)_+\psi$ simultaneously admits solutions.

Theorem 5.7. *Let R be an open rectangle in $\mathbb{R}^2$, $(x_0, t_0) \in R$. Suppose q_j, $0 \le j \le n-2$ satisfy Hypothesis 5.1 and the Gelfand-Dickey equations $GD_{n,r,j}(q_0, ..., q_{n-2}) = 0$, $0 \le j \le n-2$ on R. Then, for any choice of $(c_1, ..., c_n) \in \mathbb{C}^n$, there exists in R a unique solution $\psi(x,t)$ of the initial value problem*

$$L_n(t)\psi(x,t) = 0, \quad \psi_t(x,t) = (P_r)_+(t)\psi(x,t), \quad (\psi, ..., \psi^{(n-1)})(x_0,t_0) = (c_1, ..., c_n). \tag{5.25}$$

Proof. Suppose $R = (x_1, x_2) \times (t_1, t_2)$ and define in R the $n \times n$–matrix

$$A_n(x,t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -q_0 & -q_1 & \cdots & -q_{n-2} & 0 \end{pmatrix}. \tag{5.26}$$

Next, assuming that $\tilde{\psi}(x,t)$ satisfies,

$$L_n(t)\tilde{\psi}(x,t) = 0, \tag{5.27}$$

we may use $\deg(L_n) = n$, $\deg(P_r) = r$, and $\deg(\tilde{\psi}^{(l)}) = l$ to write

$$\frac{\partial}{\partial x^{k-1}}\Big((P_r)_+(t)\tilde{\psi}(x,t)\Big) = \sum_{k'=1}^{n} B_{r,k,k'}(x,t)\tilde{\psi}^{(k'-1)}(x,t). \tag{5.28}$$

The $B_{r,k,k'}$ are differential polynomials in the q_j homogeneous of degree $r+k-k'$ and they define in R the $n \times n$–matrix

$$B_r(x,t) = [B_{r,k,k'}(x,t)]_{k,k'=1}^{n}. \tag{5.29}$$

Note that both B_r and $B_{r,x}$ are continuous matrices since the highest x–derivative of q_j occurring in B_r is of order $r+j-1$ which can be proven by simple grading arguments like those in the proof of Lemma 5.5. Also A_n and $A_{n,t}$ are continuous since according to (5.5) and Lemma 5.5 the $q_{j,t}$ involve only x–derivatives of $q_{j'}$ up to the order $r+j'-j$. We shall construct C^1-solutions of the system

$$u_x(x,t) = A_n(x,t)u(x,t). \tag{5.30}$$

According to Theorem 7.5 of Chapter 1 in [5] (for the complex-valued case see the remarks in Section 1.8) there exists a unique C^1-solution $u_m(x,t)$ of the system (5.30) under the initial condition $u_{m,l}(x_0,t) = \delta_{m,l}$ in a strip $(x_1,x_2) \times (t_0-\varepsilon, t_0+\varepsilon)$ (here $u_{m,l}$ denotes the l-th component of u_m). Due to the linearity of (5.30) in u and the fact that A_n and $A_{n,t}$ are bounded in any compact subset of the rectangle $R = (x_1,x_2) \times (t_1,t_2)$ we may extend this solution to all of R.

Next note that

$$u(x,t) = \sum_{m=1}^{n} \gamma_m(t) u_m(x,t) \tag{5.31}$$

also solves the system (5.30) and is in $C^1(R)$ if the γ_m are continuously differentiable. We now choose the γ_m such that u satisfies the initial condition given in the theorem, i.e., such that $u(x_0,t_0) = (c_1, ..., c_n)^T$. Let U be the matrix whose m-th column is u_m, i.e., U is the fundamental matrix which is initially (i.e., at $x = x_0$ and for all $t \in (t_1,t_2)$) the identity matrix. Let $\underline{\gamma}(t) = (\gamma_1(t), ..., \gamma_n(t))$ be the unique solution of the linear system of equations

$$\frac{d\underline{\gamma}}{dt}(t) = (B_r U - U_t)(x_0,t)\underline{\gamma}(t) \tag{5.32}$$

under the initial condition $\underline{\gamma}(t_0) = (c_1, ..., c_n)^T$. Then $u \in C^1(R)$ is the unique solution of (5.30) under the initial condition $u(x_0,t_0) = (c_1, ..., c_n)^T$. Moreover, by construction, the function $u(x_0,t)$ satisfies the equation

$$u_t(x_0,t) = B_r(x_0,t)u(x_0,t). \tag{5.33}$$

We now intend to prove that this function $u(x,t)$ also satisfies

$$u_t(x,t) = B_r(x,t)u(x,t). \tag{5.34}$$

Define

$$v(x,t) = u_t(x,t) - B_r(x,t)u(x,t). \tag{5.35}$$

Then $v(x_0,t) = 0$ by (5.33). In addition,

$$u_{x,t}(x,t) = A_{n,t}(x,t)u(x,t) + A_n(x,t)u_t(x,t) \tag{5.36}$$

proves that $u_{x,t} \in C^0(R)$ and hence $u_{x,t} = u_{t,x}$. Thus

$$v_x = u_{t,x} - B_{r,x}u - B_r u_x = (A_{n,t} - B_{r,x} + A_n B_r - B_r A_n)u + Av. \tag{5.37}$$

If we can show that

$$(A_{n,t} - B_{r,x} + [A_n, B_r])u = 0, \tag{5.38}$$

then $v_x = Av$ with $v(x_0,t) = 0$ yields $v(x,t) = 0$ and hence (5.34). Since $u \in C^1(R)$ the equations (5.30) and (5.34) are equivalent to the differential equations in (5.25) upon identifying

$$u(x,t) = (\psi(x,t), \dots, \psi^{(n-1)}(x,t))^T. \tag{5.39}$$

Also u satisfies the given initial condition. It remains to prove (5.38). Define the matrix differential expressions

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ \partial_x^n - L_n & 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{5.40}$$

and, introducing the abbreviation $P = (P_r)_+$,

$$\tilde{B} = \begin{pmatrix} P & 0 & 0 & \cdots & 0 \\ P' & P & 0 & \cdots & 0 \\ P'' & 2P' & P & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ P^{(n-1)} & \binom{n-1}{1}P^{(n-2)} & \binom{n-1}{2}P^{(n-3)} & \cdots & P \end{pmatrix} \tag{5.41}$$

where P' ($P^{(j)}$) denotes the differential expression obtained from $(P_r)_+$ by differentiating its coefficients once (j times) with respect to x. In particular, $P^{(j)}$ is a differential expression of order at most $r-2$. Then $\tilde{A}\Phi = A_n\Phi$ whenever $\Phi = (f, f', \dots, f^{(n-1)})^T$ and $\tilde{B}\Phi = B_r\Phi$ whenever $\Phi = (f, f', \dots, f^{(n-1)})^T$ and $L_n f = 0$. But $\tilde{B}u$ is of the type $(f, f', \dots, f^{(n-1)})^T$ and u is of the type $(\psi, \psi', ..., \psi_{(n-1)})$ with $L_n\psi = 0$ since $u_x = A_n u$. In order to guarantee these properties we first solved the problem (5.30) and then the problem (5.34) instead of going the other way around. We now infer

$$\tilde{A}\tilde{B}u = A_n\tilde{B}u = A_nB_ru \tag{5.42}$$

and

$$\begin{aligned}(\tilde{B}_x + \tilde{B}\tilde{A})u &= \tilde{B}_x u + \tilde{B}u_x = (\tilde{B}u)_x = (B_r u)_x \\ &= B_{r,x}u + B_r u_x = (B_{r,x} + B_r A_n)u. \end{aligned} \tag{5.43}$$

Thus (5.38) is equivalent to

$$(A_{n,t} - \tilde{B}_x + [\tilde{A}, \tilde{B}])u = 0. \tag{5.44}$$

Calculating $-\tilde{B}_x + [\tilde{A}, \tilde{B}]$ yields that only its last row contains nonzero entries. This last row is given by

$$\Big([P, L_n] - [P, \partial_x^n] - P^{(n)}, -nP^{(n-1)}, \dots, -\left(\tbinom{n-1}{j} + \tbinom{n-1}{j-1}\right) P^{(n-j)}, \dots, -nP'\Big). \tag{5.45}$$

Since $\binom{n-1}{j} + \binom{n-1}{j-1} = \binom{n}{j}$, the row (5.45) applied to $u = (\psi, \dots, \psi^{(n-1)})^T$ yields $[(P_r)_+, L_n]\psi$. Since the last component of $A_{n,t}u$ is given by

$$(A_{n,t}u)_n = -\Big(\frac{d}{dt}L_n\Big)\psi, \tag{5.46}$$

we finally infer

$$\begin{aligned} (A_{n,t} - B_{r,x} + [A_n, B_r])u &= (A_{n,t} - \tilde{B}_x + [\tilde{A}, \tilde{B}])u \\ &= \Big(0, \dots, 0, \Big(-\frac{d}{dt}L_n + [(P_r)_+, L_n]\Big)\psi\Big)^T = 0, \end{aligned} \tag{5.47}$$

thus concluding the proof of the theorem. □

We now extend the proof of Theorem 5.7 to arbitrary simply connected open subsets R of $\mathbb{R}^2$.

Theorem 5.8. *Suppose q_j, $0 \le j \le n-2$ satisfy Hypothesis 5.3 and the Gelfand-Dickey equations $GD_{n,r,j}(q_0, ..., q_{n-2}) = 0$, $0 \le j \le n-2$ on Ω. Let $(x_0, t_0) \in R$. Then, for any choice of $(c_1, ..., c_n) \in \mathbb{C}^n$, there exists in Ω a unique solution $\psi(x,t)$ of the initial value problem*

$$L_n(t)\psi(x,t) = 0, \quad \psi_t(x,t) = (P_r)_+(t)\psi(x,t), \quad (\psi, ..., \psi^{(n-1)})(x_0, t_0) = (c_1, ..., c_n). \tag{5.48}$$

Proof. First note that any point $(x,t) \in \Omega$ and the point (x_0, t_0) may be joined by a compact arc contained in Ω. This arc may be covered by a finite number of open rectangles all of which lie entirely in Ω. In particular there is a sequence of open rectangles $R_0, ..., R_N$ such that $(x_0, t_0) \in R_0$, $(x,t) \in R_N$ and $R_{j-1} \cap R_j \neq \emptyset$, $j = 1, ..., N$. This follows since Ω is open and connected. Now we define $u(x,t) = (\psi(x,t), ..., \psi^{(n-1)}(x,t))^T$ for any $(x,t) \in \Omega$ in the following way. According to the last theorem we may define uniquely a function u_0 on R_0 by solving the system of differential equations

$$u_x = A_n u, \quad u_t = B_r u \tag{5.49}$$

under the initial condition $u_0(x_0, t_0) = (c_1, ..., c_n)^T$. Next we define u_1 on R_1 as solution of (5.49) under the initial condition $u_1(x_1, t_1) = u_0(x_1, t_1)$ where $(x_1, t_1) \in R_0 \cap R_1$. Note that due to uniqueness of solutions of initial value problems for (5.49) we infer $u_1|_{R_0 \cap R_1} = u_0|_{R_0 \cap R_1}$ and that the definition of u_1 does not depend on the particular choice of $(x_1, t_1) \in R_0 \cap R_1$. Repeating this construction we define successively the function u_j on R_j and finally u_N on R_N. Now let $u(x,t) = u_N(x,t)$. We only have to prove that this definition does not actually depend on the choice of the rectangles joining (x_0, t_0) and (x,t). This can be done by imitating the proof of the monodromy theorem for the continuation of analytic functions (see, e.g., [26], Theorem 8.5 in Section III.40). We note the requirement that Ω is simply connected enters here. In the following we give an outline of the procedure. Consider two distinct points joined by a curve Λ consisting of line segments parallel to the axes. Any cover of Λ with rectangles will give the same value of u at one end of Λ for a given value of u at the other end. We therefore may use the term "continuation of u along Λ". Suppose now that (x_0, t_0) and (x,t)

are joined by two curves Λ_1 and Λ_2 of the above type. Let $u_1(x,t)$ be the continuation of $u(x_0,t_0)$ along Λ_1 and $u_2(x,t)$ the continuation of $u(x_0,t_0)$ along Λ_2. In contradiction to our claim that $u(x_0,t_0)$ defines $u(x,t)$ uniquely, suppose that $u_1(x,t) \neq u_2(x,t)$. Continuing, therefore, $u_1(x,t)$ along $-\Lambda_2$ (i.e., reversing the direction of Λ_2) we find a value of u in (x_0,t_0) which is different from $u(x_0,t_0)$. Without loss of generality we assume in the following that the curve $\Lambda = \Lambda_1 - \Lambda_2$ is not self-intersecting (except that the initial and the final point coincide). Since Ω is simply connected the polygon described by this curve is a subset of Ω. We now divide the closure of this polygon into a finite number of closed rectangles which intersect only in their edges. Consider now two vertices of these rectangles, (x_1,t_1) and (x_2,t_2), which are joined by a part of Λ as well as by some edges lying inside Λ (call this part Λ^*). Continuing now $u(x_1,t_1)$ along Λ and along Λ^* we obtain two values of u in (x_2,t_2) which may or may not coincide. In the former case we replace in the following discussion the part of Λ joining (x_1,t_1) and (x_2,t_2) by Λ^* thereby obtaining a new smaller polygon (whose boundary is in the following also denoted by Λ). In the latter case the object of our following considerations is the polygon whose boundary is the part of Λ joining (x_1,t_1) and (x_2,t_2) and the curve $-\Lambda^*$. In any case we have obtained a smaller polygon with the property that continuation of u from a point (x,t) along the boundary back to (x,t) does not coincide with the initial value. We may repeat this process until we have obtained just a single rectangle as our polygon having the property that continuing u from one corner to the corner diagonal to it gives two different results as one goes in positive or negative direction. This, however, is impossible, since the value of u at one point determines u everywhere in the rectangle uniquely. Thus our assumption that u is somewhere not uniquely defined leads to a contradiction and thus the theorem is established. $\square$

Remark 5.9. Since ψ, and hence u, in (5.47) satisfies arbitrary initial values at arbitrary $(x_0,t_0) \in \Omega$, we actually infer the so called zero–curvature representation of the $\mathrm{GD}_{n,r}$ equations in the form

$$A_{n,t} - B_{r,x} + [A_n, B_r] = 0. \tag{5.50}$$

Lemma 5.10. *Suppose q_j, $0 \le j \le n-2$ satisfy Hypothesis 5.3 and $\mathrm{GD}_{n,r,j}(q_0,\dots,q_{n-2}) = 0$, $0 \le j \le n-2$. Let $\psi_1,\dots,\psi_n$ be a system of solutions of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$. Then the Wronskian $W(\psi_1,\dots,\psi_n)$ is constant (possibly zero) in Ω.*

Proof. Denoting $W(\psi_1,\dots,\psi_n)$ by W_n, we have

$$W_{n,x} = \operatorname{tr}(A_n)W_n = 0 \tag{5.51}$$

and

$$W_{n,t} = \operatorname{tr}(B_r)W_n. \tag{5.52}$$

By (5.50),

$$\operatorname{tr}(B_{r,x}) = \partial_x \operatorname{tr}(B_r) = 0 \tag{5.53}$$

and hence $\operatorname{tr}(B_r(x,t)) = f(t)$, is independent of x. On the other hand, since each $B_{r,k,k}(x,t)$ is homogeneous of degree $r \geq 1$, so is

$$\operatorname{tr}(B_r(x,t)) = \sum_{k=1}^{n} B_{r,k,k}(x,t) = f(t). \tag{5.54}$$

Since $f(t)$, being constant in x, has degree zero, $f(t)$ must vanish implying $\operatorname{tr}(B_r) = 0$ and hence $W_{n,t} = 0$ by (5.52). $\square$

Next let $\psi_1, \ldots, \psi_n$ be a fundamental system of $L_n\psi = 0$. In order to factorize L_n as in (5.12) we define

$$W_0 = 1,\ W_k = W(\psi_1, \ldots, \psi_k), \quad 1 \leq k \leq n \tag{5.55}$$

and

$$\phi_k = (\ln[W_{k-1}/W_k])_x, \quad 1 \leq k \leq n \tag{5.56}$$

whenever the latter is nonsingular. (5.56) is well known to yield the factorization (see e.g. [29], p. 108)

$$L_n = (\partial_x + \phi_n) \cdots (\partial_x + \phi_2)(\partial_x + \phi_1) \tag{5.57}$$

and clearly

$$\sum_{k=1}^{n} \phi_k = (\ln[W_0/W_n])_x = 0. \tag{5.58}$$

For factorizations in the special cases $n = 2$ and $n = 3$ see, e.g., [16] , [28] and [10], [11], [15] and the references therein. The general case $n \geq 2$ is discussed e.g. in [2], [4], [10], [18], [23], [29], [33], [36].

We shall prove in Theorem 5.13 that if in addition to $L_n\psi = 0$ the ψ_k satisfy $\psi_t = (P_r)_+\psi$ then $\phi_1, \ldots, \phi_n$ satisfy the $\mathrm{DS}_{n,r}$ equations (5.17). But first we prove

Lemma 5.11. *Suppose q_j, $0 \leq j \leq n-2$ satisfy Hypothesis 5.3 and $\mathrm{GD}_{n,r,j}(q_0, \ldots, q_{n-2}) = 0$, $0 \leq j \leq n-2$. Assume that $\psi_1, \ldots, \psi_n$ is a system of solutions of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$. Let Ω_0 be any open connected subset of Ω such that the Wronskians $W(\psi_1, \ldots, \psi_k)$ are different from zero on Ω_0 for $k = 1, ..., n$. Then the ϕ_k defined in (5.56) satisfy*

$$\phi_k^{(l)} \in C_0(\Omega_0), \quad 1 \leq k \leq n, \quad 0 \leq l \leq n + r - 1. \tag{5.59}$$

Moreover, given the ϕ_k, define $q_{j,k}$ as in (5.12). Then

$$q_{j,k}^{(l)} \in C^0(\Omega_0), \quad 0 \leq l \leq r + k, \quad 0 \leq j \leq n-2, \quad 1 \leq k \leq n. \tag{5.60}$$

In other words the ϕ_k satisfy Hypothesis 5.4 and the $q_{j,k}$ satisfy Hypothesis 5.3 on the set Ω_0.

Proof. The ϕ_k, $k = 1, ..., n$ are well defined in Ω_0. Equation (5.59) follows from Lemma 5.6. The $q_{k,j}$ involve x–derivatives of $\phi_1, \dots, \phi_n$ up to the order $n - j - 1$ by Lemma 5.5. This proves (5.60). □

An analytic version of Theorem 4.1 then reads

Theorem 5.12. *Suppose ϕ_k, $1 \le k \le n$ satisfy Hypothesis 5.4 and* $\mathrm{DS}_{n,r,k}(\phi_1, \dots, \phi_n) = 0$, $1 \le k \le n$. *Define $q_{j,k}$ by (5.12). Then the $q_{j,k}$ satisfy Hypothesis 5.3 and*

$$\mathrm{GD}_{n,r,j}(q_{0,k}, \dots, q_{n-2,k}) = 0, \quad 0 \le j \le n-2, \quad 1 \le k \le n. \tag{5.61}$$

Proof. Combine Lemma 5.11 and the proof of the Theorem 4.1. □

Next we shall derive our main result which amounts to a converse of Theorem 5.12. Given a solution $(q_0, ..., q_{n-2}) = (q_{1,0}, ..., q_{1,n-2})$ of the $\mathrm{GD}_{n,r}$ equations, we shall construct a solution $(\phi_1, ..., \phi_n)$ of the $\mathrm{DS}_{n,r}$ equations (5.17) and $n-1$ further solutions $(q_{0,k}, ..., q_{n-2,k})$ for $2 \le k \le n$ of the $\mathrm{GD}_{n,r}$ equations (5.6) which are all linked to the ϕ_k by the generalized Miura transformations contained in (5.12), (5.13).

Theorem 5.13. *Suppose q_j, $0 \le j \le n-2$ satisfy Hypothesis 5.3 and $q_{j,t} \in C^0(\Omega)$, $0 \le j \le n-2$. Let $\psi_1(x,t), \dots, \psi_n(x,t)$ be a fundamental system of solutions $L_n(t)\psi(x,t) = 0$ in Ω considering t as a parameter. Let Ω_0 be any open subset of Ω such that $W_k = W(\psi_1, ..., \psi_k) \neq 0$ in Ω_0 for $k = 1, ..., n-1$. Define ϕ_k, $1 \le k \le n$ according to (5.56) on Ω_0. Then*

$$L_n = A_n \cdots A_2 A_1, \tag{5.62}$$

where

$$A_k = \partial_x + \phi_k, \quad 1 \le k \le n. \tag{5.63}$$

Moreover, $(q_0, ..., q_{n-2})$ satisfies the equations $\mathrm{GD}_{n,r,j}(q_0, \dots, q_{n-2}) = 0, 0 \le j \le n-2$ *on Ω iff*

$$L_n(\partial_t - (P_r)_+)\psi_k = 0, \quad 1 \le k \le n-1, \tag{5.64}$$

on Ω or, equivalently, iff

$$(\partial_t - (P_r)_+)\psi_k = \sum_{l=1}^{n} \alpha_{k,l}\psi_l, \quad 1 \le k \le n-1, \tag{5.65}$$

on Ω, where $\alpha_{k,l}$ are (in general t–dependent) constants. Finally, assuming $GD_{n,r,j}(q_0, ..., q_{n-2}) = 0$, $0 \le j \le n-2$, we find that $(\phi_1, \dots, \phi_n)$ satisfies the equations $\mathrm{DS}_{n,r,k}(\phi_1, \dots, \phi_n) = 0$, $1 \le k \le n$ *on Ω_0 iff*

$$\alpha_{h,l} = 0, \quad h+1 \le l \le n, \quad 1 \le h \le n-1. \tag{5.66}$$

Proof. (5.62) has been discussed in (5.55)–(5.58). Since $L_n\psi_k = 0$ yields $(\frac{d}{dt}L_n)\psi_k = -L_n\psi_{k,t}$ equation (5.7) implies

$$\Big(\frac{d}{dt}L_n\Big)\psi_k - [(P_r)_+, L_n]\psi_k = -L_n(\partial_t - (P_r)_+)\psi_k = \sum_{j=0}^{n-2} \mathrm{GD}_{n,r,j}(q_0, \dots, q_{n-2})\psi_k^{(j)}. \tag{5.67}$$

Thus $\mathrm{GD}_{n,r,j}(q_0, \dots, q_{n-2}) = 0$, $0 \le j \le n-2$ implies $L_n(\partial_t - (P_r)_+)\psi_k = 0$, $1 \le k \le n-1$. Conversely, if $L_n(\partial_t - (P_r)_+)\psi_k = 0, 1 \le k \le n-1$ then (5.67) for $k = 1, ..., n-1$ represents a homogeneous system of equations for $\mathrm{GD}_{n,r,j}$. The determinant of the matrix associated to this system is $W_{n-1}(x,t)$. Hence

$$\mathrm{GD}_{n,r,j}(q_0, \dots, q_{n-2}) = 0, \quad 0 \le j \le n-2 \tag{5.68}$$

on the set where $W_{n-1}(x,t) \neq 0$. Next we show that W_{n-1} has only simple zeros as a function of x (if any). Suppose that

$$W_{n-1}(x_0, t_0) = (\frac{\partial W_{n-1}}{\partial x})(x_0, t_0) = 0. \tag{5.69}$$

By hypothesis we have

$$0 \neq W_n(\psi_1, \dots, \psi_n)(x_0, t_0) = \sum_{j=0}^{n-1} (-1)^{n+1+j}\psi_n^{(j)}(x_0, t_0) \begin{vmatrix} \psi_1 & \cdots & \psi_{n-1} \\ \vdots & & \vdots \\ \psi_1^{(j-1)} & \cdots & \psi_{n-1}^{(j-1)} \\ \psi_1^{(j+1)} & \cdots & \psi_{n-1}^{(j+1)} \\ \vdots & & \vdots \\ \psi_1^{(n-1)} & \cdots & \psi_{n-1}^{(n-1)} \end{vmatrix} \tag{5.70}$$

Since

$$\frac{\partial W_{n-1}}{\partial x}(x_0, t_0) = \begin{vmatrix} \psi_1 & \cdots & \psi_{n-1} \\ \vdots & & \vdots \\ \psi_1^{(n-3)} & \cdots & \psi_{n-1}^{(n-3)} \\ \psi_1^{(n-1)} & \cdots & \psi_{n-1}^{(n-1)} \end{vmatrix} = 0, \tag{5.71}$$

we may replace the last line in (5.70) by

$$\Big(\sum_{j=0}^{n-3} c_j\psi_1^{(j)}, \dots, \sum_{j=0}^{n-3} c_j\psi_{n-1}^{(j)}\Big). \tag{5.72}$$

Inserting (5.72) into (5.70) yields

$$\begin{aligned}
0 \;\neq\; & \sum_{j=0}^{n-3}(-1)^{n+1+j}\psi_n^{(j)}(x_0,t_0)\begin{vmatrix} \psi_1 & \cdots & \psi_{n-1} \\ \vdots & & \vdots \\ \psi_1^{(j-1)} & \cdots & \psi_{n-1}^{(j-1)} \\ \psi_1^{(j+1)} & \cdots & \psi_{n-1}^{(j+1)} \\ \vdots & & \vdots \\ \psi_1^{(n-2)} & \cdots & \psi_{n-1}^{(n-2)} \\ \psi_1^{(j)} & \cdots & \psi_{n-1}^{(j)} \end{vmatrix} \\
& -\psi_n^{(n-2)}(x_0,t_0)(\frac{\partial W_{n-1}}{\partial x})(x_0,t_0) + \psi_n^{(n-1)}(x_0,t_0)W_{n-1}(x_0,t_0) \\
& = -\sum_{j=0}^{n-3}\psi_n^{(j)}(x_0,t_0)W_{n-1}(x_0,t_0) \\
& -\psi_n^{(n-2)}(x_0,t_0)(\frac{\partial W_{n-1}}{\partial x})(x_0,t_0) + \psi_n^{(n-1)}(x_0,t_0)W_{n-1}(x_0,t_0). \qquad (5.73)
\end{aligned}$$

Since the right–hand–side of (5.73) equals zero by assumption (5.69), this contradiction proves that $W_{n-1}(x,t)$ has only simple zeros for fixed $t \in \mathbb{R}$. Thus (5.68) extends to all of Ω by continuity of the functions $\mathrm{GD}_{n,r,j}(q_0,\ldots,q_{n-2})(x,t)$ viewed as functions of x. We also remark that therefore

$$L_n(\partial_t - (P_r)_+)\psi_n = -\sum_{j=0}^{n-2}\mathrm{GD}_{n,r,j}(q_0,\ldots,q_{n-2})\psi_n^{(j)} = 0 \qquad (5.74)$$

on Ω, i.e.,

$$L_n(\partial_t - (P_r)_+)\psi_k = 0, \quad 1 \le k \le n \qquad \text{on} \quad \Omega. \qquad (5.75)$$

In order to prove (5.66) we consider

$$\begin{aligned}
& \Big\{\frac{d}{dt}\mathcal{M}_n - [(\mathcal{Q}_r)_+,\mathcal{M}_n]\Big\}(\psi_1, A_1\psi_2,\ldots,A_{n-1}\ldots A_2A_1\psi_n)^T \\
& \quad = (\mathrm{DS}_{n,r,n}(\phi_1,\ldots,\phi_n)A_{n-1}\cdots A_2A_1\psi_n,\ \mathrm{DS}_{n,r,1}(\phi_1,\ldots,\phi_n)\psi_1,\ \mathrm{DS}_{n,r,2}(\phi_1,\ldots,\phi_n)A_1\psi_2, \\
& \quad \cdots, \mathrm{DS}_{n,r,n-1}(\phi_1,\ldots,\phi_n)A_{n-2}\cdots A_2A_1\psi_{n-1})^T, \qquad (5.76)
\end{aligned}$$

where we used

$$\begin{aligned}
& \frac{d}{dt}\mathcal{M}_n - \Big[(\mathcal{Q}_r)_+,\mathcal{M}_n\Big] \\
& = \begin{pmatrix} 0 & 0 & \cdots & 0 & \mathrm{DS}_{n,r,n}(\phi_1,\ldots,\phi_n) \\ \mathrm{DS}_{n,r,1}(\phi_1,\ldots,\phi_n) & 0 & & & 0 \\ 0 & \mathrm{DS}_{n,r,2}(\phi_1,\ldots,\phi_n) & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathrm{DS}_{n,r,n-1}(\phi_1,\ldots,\phi_n) & 0 \end{pmatrix}
\end{aligned}$$

$$(5.77)$$

with $\mathcal{M}_n$ and $\mathcal{Q}_r$ defined in (5.8) and (5.14). (Here we are using the notation employed in (5.12) and (5.14) with $L_n = L_{n,1}$, $P_r = P_{r,1}$.) Using

$$\begin{aligned} \mathrm{DS}_{n,r,k}(\phi_1,\dots,\phi_n) &= \phi_t - (P_{r,k+1})_+ A_k + A_k (P_{r,k})_+ \\ &= \Big(\partial_t - (P_{r,k+1})_+\Big) A_k - A_k \Big(\partial_t - (P_{r,k})_+\Big), \quad 1 \le k \le n \end{aligned} \tag{5.78}$$

(we recall $A_{k+n} = A_k$, $1 \le k \le n$) and

$$A_1 \psi_1 = 0 \tag{5.79}$$

(cf. (5.55), (5.56), and (5.65)), one computes

$$\begin{aligned} \mathrm{DS}_{n,r,1}(\phi_1,\dots,\phi_n)\psi_1 &= (\partial_t - (P_{r,2})_+) A_1 \psi_1 - A_1(\partial_t - (P_{r,1})_+)\psi_1 \\ &= -A_1 \sum_{k=2}^{n} \alpha_{1,k} \psi_k \end{aligned} \tag{5.80}$$

and hence

$$\mathrm{DS}_{n,r,1}(\phi_1,\dots,\phi_n) = 0 \text{ iff } \alpha_{1,k} = 0, \quad 2 \le k \le n \tag{5.81}$$

since $A_1\psi_k$, $2 \le k \le n$, are linearly independent by construction. Next, assuming $\alpha_{1,k} = 0$, $2 \le k \le n$, which implies (see (5.78))

$$\Big(\partial_t - (P_{r,2})_+\Big) A_1 = A_1 \Big(\partial_t - (P_{r,1})_+\Big), \tag{5.82}$$

one computes from (5.65), (5.78), (5.79), and

$$A_2 A_1 \psi_2 = 0 \tag{5.83}$$

(cf. (5.55), (5.56)) that

$$\begin{aligned} \mathrm{DS}_{n,r,2}(\phi_1,\dots,\phi_n) A_1\psi_2 &= \Big(\partial_t - (P_{r,3})_+\Big) A_2 A_1 \psi_2 - A_2 \Big(\partial_t - (P_{r,2})_+\Big) A_1 \psi_2 \\ = -A_2 \Big(\partial_t - (P_{r,2})_+\Big) A_1 \psi_2 &= -A_2 A_1 \Big(\partial_t - (P_{r,1})_+\Big)\psi_2 = -A_2 A_1 \sum_{k=3}^{n} \alpha_{2,k}\psi_k. \end{aligned} \tag{5.84}$$

Thus

$$\mathrm{DS}_{n,r,2}(\phi_1,\dots,\phi_n) = 0 \text{ iff } \alpha_{2,k} = 0, \quad 3 \le k \le n \tag{5.85}$$

since $A_2A_1\psi_k$, $3 \le k \le n$ are linearly independent. An iteration of this argument, assuming $\alpha_{h,l} = 0$ for $h+1 \le l \le n$ and $1 \le h \le n-1$, which implies

$$\mathrm{DS}_{n,r,k}(\phi_1, \dots, \phi_n) = \Big(\partial_t - (P_{r,k+1})_+\Big)A_k - A_k\Big(\partial_t - (P_{r,k})_+\Big) = 0, \quad 1 \le k \le n-1, \tag{5.86}$$

finally yields

$$\mathrm{DS}_{n,r,n}(\phi_1, \dots, \phi_n)A_{n-1}\cdots A_2A_1\psi_n = -L_{n,1}\Big(\partial_t - (P_{r,1})_+\Big)\psi_n = 0 \tag{5.87}$$

since $\frac{d}{dt}L_{n,1} = \Big[L_{n,1}, (P_{r,1})_+\Big]$ implies $L_{n,1}\Big(\partial_t - (P_{r,1})_+\Big)\psi = 0$ for any solution ψ of $L_{n,1}\psi = 0$. $\square$

The following result details the auto–Bäcklund transformations for the $\mathrm{GD}_{n,r}$ equations in terms of factorizations of L_n.

Corollary 5.14. *In addition to the hypotheses in Theorem 5.13 assume that $\psi_{k,t} = (P_r)_+\psi_k$ for $1 \le k \le n-1$ (instead of (5.64)). Then, by (5.12) and (5.13), the solution $(\phi_1, \dots, \phi_n)$ of the $\mathrm{DS}_{n,r}$ equations (5.17) constructed in Theorem 5.13 yields $(n-1)$ further solutions $(q_{0,k}, \dots, q_{n-2,k})$, $2 \le k \le n$ of the $\mathrm{GD}_{n,r}$ equations (5.6), i.e., $q_{j,k}$ satisfy Hypothesis 5.3 and*

$$\mathrm{GD}_{n,r,j}(q_{0,k}, \dots, q_{n-2,k}) = 0, \quad 0 \le j \le n-2, \quad 2 \le k \le n. \tag{5.88}$$

Remark 5.15. As shown in Remark 3.9, the $\mathrm{DS}_{n,r,k}$ equations for $k = 1, ..., n-1$ imply the $\mathrm{DS}_{n,r,n}$ equation, i.e.,

$$\mathrm{DS}_{n,r,k}(\phi_1, \dots, \phi_n) = 0, \quad 1 \le k \le n-1 \tag{5.89}$$

implies

$$\mathrm{DS}_{n,r,n}(\phi_1, \dots, \phi_n) = 0. \tag{5.90}$$

Remark 5.16 To know $n-1$ linearly independent solutions of $L_n(t)\psi = 0$ is, in fact, sufficient in the Hypothesis of Theorem 5.13 according to the well-known fact that another one may always be obtained by a quadrature (see e.g. [18], p.122–123). In fact, integrating $W_n = W(\psi_1, \dots, \psi_n) = C$ one obtains

$$\psi_n = \sum_{k=1}^{n-1}(c_k + \Phi_k)\psi_k, \tag{5.91}$$

where

$$\Phi_k = (-1)^{n+k+1}C\int^x dx' W_{n-1}(x')^{-2}W(\psi_1, \dots, \psi_{k-1}, \psi_{k+1}, \dots, \psi_{n-1})(x'), \tag{5.92}$$

C, c_k are (in general t–dependent) constants, and $W_{n-1} = W(\psi_1, \dots, \psi_{n-1})$.

Remark 5.17. The final part of the proof of Theorem 5.13 following identity (5.78) can be slightly modified. Let Ψ_k be the solutions of

$$A_k \Psi_k = 0, \qquad 1 \leq k \leq n-1, \tag{5.93}$$

i.e.,

$$\Psi_k = \frac{W_k}{W_{k-1}}, \quad \text{or equivalently,} \quad \Psi_k = A_{k-1} \cdots A_1 \psi_k, \qquad 1 \leq k \leq n-1. \tag{5.94}$$

Then (see (5.56))

$$\phi_k = -\partial_x \ln \Psi_k \tag{5.95}$$

and

$$\mathrm{DS}_{n,r,k}\left(-\partial_x \ln \Psi_1, \dots, -\partial_x \ln \Psi_{n-1}, \partial_x \sum_{l=1}^{n-1} \ln \Psi_l\right) = -\partial_x \left(\frac{\Psi_{k,t_r} - (P_{r,k})_+ \Psi_k}{\Psi_k}\right), \quad 1 \leq k \leq n-1, \tag{5.96}$$

where the coefficients of $(P_{r,k})_+$ are expressed in terms of $\Psi_k, 1 \leq k \leq n-1$. Identity (5.96), which is of interest of its own, follows by multiplying (5.78) from the right with Ψ_k and from the left with Ψ_k^{-1}. This immediately yields

$$\begin{aligned}
&-\partial_x \left(\frac{\Psi_{k,t_r} - (P_{r,k})_+ \Psi_k}{\Psi_k}\right) = -\partial_x \left(\frac{(\partial_{t_r} - (P_{r,k})_+)\, A_{k-1} \cdots A_1 \psi_k}{A_{k-1} \cdots A_1 \psi_k}\right) \\
&= -\partial_x \left(\frac{A_{k-1} \cdots A_1\, (\partial_{t_r} - (P_{r,1})_+)\, \psi_k}{A_{k-1} \cdots A_1 \psi_k}\right) \\
&= -\partial_x \left((A_{k-1} \cdots A_1 \psi_k)^{-1} A_{k-1} \cdots A_1 \sum_{l=1}^{n} \alpha_{k,l} \psi_l\right) \\
&= -\partial_x \left(\alpha_{k,k} + (A_{k-1} \cdots A_1 \psi_k)^{-1} \sum_{l=k+1}^{n} \alpha_{k,l} (A_{k-1} \cdots A_1) \psi_l\right) = 0, \quad 1 \leq k \leq n-1.
\end{aligned} \tag{5.97}$$

We emphasize that despite the simplicity of the proofs in Theorem 5.13 (and Remark 5.17), the results obtained are valid under extremely general conditions on the coefficients q_j. In particular, in contrast to other possible approaches based on bi–Hamiltonian structures or inverse scattering techniques [4], [7], [31], we do not require (almost) periodicity or decay conditions on the coefficients q_j as $|x| \to \infty$.

Finally, we further clarify the relation between solutions $(\phi_1, \dots, \phi_n)$ of the $\mathrm{DS}_{n,r}$ equations and factorizations of L_n. Clearly a fundamental system of solutions of

$$L_n\psi = 0, \quad (P_r)_+\psi = \psi_t \tag{5.98}$$

uniquely defines a factorization of the differential expressions $L_n(t)$ for any value of t (provided $[L_n, P_r] = 0$). In our final theorem we show that the knowledge of a factorization of $L_n(t)$ also uniquely defines an equivalence class of fundamental systems of solutions of (5.98) corresponding to a given factorization. The significance of this statement is that it shows that one only needs to consider solutions of the system (5.98) in order to get all possible auto–Bäcklund transformations for the Lax pair (P_r, L_n) our approach might produce, since one obtains all possible factorizations of L_n associated with the given P_r.

Definition 5.18. *Two fundamental systems of solutions of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$ are called equivalent if and only if they define the same set of functions $(\phi_1, \dots, \phi_n)$ in (5.56).*

It is easily seen that this notion of equivalence defines an equivalence relation on the set of fundamental systems of solutions of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$. The system of solutions obtained by replacing any ψ_k by the sum of linear combinations of the $\psi_1, \dots, \psi_{k-1}$ (with t–independent coefficients) and a nonzero (t–independent) multiple of ψ_k represents the same equivalence class as $\psi_1, \dots, \psi_{n-1}$. Because of Remark 5.16 a choice of $\psi_1, \dots, \psi_{n-1}$ already characterizes an equivalence class.

Theorem 5.19. *Suppose q_j, $0 \le j \le n-2$ satisfy Hypothesis 5.3 and the $GD_{n,r}$ equations. Consider solutions $\psi_1, \dots, \psi_{n-1}$ of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$ and factorizations of L_n of the form*

$$L_n = (\partial_x + \phi_n) \cdots (\partial_x + \phi_2)(\partial_x + \phi_1) \tag{5.99}$$

with

$$\sum_{k=1}^{n} \phi_k = 0 \tag{5.100}$$

as constructed in (5.55)–(5.58). Then, on any open set $\Omega_0 \subseteq \Omega$ where $W_k = W(\psi_1, \dots, \psi_k) \neq 0, 1 \le k \le n-1$ the functions ϕ_k in (5.99) are well defined, satisfy Hypothesis 5.4 and the $DS_{n,r}$ equations. Conversely, on any open set $\Omega_0 \subseteq \Omega$ where a factorization of L_n is given by (5.99) with functions ϕ_k satisfying Hypothesis 5.4 and the $DS_{n,r}$ equations on Ω_0 there is an equivalence class of fundamental systems of solutions of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$ associated with the given factorization.

In other words, on Ω_0 there is a one–to–one correspondence between equivalence classes of solutions of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$ and factorizations of L_n of the form (5.99) such that $(\phi_1, \dots, \phi_n)$ satisfies Hypothesis 5.4 and the $\mathrm{DS}_{n,r}$ *equations.*

Proof. By definition, the mapping from the equivalence classes of fundamental systems of solutions of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$ to factorizations (5.99) of L_n given by (5.55)–(5.58) is injective. The functions ϕ_k defined this way satisfy Hypothesis 5.4 according to Lemma 5.11 and the $\mathrm{DS}_{n,r}$ equations according to Theorem 5.13. We shall prove that this mapping is also surjective. Specifically, suppose a given solution $(\phi_1, \dots, \phi_n)$ of the $\mathrm{DS}_{n,r}$ equations satisfies Hypothesis 5.4 on a set Ω_0. Then there exists an equivalence class of linearly independent functions $\psi_1, \dots, \psi_{n-1}$ which solve $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$ on Ω_0. Moreover, using (5.55)–(5.58), they yield the factorization (5.99) with the ϕ_k being the given solution of the $\mathrm{DS}_{n,r}$ equations. Choose $(x_0, t_0) \in \Omega_0$. Let $W_0 = 1$ and define recursively on any open rectangle $R = (x_1, x_2) \times (t_1, t_2) \subseteq \Omega_0$ containing (x_0, t_0)

$$W_k(x,t) = W_{k-1}(x,t)a_k(t) \exp\Big[- \int_{x_0}^{x} dx' \phi_k(x',t)\Big], \tag{5.101}$$

where a_k is the unique solution of the first order linear ordinary differential equation

$$\begin{aligned}\frac{d}{dt}a_k(t) \quad &= \quad a_k(t)\Big\{ \int_{x_0}^{x} dx' \phi_{k,t}(x',t) \\ &\quad + \exp\Big[\int_{x_0}^{x} dx'\phi_k(x',t)\Big]\Big((P_{r,k})_+ \exp\Big[-\int_{x_0}^{x} dx''\phi_k(x'',t)\Big]\Big)\Big\}\end{aligned} \tag{5.102}$$

under the initial condition $a_k(t_0) = 1$. Equation (5.102) is well defined since the bracket $\{\cdots\}$ does not depend on x. In fact, computing $\partial_x\{\cdots\}$, one infers

$$\partial_x\{\dots\} = \phi_{k,t}(x,t) + \exp\Big[\int_{x_0}^{x} dx'\phi_k(x',t)\Big]\Big(A_k(P_{r,k})_+ \exp\Big[-\int_{x_0}^{x} dx''\phi_k(x'',t)\Big]\Big). \tag{5.103}$$

Since by the $\mathrm{DS}_{n,r}$ equations (see (5.78)) $\phi_{k,t} = (P_{r,k+1})_+A_k - A_k(P_{r,k})_+$ and since

$$A_k \exp\Big(-\int_{x_0}^{x} dx''\phi_k(x'',t)\Big) = 0, \tag{5.104}$$

we find that (5.103) indeed equals zero. Thus we have defined functions W_k which do not vanish in R. Moreover, we note that $W_k^{(l)} \in C^0(R), 0 \le l \le n+r$ and $W_{k,t}^{(m)} \in C^0(R), 0 \le m \le n$. Next we put

$$\psi_1 = \tilde{\psi}_1 = W_1 \tag{5.105}$$

and remark that $\tilde{\psi}_1$ is an element of the set

$$S = \{\psi |\ \psi^{(l)} \in C^0(R),\ 0 \le l \le n+r, \quad \psi_t^{(m)} \in C^0(R),\ 0 \le m \le n\}. \tag{5.106}$$

Next, suppose that we have defined $\tilde{\psi}_2, \dots, \tilde{\psi}_{k-1} \in S$. Then

$$W(\tilde{\psi}_1, \dots, \tilde{\psi}_{k-1}, \psi) = W_k \tag{5.107}$$

is a linear inhomogeneous ordinary differential equation for ψ of order $k-1$ with t as a parameter. By Theorem 7.5 in Section 1.7 of [5] (and its generalization in Section 1.8) there exists for the associated first order system a unique C^1-solution $(\tilde{\psi}_k, ..., \tilde{\psi}_k^{(k-2)})$ under the initial condition $(\tilde{\psi}(x_0,t), ..., \tilde{\psi}^{(k-2)}(x_0,t)) = (1,0,...,0)$ on a strip $(x_1,x_2)\times(t_0-\delta, t_0+\delta)$ for some $\delta > 0$. Due to the linearity of the equation in ψ and to the fact that the coefficients of the equation and their t-derivatives are bounded on any compact subset of the rectangle R, we can in fact continue the solution into all of R. Using the differentiability properties of W_k and of the functions $\tilde{\psi}_j$ one actually infers that $\tilde{\psi}_k \in S$. We remark that this choice of the initial condition is for convenience only since the general solution of equation (5.107) is given by the sum of $\tilde{\psi}_k$ and a linear combination (with t-dependent coefficients) of $\tilde{\psi}_1$, ..., $\tilde{\psi}_{k-1}$. This process determines a linearly independent system of functions $\tilde{\psi}_1, \ldots, \tilde{\psi}_{n-1}$ on R. From (5.105) one gets

$$A_1\psi = W(\tilde{\psi}_1, \psi)/W(\tilde{\psi}_1) \tag{5.108}$$

for all sufficiently smooth ψ. We prove by induction on k that for $k = 1, ..., n$

$$A_k \cdots A_2 A_1 = W(\tilde{\psi}_1, \ldots, \tilde{\psi}_k, \cdot)/W(\tilde{\psi}_1, \ldots, \tilde{\psi}_k). \tag{5.109}$$

Both differential expressions are of the same order with leading coefficients 1. Suppose (5.109) holds for $k-1$. Then both differential expressions have $\tilde{\psi}_1, \ldots, \tilde{\psi}_{k-1}$ in their kernels. $\tilde{\psi}_k$ is of course in the kernel of the expression on the right hand side of (5.109). Since also

$$A_k \cdots A_2 A_1 \tilde{\psi}_k = A_k(W_k/W_{k-1}) = 0 \tag{5.110}$$

by (5.101) and (5.104) the kernels of the differential expressions and therefore the differential expressions in (5.109) themselves are identical. From $L_n = A_n \cdots A_2 A_1$ one infers $L_n\tilde{\psi}_k = 0$, $1 \le k \le n-1$ in R. Since (5.102) is equivalent to

$$(W_k/W_{k-1})_t = (P_{r,k})_+(W_k/W_{k-1}) \tag{5.111}$$

one gets

$$\begin{aligned} 0 &= (\partial_t - (P_{r,k})_+)(W_k/W_{k-1}) \\ &= -(P_{r,k})_+ A_{k-1} \cdots A_2 A_1 \tilde{\psi}_k + \Big(\frac{d}{dt}A_{k-1}\Big) A_{k-2} \cdots A_2 A_1 \tilde{\psi}_k \\ &\quad + A_{k-1}(A_{k-2} \cdots A_2 A_1 \tilde{\psi}_k)_t. \end{aligned} \tag{5.112}$$

From $(P_{r,k})_+ A_{k-1} = A_{k-1}(P_{r,k-1})_+ + \phi_{k-1,t}$ and $\Big(\frac{d}{dt}A_{k-1}\Big) = \phi_{k-1,t}$ we then obtain

$$0 = \Big(\partial_t - (P_{r,k})_+\Big)(W_k/W_{k-1}) = A_{k-1}\Big(\partial_t (P_{k-1})_+\Big) A_{k-2} \cdots A_2 A_1 \tilde{\psi}_k. \tag{5.113}$$

Iterating this procedure $k-1$ times yields

$$A_{k-1} \cdots A_2 A_1 \Big(\partial_t - (P_{r,1})_+\Big)\tilde{\psi}_k = 0, \quad 1 \le k \le n-1. \tag{5.114}$$

Thus

$$\Big(\partial_t - (P_r)_+\Big)\tilde{\psi}_k = \sum_{i=1}^{k-1} \alpha_{k,i}\tilde{\psi}_i, \tag{5.115}$$

where the $\alpha_{k,i}$ are (in general t–dependent) constants. Differentiating (5.115) $k-2$ times with respect to x yields

$$\begin{pmatrix} \alpha_{k,1} \\ \vdots \\ \alpha_{k,k-1} \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_1 & \cdots & \tilde{\psi}_{k-1} \\ \vdots & & \vdots \\ \tilde{\psi}_1^{(k-2)} & \cdots & \tilde{\psi}_{k-1}^{(k-2)} \end{pmatrix}^{-1} \begin{pmatrix} \Big(\partial_t - (P_r)_+\Big)\tilde{\psi}_k \\ \vdots \\ \Big(\Big(\partial_t - (P_r)_+\Big)\tilde{\psi}_k\Big)^{(k-2)} \end{pmatrix}. \tag{5.116}$$

Here the inverse matrix on the right exists since $W_{k-1} \neq 0$. The fact that the $\tilde{\psi}_j$ are elements of the set S shows that the $\alpha_{k,i}$, $i = 1, ..., k-1$, $k = 1, ..., n-1$ are continuous in t. Finally, we define the functions ψ_k, i.e., the solutions of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$, as a linear combination of the functions $\tilde{\psi}_j$ with t-dependent coefficients. Any such linear combination will satisfy $L_n\psi = 0$. Let

$$\psi_k(x,t) = b_{k,k}\tilde{\psi}_k(x,t) + \sum_{k'=1}^{k-1} b_{k,k'}(t)\tilde{\psi}_{k'}(x,t). \tag{5.117}$$

Then the equation $0 = (\partial_t - (P_r)_+)\psi_k$ is equivalent to

$$0 = \sum_{k'=1}^{k} \left(b_{k,k',t}\tilde{\psi}_{k'} + b_{k,k'} \sum_{j=1}^{k'-1} \alpha_{k',j}\tilde{\psi}_j \right) = \sum_{k'=1}^{k-1} \left(b_{k,k',t} + \sum_{j=k'+1}^{k} b_{k,j}\alpha_{j,k'} \right) \tilde{\psi}_{k'} \tag{5.118}$$

where $b_{k,k}$ is a constant different from zero. From this equation we may obtain recursively (going backwards) the functions $b_{k,k'}(t)$ for $k' = k-1, ..., 1$ by quadratures using the linear independence of the $\tilde{\psi}_j$. Therefore, by construction, the functions ψ_j will satisfy $\psi_t = (P_r)_+\psi$ in R. Any choice of the integration constants and of the constant $b_{k,k}$ yields a representative of the equivalence class of solutions of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$ which reconstructs the original $\mathrm{DS}_{n,r}$-solution $(\phi_1, \dots, \phi_n)$ in R. Note that starting from a different point (x_0, t_0) in the same rectangle yields the same class of solutions $(\psi_1, ..., \psi_{n-1})$. The construction may now be performed in any open rectangle contained in Ω_0. If two of those rectangles have a nonempty intersection then the last argument shows that the classes coincide on this intersection. Therefore these equivalence classes of solutions are well defined in all of Ω_0. □

Remark 5.20. With some extra effort one can show that for a fundamental system of solutions $\psi_1, ..., \psi_n$ of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$ the Wronskian $W(\psi_1, ..., \psi_k)$, viewed as a function of x, has, at most, zeros of order $n-k$ for $k = 1, ..., n$. In particular these zeros are isolated. This implies that the solutions of the $\mathrm{DS}_{n,r}$ equations constructed in Theorem 5.13 have at most simple poles. One may therefore extend the definition of the class of solutions of $L_n\psi = 0$ and $\psi_t = (P_r)_+\psi$ in the last theorem to sets Ω on which there exist solutions ϕ_k of the $\mathrm{DS}_{n,r}$ equations which have, as functions of x, only simple poles.

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