ISOSPECTRAL DEFORMATIONS FOR STURM–LIOUVILLE AND DIRAC-TYPE OPERATORS AND ASSOCIATED NONLINEAR EVOLUTION EQUATIONS

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We give a systematic account of isospectral deformations for Sturm–Liouville and Dirac-type operators and associated hierarchies of nonlinear evolution equations. In particular, we study generalized KdV and modified KdV-hierarchies and their reduction to the standard (m)KdV-hierarchy. As an example we discuss the Harry Dym equation in some detail and relate its solutions to KdV-solutions and to Hirota's τ-functions.

1. Introduction

In this note we attempt to give a systematic treatment of certain isospectral deformations for Sturm–Liouville and Dirac-type operators and nonlinear evolution equations associated with them. The differential expressions we are most interested in are of the type

$$l(t) = -\frac{d}{dx}p(t,x)^2\frac{d}{dx} + q(t,x)\,, \tag{1.1}$$

and

$$m(t) = \begin{pmatrix} 0 & -p(t,x)\frac{d}{dx} - p_x(t,x) + \phi(t,x) \\ p(t,x)\frac{d}{dx} + \phi(t,x) & 0 \end{pmatrix} \tag{1.2}$$

where $t \in \boldsymbol{R}$, x varies on a (finite or infinite) interval (a,b), and p, q, ϕ satisfy appropriate conditions.

In Section 2 we recall the Liouville transformation which transforms (1.1) into

$$\tilde{l}(s) = -\frac{d^2}{dy^2} + \widetilde{v}(s,y), \qquad (s,y) \in \boldsymbol{R}^2 \tag{1.3}$$

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and (1.2) into

$$\widetilde{m}(s) = \begin{pmatrix} 0 & -\frac{d}{dy} + \widetilde{\varphi}(s,y) \\ \frac{d}{dy} + \widetilde{\varphi}(s,y) & 0 \end{pmatrix}, \tag{1.4}$$

for appropriate coefficients $\widetilde{v}$ and $\widetilde{\varphi}$. In Section 3 we study the differential expression $l(t)$ in (1.1) and a hierarchy of Lax differential expressions $b_n(t)$, $n \in \boldsymbol{N}_0$. The Lax equations

$$\frac{dl}{dt} = [b_n, l], \qquad n \in \boldsymbol{N}_0 \tag{1.5}$$

then yield a hierarchy of coupled nonlinear evolution equations (3.11), (3.12). In the remainder of Section 3 we then show how to reduce these generalized hierarchies to the standard Korteweg–de Vries (KdV)-hierarchy by means of the Liouville transformation of Section 2. As special cases of these generalized hierarchies we isolate various examples, most notably the Harry Dym (HD)-hierarchy. In Section 4 we study the modified versions of the hierarchies introduced in Section 3 and the analog of Miura-type transformations that link solutions of the (generalized) KdV and (generalized) modified Korteweg–de Vries (mKdV)-hierarchy. This modified hierarchy is defined in terms of the Lax equations

$$\frac{dm}{dt} = [d_n, m], \qquad n \in \boldsymbol{N}_0, \tag{1.6}$$

where m is the Dirac-type differential expression (1.2) and d_n are appropriate (matrix-valued) Lax differential expressions. Section 5 finally gives a systematic treatment of the Harry Dym equation within our approach. In particular, we provide a detailed discussion of how to generate solutions of the HD-equation with the help of solutions of the KdV-equation extending various earlier results on this subject [4, 8, 12, 13, 15, 16, 17, 20, 21, 22, 25] (see also the references therein). As shown by several illustrations involving solitons and quasi-periodic finite-gap solutions of the KdV-equation, our approach to the HD-equation is most effectively combined with Hirota's τ-function methods. We conclude with two appendices summarizing Hirota's τ-functions as needed in Section 5 and the construction of a typical example of a differential operator on a finite interval with a nontrivial absolutely continuous component in its spectrum. (Such spectral properties, although perhaps unexpected at first sight, turn out to be quite typical in connection with the HD-equation.)

2. Liouville-type transformations for Schrödinger and Dirac operators

In this section we briefly recall the well known Liouville transformation for one-dimensional Schrödinger and Dirac operators needed later on.

Assuming hypothesis

(H.2.1). $p, r > 0, p, r, q \in C^\infty(\boldsymbol{R} \times (a,b))$, $\frac{r}{p} \notin L^1((x_0, b); dx)$, $\frac{r}{p} \notin L^1((a, x_0); dx)$ for some $x_0 \in (a,b)$

we introduce on (a, b) ($a = -\infty$ and/or $b = +\infty$ included) the differential expression

$$l(t) = r(t,x)^{-2}\left[-\frac{d}{dx}p(t,x)^2\frac{d}{dx} + q(t,x)\right], \quad t \in \boldsymbol{R}, \quad x \in (a,b) \tag{2.1}$$

and the associated maximal Sturm–Liouville operator in $L^2((a,b); r(t,x)^2 dx)$

$$\begin{aligned} &L(t)f = l(t)f, \\ &f \in \mathcal{D}(L(t)) = \{g \in L^2((a,b); r(t,x)^2dx) | g, g' \in \mathrm{AC}_{\mathrm{loc}}((a,b)); \\ &\qquad l(t)g \in L^2((a,b); r(t,x)^2dx)\}, \quad t \in \boldsymbol{R}. \end{aligned} \tag{2.2}$$

(Here $\mathrm{AC}_{\mathrm{loc}}(\Omega)$ denotes the set of locally absolutely continuous functions on $\Omega \subseteq \boldsymbol{R}$, open.) $L(t)$ is well known to be a densely defined and closed operator. In addition we require

(H.2.2). $l(t)$ is in the limit point case at a and b (i.e., $L(t)$ is self-adjoint) for all $t \in \boldsymbol{R}$.

Next we recall the Liouville transformation from the variables (t, x) to (s, y), (see, e.g., [6], page 1500), where

$$s = t, \quad y = y(t,x) = \int_{x_0}^{x} \frac{r(t,x')}{p(t,x')}\,dx' + \eta(t), \quad x_0 \in (a,b), \quad \eta \in C^\infty(\boldsymbol{R}). \tag{2.3}$$

Since y is strictly monotone in x, the inverse function $x = x(s, y)$ exists and one introduces

$$\begin{aligned} &\widetilde{r}(s,y) = r(t,x(s,y)), \quad \widetilde{p}(s,y) = p(t,x(s,y)), \\ &\widetilde{q}(s,y) = q(t,x(s,y)), \quad \widetilde{v}(s,y) = v(t,x(s,y)), \\ &v(t,x) = v = \frac{q}{r^2} + \frac{1}{4r^2}\left(p_x^2 + 2pp_{xx} + 2pp_x\frac{r_x}{r} + 2p^2\frac{r_{xx}}{r} - 3p^2\frac{r_x^2}{r^2}\right) \end{aligned} \tag{2.4}$$

and the family of unitary operators

$$U(s)\colon\ L^2((a,b); r^2dx) \to L^2(\boldsymbol{R}; dy), \quad (U(s)f)(y) = \sqrt{\widetilde{r}(s,y)\widetilde{p}(s,y)}\, f(x(s,y)). \tag{2.5}$$

A straightforward computation then yields for the differential expression $\widetilde{l}(s)$ associated with $\widetilde{L}(s) = U(s)L(s)U^{-1}(s)$ in $L^2(\boldsymbol{R}; dy)$

$$\widetilde{l}(s) = -\frac{d^2}{dy^2} + \widetilde{v}(s,y), \quad \widetilde{v} = \frac{\widetilde{q}}{\widetilde{r}^2} + \frac{1}{(\widetilde{r}\widetilde{p})^2}\left(\frac{1}{2}\widetilde{r}\widetilde{p}(\widetilde{r}\widetilde{p})_{yy} - \frac{1}{4}(\widetilde{r}\widetilde{p})_y^2\right). \tag{2.6}$$

Thus the nonlinear evolution equations that leave the spectrum of $\widetilde{L}(s)$ (and hence of $L(t)$) invariant are given by the KdV-hierarchy for $\widetilde{v}$. Since $\widetilde{v} = \widetilde{v}(\widetilde{r}, \widetilde{p}, \widetilde{q})$, two of the three functions can be chosen freely.

EXAMPLE 2.3. (i) If $r = p = \sqrt{k}$, $a = -\infty$, $b = \infty$, $x_0 = 0$, $\eta = 0$, $q = \widehat{q}r^2$ then $x = y$ and

$$l = -\frac{1}{k}\frac{d}{dx}k\frac{d}{dx} + \widehat{q}, \quad \widetilde{l} = -\frac{d^2}{dy^2} + \widetilde{v}, \quad \widetilde{v} = \widehat{q} + \frac{1}{2}\frac{k_{yy}}{k} - \frac{1}{4}\frac{k_y^2}{k^2} = \widehat{q} + \frac{p_{yy}}{p}. \tag{2.7}$$

l turns out to be the differential expression of the impedance equation [5].

(ii) If $r^{-1} = p = \sqrt{k}$ then

$$l = k\left(-\frac{d}{dx}k\frac{d}{dx} + q\right), \qquad \tilde{l} = -\frac{d^2}{dy^2} + \widetilde{v}, \qquad \widetilde{v} = \widetilde{q}\widetilde{k}. \tag{2.8}$$

In particular, if $q = 0$ then $\widetilde{v} = 0$.

(iii) If $r = 1$, $p = s^2$, $q = 0$ then

$$l = -\frac{d}{dx}s^4\frac{d}{dx}, \qquad \tilde{l} = -\frac{d^2}{dy^2} + \frac{\widetilde{s}_{yy}}{\widetilde{s}}. \tag{2.9}$$

Next we turn to certain Dirac-type operators. Assuming

(H.2.4.). $\phi \in C^\infty((\boldsymbol{R} \times (a,b))$ real-valued

in addition to (H.2.1) we define the minimal operator

$$\widehat{A}(t) = r(t,\cdot)^{-2}[r(t,\cdot)p(t,\cdot)\frac{d}{dx} + \phi(t,\cdot)], \tag{2.10}$$

$$\mathcal{D}(\widehat{A}(t)) = \{g \in L^2((a,b); r(t,x)^2dx) | g \in \mathrm{AC_{loc}}((a,b)), \mathrm{supp}(g) \subset (a,b) \,\text{compact}\}, \quad t \in \boldsymbol{R}$$

and let $A(t)$ be the closure of $\widehat{A}(t)$, $t \in \boldsymbol{R}$. Then introducing the self-adjoint Dirac-type operator in $[L^2((a,b); r(t,x)^2dx)]^2$

$$M(t) = \begin{pmatrix} 0 & A(t)^* \\ A(t) & 0 \end{pmatrix}, \qquad \mathcal{D}(M(t)) = \mathcal{D}(A(t)) \oplus \mathcal{D}(A(t)^*), \tag{2.11}$$

one infers

$$M(t)^2 = \begin{pmatrix} L_1(t) & 0 \\ 0 & L_2(t) \end{pmatrix}, \qquad t \in \boldsymbol{R}, \tag{2.12}$$

where

$$L_1(t) = A(t)^*A(t), \qquad L_2(t) = A(t)A(t)^*. \tag{2.13}$$

Here $L_1(t)$ and $L_2(t)$ are generated by differential expressions of the type

$$l_1(t) = r(t,x)^{-2}\left\{ -\frac{d}{dx}p(t,x)^2\frac{d}{dx} - [r(t,x)^{-1}p(t,x)\phi(t,x)]_x + r(t,x)^{-2}\phi(t,x)^2\right\}, \tag{2.14}$$

$$\begin{aligned} l_2(t) = r(t,x)^{-2}\Big\{ & -\frac{d}{dx}p(t,x)^2\frac{d}{dx} + r(t,x)^{-1}p(t,x)\phi_x(t,x) - r(t,x)^{-1}p_x(t,x)\phi(t,x) \\ & - 3r(t,x)^{-2}r_x(t,x)p(t,x)\phi(t,x) - p(t,x)p_{xx}(t,x) - r(t,x)^{-1}r_{xx}(t,x)p(t,x)^2 \\ & + 2r(t,x)^{-2}r_x(t,x)^2p(t,x)^2 + r(t,x)^{-2}\phi(t,x)^2\Big\} \end{aligned} \tag{2.15}$$

and $M(t)$ is generated by the matrix differential expression

$$m(t) = \begin{pmatrix} 0 & a(t)^* \\ a(t) & 0 \end{pmatrix}, \tag{2.16}$$

$$a(t) = r(t,x)^{-2}\left[r(t,x)p(t,x)\frac{d}{dx} + \phi(t,x)\right], \tag{2.17}$$

$$a(t)^* = r(t,x)^{-2}\Big[-r(t,x)p(t,x)\frac{d}{dx} - (r(t,x)p(t,x))_x + \phi(t,x)\Big]. \tag{2.18}$$

Given (2.3) and (2.4) and

$$\begin{gathered}\widetilde{\phi}(s,y) = \phi(t,x(s,y)), \qquad \widetilde{\varphi}(s,y) = \varphi(t,x(s,y)),\\ \varphi(t,x) = \frac{\phi(t,x)}{r(t,x)^2} - \frac{1}{2}(r(t,x)p(t,x))_x\end{gathered} \tag{2.19}$$

we introduce the following family of unitary operators

$$\begin{gathered}W(s):\ L^2((a,b);dx)^2 \to L^2(\boldsymbol{R};dy)^2, \qquad W(s) = U(s)\cdot 1_2,\\ (W(s)f)(y)_j = \sqrt{\widetilde{r}(s,y)\widetilde{p}(s,y)}f(x(s,y))_j, \qquad j=1,2.\end{gathered} \tag{2.20}$$

A computation analogous to (2.6) then yields

$$\widetilde{M}(s) = W(s)M(s)W^{-1}(s) = \begin{pmatrix} 0 & \widetilde{A}(s)^* \\ \widetilde{A}(s) & 0 \end{pmatrix}, \tag{2.21}$$

$$\widetilde{A}(s) = U(s)A(s)U(s)^{-1}, \qquad \widetilde{A}(s)^* = U(s)A(s)^*U(s)^{-1}, \tag{2.22}$$

where $\widetilde{A}(s)$ and $A(s)^*$ are generated by the differential expressions

$$a(s) = \frac{d}{dy} + \widetilde{\varphi}(s,y), \qquad a(s)^* = -\frac{d}{dy} + \widetilde{\varphi}(s,y), \tag{2.23}$$

$$\widetilde{\varphi} = \frac{\widetilde{\phi}}{\widetilde{r}^2} - \frac{1}{2\widetilde{r}\widetilde{p}}(\widetilde{r}\widetilde{p})_y \tag{2.24}$$

and hence $\widetilde{A}(s)^*\widetilde{A}(s)$, $\widetilde{A}(s)\widetilde{A}(s)^*$ and $\widetilde{M}(s)$ are generated by

$$\begin{aligned}\tilde{l}_1(s) &= \widetilde{a}(s)^*\widetilde{a}(s) = -\frac{d^2}{dy^2} + \widetilde{v}_1(s,y),\\ \tilde{l}_2(s) &= \widetilde{a}(s)\widetilde{a}(s)^* = -\frac{d^2}{dy^2} + \widetilde{v}_2(s,y),\end{aligned} \tag{2.25}$$

$$\widetilde{v}_j(s,y) = \widetilde{\varphi}(s,y)^2 + (-1)^j\widetilde{\varphi}_y(s,y), \qquad j=1,2, \tag{2.26}$$

$$\widetilde{m}(s) = \begin{pmatrix} 0 & -\frac{d}{dy} + \widetilde{\varphi}(s,y) \\ \frac{d}{dy} + \widetilde{\varphi}(s,y) & 0 \end{pmatrix}. \tag{2.27}$$

EXAMPLE 2.5. (i) If $p=1$, $r=1$, $a=-\infty$, $b=\infty$, $x_0=0$, $\eta=0$, then $q=v$, $\phi=\varphi$, $x=y$ and

$$q_j = \varphi^2 + (-1)^j\varphi_x, \qquad j=1,2 \tag{2.28}$$

is the well known Miura transformation for the KdV-hierarchy.

(ii) If $r=1, \phi=0$ we get

$$\widetilde{v}_1 = \frac{1}{2}\frac{\widetilde{p}_{yy}}{\widetilde{p}} - \frac{1}{4}\frac{\widetilde{p}_y^2}{\widetilde{p}^2}, \qquad \widetilde{v}_2 = -\frac{1}{2}\frac{\widetilde{p}_{yy}}{\widetilde{p}} + \frac{3}{4}\frac{\widetilde{p}_y^2}{\widetilde{p}^2}. \tag{2.29}$$

By the transformation $\widetilde{p} \to \frac{1}{\widetilde{\rho}}, \widetilde{v}_j, j = 1, 2$ transform into

$$\widetilde{v}_1 \to -\frac{1}{2}\frac{\widetilde{\rho}_{yy}}{\widetilde{\rho}} + \frac{3}{4}\frac{\widetilde{\rho}_y^2}{\widetilde{\rho}^2}, \qquad \widetilde{v}_2 \to \frac{1}{2}\frac{\widetilde{\rho}_{yy}}{\widetilde{\rho}} - \frac{1}{4}\frac{\widetilde{\rho}_y^2}{\widetilde{\rho}^2}. \tag{2.30}$$

3. A generalized KdV-hierarchy for the case $r(t, x) = 1$

In this section we will concentrate on the special case $r(t, x) = 1$ and study a hierarchy of nonlinear evolution equations associated with L in (2.2). (The case $r \neq 1$, $p = 1$ is discussed in detail in [1] using the inverse scattering method (see also [19, 23, 24, 26]).)

At the end of this section we illustrate a reduction of this hierarchy to the KdV-hierarchy by means of the Liouville transformation of Section 2.

Throughout this section we shall use hypothesis

(H.3.1). Assume Hypotheses (H.2.1) and (H.2.2) with $r(t, x) = 1$.

Introducing v by

$$v = q + \frac{p_x^2}{4} + \frac{pp_{xx}}{2} \tag{3.1}$$

we can rewrite $l(t)$ in the form

$$l = -\frac{d}{dx}p^2\frac{d}{dx} + v - \frac{p_x^2}{4} - \frac{pp_{xx}}{2}, \qquad t \in \boldsymbol{R}, x \in (a, b). \tag{3.2}$$

Then

$$\frac{d}{dt}l = -\frac{d}{dx}2pp_t\frac{d}{dx} - \frac{p_x p_{xt}}{2} - \frac{p_t p_{xx}}{2} - \frac{pp_{xxt}}{2} + v_t. \tag{3.3}$$

For the Lax differential expressions $b_n(t)$ we make the usual ansatz

$$b_n(t) = \sum_{l=1}^{n}\left(\beta_{2l-1}(t, x)\frac{d^{2l-1}}{dx^{2l-1}} + \frac{d^{2l-1}}{dx^{2l-1}}\beta_{2l-1}(t, x)\right), \qquad b_0(t) = \beta_0(t),$$
$$t \in \boldsymbol{R},\ x \in (a, b),\ \beta_m \in C^\infty(\boldsymbol{R} \times (a, b)), \qquad m \in \boldsymbol{N}_0. \tag{3.4}$$

In order to illustrate some of the nonlinear equations covered by this ansatz we present a few special examples:

EXAMPLE 3.2. (i) $\beta_1 = -\frac{1}{2}p(\beta - 2)$ yields

$$b_1 = -\tfrac{1}{2}\left(p(\beta - 2)\frac{d}{dx} + \frac{d}{dx}(\beta - 2)p\right), \tag{3.5}$$

$$[b_1, l] = -\frac{d}{dx}2p^3\beta_x\frac{d}{dx} - $$
$$-(2pp_x^2\beta_x + \tfrac{3}{2}p^2p_{xx}\beta_x + \tfrac{5}{2}p^2p_x\beta_{xx} + \tfrac{1}{2}p^3\beta_{xxx} + p(\beta - 2)v_x). \tag{3.6}$$

The requirement $\frac{dl}{dt} = [b_1, l]$ then gives the evolution equations

$$p_t = p^2\beta_x, \tag{3.7}$$

$$v_t = 2pv_x - \beta pv_x, \tag{3.8}$$

where the smooth function $\beta = \beta(p, p_x, p_{xx}, ...)$ can be chosen freely.

(ii) $\beta_1 = -\frac{1}{2}\beta p + 6pv + 23pp_x^2 + 8p^2p_{xx}, \beta_3 = -4p^3$ yields

$$\begin{aligned} b_2 = & -8p^3\frac{d^3}{dx^3} - 36p^2p_x\frac{d^2}{dx^2} + (-\beta p + 12pv - 26pp_x^2 - 20p^2pp_{xx})\frac{d}{dx} \\ & -\frac{1}{2}p\beta_x - \frac{1}{2}\beta p_x + 6vp_x - p_x^3 + 6pv_x - 10pp_xp_{xx} - 4p^2p_{xxx}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} [b_2, l] = & -\frac{d}{dx}2p^3\beta_x\frac{d}{dx} - p\beta v_x - 2pp_x^2\beta_x + 12pvv_x - 2pp_x^2v_x - \frac{3}{2}p^2p_{xx}\beta_x \\ & -\frac{5}{2}p^2p_x\beta_{xx} - 2p^2p_{xx}v_x - 6p^2p_xv_{xx} - \frac{1}{2}p^3\beta_{xxx} - 2p^3v_{xxx}. \end{aligned} \tag{3.10}$$

$\frac{dl}{dt} = [b_2, l]$ then yields

$$p_t = p^2\beta_x, \tag{3.11}$$
$$v_t = 12pvv_x - (2p^3v_{xx})_x - (2p_x^2 + 2pp_{xx} + \beta)pv_x, \tag{3.12}$$

where again the smooth function $\beta = \beta(p, p_x, p_{xx}, ...)$ can be chosen freely.

Consequently, we define the generalized Korteweg–de Vries (gKdV)-equation by

$$\text{gKdV}(v) = v_t - 12pvv_x + (2p^3v_{xx})_x + (2p_x^2 + 2pp_{xx} + \beta)pv_x = 0. \tag{3.13}$$

Remark 3.3: The freedom in the choice of the function β just expresses the fact that we have two functions p, v and one can be chosen freely.

Remark 3.4: In the special case where $v(t, x) = 0$ (and hence $\widetilde{v}(s, y) = 0$ in (2.6)), any smooth solution $p(t, x)$ of (3.11) leaves the spectrum of $L(t)$ invariant. Actually, one infers quite generally that in this case (independently of (3.11))

$$\sigma(L(t)) = \sigma_{ac}(L(t)) = [0, \infty) \tag{3.14}$$

since

$$f_\pm(\lambda, t, x) = p(t, x)^{-1/2}e^{\pm i\sqrt{\lambda}\int_{x_0}^{x} p(t,x')^{-1}dx'}, \quad \lambda \geq 0 \tag{3.15}$$

are the generalized eigenfunctions of $L(t)$. (Here $\sigma(\cdot), \sigma_{ac}(\cdot)$ denote the spectrum and the absolutely continuous spectrum respectively.)

Remark 3.5: Imposing conditions on v (or q) fixes the choice of β. E.g., $q = 0$ is equivalent to $v = \frac{1}{4}p_x^2 + \frac{1}{2}pp_{xx}$ which implies $\beta = -2pp_{xx} + p_x^2$ and p must now fulfill the Harry Dym (HD)-equation

$$p_t = -2p^3p_{xxx}. \tag{3.16}$$

Also mixed types are possible, giving other forms of evolution equations:

EXAMPLE 3.6. (i) Setting $v = p$ in (3.8) we get $(\beta - 2) = -p^{-1}$ and hence

$$p_t = p_x. \tag{3.17}$$

(ii) Setting $v = p$ in (3.12) we get $\beta = 6p - 2pp_{xx} - 2p_x^2$ and hence

$$p_t = 6p^2p_x - 6p^2p_xp_{xx} - 2p^3p_{xxx}. \tag{3.18}$$

(This equation is sometimes called the "modified" magma equation.) By (2.3) and (3.55) this equation is also transformed into the KdV-equation.

(iii) Setting $v = p^2$ in (3.12) yields $b = 4p^2 - 2pp_{xx} - 8p_x^2 + 6p^{-1}\int_{x_0}^x pp_{x'}p_{x'x'}\,dx'$ and hence

$$p_t = 8p^3p_x - 12p^2p_xp_{xx} - 2p^3p_{xxx} - 6p_x\int_{x_0}^{x} pp_{x'}p_{x'x'}\,dx'. \tag{3.19}$$

Next we shall describe a hierarchy of nonlinear evolution equations associated with (3.2) and (3.4) in two different ways.

The first way is to construct the Lax pairs (l, b_n) from the corresponding Lax pairs $(\tilde{l}, \tilde{b}_n)$ of the Korteweg–de Vries (KdV)-equation. Consider

$$\frac{d\tilde{l}}{ds} - [\tilde{b}_n, \tilde{l}], \tag{3.20}$$

$$\tilde{l} = -\frac{d^2}{dy^2} + \tilde{v}, \tag{3.21}$$

where $(\tilde{l}, \tilde{b}_n)$ are the Lax pairs of the KdV-hierarchy (see, e.g., [18])

$$\tilde{b}_n = \sum_{m=1}^{n}\left(2\frac{\delta F_{m-1}}{\delta\tilde{v}}\partial_y - X_{m-1}(\tilde{v})\right)(4\tilde{l})^{n-m}, \qquad n \in \boldsymbol{N}, \qquad \tilde{b}_0(t) = \beta_0(t), \tag{3.22}$$

with the sequence $\dfrac{\delta F_n}{\delta\tilde{v}}$ defined by

$$\partial_y\frac{\delta F_n}{\delta\tilde{v}} = (4\tilde{v}\partial_y + 2\tilde{v}_y - \partial_y^3)\frac{\delta F_{n-1}}{\delta\tilde{v}}, \qquad \frac{\delta F_0}{\delta\tilde{v}} = 1, \tag{3.23}$$

$$X_n(\tilde{v}) = \partial_y\frac{\delta F_n}{\delta\tilde{v}} = (4\tilde{v} + 2\tilde{v}_y\partial_y^{-1} - \partial_y^2)X_{n-1}(\tilde{v}), \tag{3.24}$$

$$\frac{d\tilde{l}}{ds} - [\tilde{b}_n, \tilde{l}] = \tilde{v}_s - \partial_y\frac{\delta F_n}{\delta\tilde{v}}. \tag{3.25}$$

Hence we get

$$\frac{\delta F_1}{\delta\tilde{v}} = 2\tilde{v}, \qquad \frac{\delta F_2}{\delta\tilde{v}} = 6\tilde{v}^2 - 2\tilde{v}_{yy}, \qquad X_0 = 0, \qquad X_1 = 2\tilde{v}_y, \qquad X_2 = 12\tilde{v}\tilde{v}_y - 2\tilde{v}_{yyy},$$

$$\tilde{b}_1 = 2\partial_y, \qquad \tilde{b}_2 = -\partial_y^3 + 12\tilde{v}\partial_y + 6\tilde{v}_y. \tag{3.26}$$

Considering first the special case where $p_t = 0$, we formally transform by U in (2.5) and get

$$U^{-1}\left(\frac{d\tilde{l}}{dt} - [\tilde{b}_n, \tilde{l}]\right)U = \frac{dl}{dt} - [b_n, l], \tag{3.27}$$

where

$$b_n = U^{-1}\widetilde{b}_n U. \tag{3.28}$$

The b_n are the transformed Lax differential expressions of the KdV-hierarchy. We have

$$\frac{dl}{dt} - [b_n, l] = -\frac{d}{dx}2pp_t\frac{d}{dx} - \frac{p_x p_{xt}}{2} - \frac{p_t p_{xx}}{2} - \frac{pp_{xxt}}{2} + v_t - [b_n, l]. \tag{3.29}$$

Now

$$\frac{dl}{dt} - [b_n, l] = 0 \tag{3.30}$$

implies (the commutator is still a multiplication operator!)

$$p_t = 0, \tag{3.31}$$

$$v_t = [b_n, l], \qquad n \in \boldsymbol{N}_0. \tag{3.32}$$

The second way to obtain the Lax differential expressions b_n is essentially due to [2]. According to our conventions we define

$$A = -\left(p^3\frac{d^3}{dx^3} + 3p^2 p_x\frac{d^2}{dx^2} + (-4pv + pp_x^2 + p^2 p_{xx})\frac{d}{dx} - 2pv_x\right), \tag{3.33}$$

$$J = p\frac{d}{dx}, \tag{3.34}$$

$$G_0 = 1, \qquad JG_{n+1} = AG_n, \qquad n \in \boldsymbol{N}_0. \tag{3.35}$$

Then this sequence is well defined [2] and the evolution equations are given by

$$p_t = 0, \tag{3.36}$$

$$v_t = JG_n, \qquad n \in \boldsymbol{N}_0. \tag{3.37}$$

This yields the same b_n as in (3.28) by

$$[b_n, l] = JG_n, \qquad n \in \boldsymbol{N}_0. \tag{3.38}$$

In order to include the time dependence of $p, p_t \neq 0$, we extend the formalism of [2] by setting

$$\bar{b}_n = b_n + b,$$

$$b = -\frac{1}{2}\left(p\beta\frac{d}{dx} + \frac{d}{dx}\beta p\right), \qquad \beta = \beta(p, p_x, p_{xx}, \ldots) \tag{3.39}$$

and therefore get (since $[\bar{b}_n, l] = [b_n, l] + [b, l] = JG_n + [b, l]$),

$$\frac{dl}{dt} - [\bar{b}_n, l] = -\frac{d}{dx}2pp_t\frac{d}{dx} - \frac{p_x p_{xt}}{2} - \frac{p_t p_{xx}}{2} - \frac{pp_{xxt}}{2} + v_t + \frac{d}{dx}2p^3\beta_x\frac{d}{dx}$$
$$+(2pp_x^2\beta_x + \frac{3}{2}p^2 p_{xx}\beta_x + \frac{5}{2}p^2 p_x\beta_{xx} + \frac{1}{2}p^3\beta_{xxx} + p\beta v_x) - JG_n. \tag{3.40}$$

Requiring $\frac{dl}{dt} = [\bar{b}_n, l]$ then yields the pair of equations

$$p_t = p^2\beta_x, \tag{3.41}$$

$$v_t = JG_n - p\beta v_x, \qquad n \in \boldsymbol{N}_0. \tag{3.42}$$

Thus we define the generalized KdV-hierarchy by

$$\text{gKdV}_n(v) = v_t - JG_n + p\beta v_x, \qquad n \in \boldsymbol{N}_0. \tag{3.43}$$

The first few equations of the sequence $\text{gKdV}_n(v) = 0$ are given by

$$G_0 = 1, \qquad G_1 = 2v, \qquad G_2 = 6v^2 - 2p^2 v_{xx} - 2pp_x v_x, \tag{3.44}$$

$$\begin{aligned} n = 0 &: v_t = -pv_x\beta, \\ n = 1 &: v_t = 2pv_x - pv_x\beta, \\ n = 2 &: v_t = 12pvv_x - (2p^3 v_{xx})_x - (2p_x^2 + 2pp_{xx} + \beta)pv_x. \end{aligned} \tag{3.45}$$

Choosing p in (3.41) which fixes β, the hierarchy for v is then determined by equation (3.42). On the other hand, choosing a relation between p and v fixes β in (3.42) and one gets a hierarchy for p by (3.41). This is well illustrated, e.g., in

EXAMPLE 3.7. Let $q = 0$, i.e. $v = \frac{p_x^2}{4} + \frac{pp_{xx}}{2}$ and define $m = n-1$. Taking $\beta = -2H_m$, where

$$H_{m+1,x} = -p(pH_m)_{xxx}, \qquad H_0 = -1, \qquad H_1 = pp_{xx} - \frac{p_x^2}{2}, \qquad G_1 = pp_{xx} + \frac{p_x^2}{2}, \tag{3.46}$$

(3.41) yields the HD-hierarchy for p

$$p_t = -2p^2 H_{m,x}, \tag{3.47}$$

$$m = 0 : p_t = 0, \tag{3.48}$$

$$m = 1 : p_t = -2p^3 p_{xxx}. \tag{3.49}$$

In this case (3.42) becomes the identity

$$p^2 H_{m,xxx} + 5pp_x H_{m,xx} + (4p_x^2 + 3pp_{xx})H_{m,x} + (2p_x p_{xx} + pp_{xxx})H_m = -G_{m+1,x} \tag{3.50}$$

as can be shown by a straightforward induction argument.

Another example illustrating (3.41), (3.42) is given by

EXAMPLE 3.8. Taking $v = p$ and $\beta = p^{-1}G_n$ we get from (3.41) and (3.42)

$$p_t = p^2(p^{-1}G_n)_x = -p_x G_n + pG_{n,x}, \tag{3.51}$$

$$n = 0 : p_t = -p_x, \tag{3.52}$$

$$n = 1 : p_t = 0, \tag{3.53}$$

$$n = 2 : p_t = 6p^2 p_x - 2p^3 p_{xxx} - 6p^2 p_x p_{xx}. \tag{3.54}$$

Having introduced the hierarchy (3.41), (3.42) with the help of the KdV-hierarchy (3.21), (3.22) we now briefly consider the converse approach, i.e. given the hierarchy (3.41), (3.42) we shall reduce it to the KdV-hierarchy. Consider the Liouville transformation (2.3), where η is defined in terms of β by

$$\eta(t) = -\int^{t} dt' \beta(t', x_0) \tag{3.55}$$

implying

$$\frac{\partial}{\partial x} = \frac{1}{\widetilde{p}}\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial s} + \widetilde{\frac{\partial y}{\partial t}}\frac{\partial}{\partial y}, \qquad \frac{\partial y}{\partial t} = -\beta \tag{3.56}$$

by (2.3) and (3.41), where

$$\widetilde{p}(s,y) = p(t, x(s,y)), \qquad \widetilde{v}(s,y) = v(t, x(s,y)), \qquad \widetilde{\beta}(s,y) = \beta(t, x(s,y)),$$
$$\dot{y} = \frac{\partial y}{\partial t}, \qquad \widetilde{\dot{y}}(s,y) = \dot{y}(t, x(s,y)), \qquad \widetilde{\dot{y}}(s,y)_y = -\frac{\widetilde{p}_s + \widetilde{p}_y\widetilde{\dot{y}}}{\widetilde{p}}. \tag{3.57}$$

Now we get for the transformed gKdV-equation (3.13) the ordinary KdV-equation

$$\mathrm{KdV}(\widetilde{v}) = \widetilde{v}_s - 12\widetilde{v}\widetilde{v}_y + 2\widetilde{v}_{yyy} = 0. \tag{3.58}$$

To transform the entire hierarchy we describe again two possibilities.

First we observe that

$$\begin{aligned} G_0 &= 1, \qquad \widetilde{G}_n(\widetilde{v}(s,y)) = G_n(v(t,x(s,y))),\\ J &= p\frac{d}{dx} = \frac{d}{dy} = \widetilde{J},\\ A &= -\left(p^3\frac{d^3}{dx^3} + 3p^2p_x\frac{d^2}{dx^2} + (-4pv + pp_x^2 + p^2p_{xx})\frac{d}{dx} - 2pv_x\right)\\ &= -\frac{d^3}{dy^3} + 2\left(\widetilde{v}\frac{d}{dy} + \frac{d}{dy}\widetilde{v}\right) = \widetilde{A}. \end{aligned} \tag{3.59}$$

Now $v_t = JG_n - p\beta v_x$ implies $\widetilde{v}_s + \widetilde{v}_y\widetilde{\frac{\partial y}{\partial t}} = \widetilde{J}\widetilde{G}_n - \widetilde{\beta}\widetilde{v}_y$ which in turn implies

$$\widetilde{v}_s = \widetilde{J}\widetilde{G}_n. \tag{3.60}$$

Thus we have reduced this problem to the KdV-hierarchy: if $\widetilde{v}(s,y)$ is a solution of the n-th KdV-equation then $v(t,x) = \widetilde{v}(s, y(t,x))$ solves the n-th gKdV-equation.

A second way is to transform the Lax-equation

$$U\left(\frac{dl}{dt} - [\bar{b}_n, l]\right)U^{-1} = \frac{d\tilde{l}}{ds} - [\widetilde{b}_n, \tilde{l}] - [\widetilde{b} + \widetilde{e}, \tilde{l}], \tag{3.61}$$

where

$$b = -\frac{1}{2}\left(p\beta\frac{d}{dx} + \frac{d}{dx}\beta p\right), \tag{3.62}$$

$$\widetilde{b} = U^{-1}bU = -\frac{1}{2}\left(\widetilde{\beta}\frac{d}{dy} + \frac{d}{dy}\widetilde{\beta}\right), \tag{3.63}$$

$$\widetilde{e} = -\frac{1}{2}\left(\widetilde{\dot{y}}\frac{d}{dy} + \frac{d}{dy}\widetilde{\dot{y}}\right). \tag{3.64}$$

Requiring $dl/dt = [\bar{b}_n, l]$, which implies $p_t = p^2\beta_x$, we infer $-\beta = \dot{y}$, $-\widetilde{\beta} = \widetilde{\dot{y}}$ and hence $\widetilde{b} + \widetilde{e} = 0$. We conclude this section with the simple example of a one-soliton solution.

EXAMPLE 3.9. Suppose p satisfies (3.41) and η is defined as in (3.55). Then

$$\mathrm{gKdV}(v_{\mathrm{sol}}) = 0, \tag{3.65}$$

$$v_{\mathrm{sol}}(t,x) = -2\kappa^2\left(\cosh\kappa\left(D + \eta(t,x_0) - 8\kappa^2 t + \int_{x_0}^{x} dx' \frac{1}{p(t,x')}\right)\right)^{-2},$$
$$\kappa, D \in \boldsymbol{R}. \tag{3.66}$$

Other solutions of the KdV-equation transform in an analogous way.

4. The modified gKdV-hierarchy for $r(t,x) = 1$

In this section we derive the modified version of the generalized KdV-hierarchy of Section 3 by invoking Miura's transformation. Throughout this section we shall use hypothesis

(H.4.1). Assume Hypotheses (H.2.1), (H.2.2), and (H.2.4) with $r(t,x) = 1$.

Consider the matrix differential expression

$$m(t) = \begin{pmatrix} 0 & a(t)^* \\ a(t) & 0 \end{pmatrix} \tag{4.1}$$

with (see (2.14), (2.15), (2.17), (2.18))

$$\varphi(t,x) = \phi(t,x) - \tfrac{1}{2}p_x(t,x), \tag{4.2}$$

$$a = p\frac{d}{dx} + \frac{p_x}{2} + \varphi, \qquad a^* = -p\frac{d}{dx} - \frac{p_x}{2} + \varphi, \qquad a_t = p_t\frac{d}{dx} + \frac{1}{2}p_{x,t} + \varphi_t, \tag{4.3}$$

$$l_1 = a^*a = -\frac{d}{dx}p^2\frac{d}{dx} - \frac{1}{4}p_x^2 - \frac{1}{2}pp_{xx} + v_1, \tag{4.4}$$

$$l_2 = aa^* = -\frac{d}{dx}p^2\frac{d}{dx} - \frac{1}{4}p_x^2 - \frac{1}{2}pp_{xx} + v_2. \tag{4.5}$$

Then Miura's transformation reads

$$v_j = \varphi + (-1)^j p\varphi_x, \qquad j = 1,2. \tag{4.6}$$

Introducing

$$d_{2,l} = \delta_{2,l,1}\frac{d}{dx} + \frac{d}{dx}\delta_{2,l,1} - 4p^3\frac{d^3}{dx^3} - \frac{d^3}{dx^3}4p^3, \qquad l = 1,2, \tag{4.7}$$

$$\delta_{2,1,1} = -\tfrac{1}{2}\delta p + 6p\varphi^2 - 6p^2\varphi_x + 23pp_x^2 + 8p^2p_{xx}, \tag{4.8}$$

$$\delta_{2,2,1} = \delta_{2,1,1} + 12p^2\varphi_x \tag{4.9}$$

and

$$d_2 = \begin{pmatrix} d_{2,1} & 0 \\ 0 & d_{2,2} \end{pmatrix} \tag{4.10}$$

we get

$$[d_2, m] = \begin{pmatrix} 0 & d_{2,1}a^* - a^*d_{2,2} \\ d_{2,2}a - ad_{2,1} & 0 \end{pmatrix}, \tag{4.11}$$

$$\begin{aligned} d_{2,2}a - ad_{2,1} &= p^2\delta_x \frac{d}{dx} - \delta p\varphi_x + 12p\varphi^2\varphi_x + \delta_x pp_x \\ &\quad -2pp_x^2\varphi_x + \tfrac{1}{2}p^2\delta_{xx} - 6p^2p_x\varphi_{xx} - 2p^2p_{xx}\varphi_x - 2p^3\varphi_{xxx}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} d_{2,1}a^* - a^*d_{2,2} &= -p^2\delta_x \frac{d}{dx} - \delta p\varphi_x + 12p\varphi^2\varphi_x - \delta_x pp_x \\ &\quad -2pp_x^2\varphi_x - \tfrac{1}{2}p^2\delta_{xx} - 6p^2p_x\varphi_{xx} - 2p^2p_{xx}\varphi_x - 2p^3\varphi_{xxx}. \end{aligned} \tag{4.13}$$

The modified nonlinear evolution equations determined by $\frac{d}{dt}m = [d_2, m]$ then read

$$p_t = p^2\delta_x, \tag{4.14}$$

$$\varphi_t = 12p\varphi^2\varphi_x - (2p^3\varphi_{xx})_x - (2p_x^2 + 2pp_{xx} + \delta)p\varphi_x. \tag{4.15}$$

Introducing the generalized modified Korteweg–de Vries functional by

$$\mathrm{gmKdV}(\varphi) = \varphi_t - 12p\varphi^2\varphi_x + (2p^3\varphi_{xx})_x + (2p_x^2 + 2pp_{xx} + \delta)p\varphi_x \tag{4.16}$$

we obtain Miura's identity in the special case where $\beta = \delta$ in (3.11) and (4.14)

$$\mathrm{gKdV}(\varphi^2 + (-1)^j p\varphi_x) = [2\varphi + (-1)^j p\partial_x]\,\mathrm{gmKdV}(\varphi), \quad j = 1, 2, \quad \beta = \delta. \tag{4.17}$$

In order to derive the hierarchy we proceed as before. Let $\widetilde{d}_n$ be the Lax differential expressions for the mKdV-hierarchy (in the variables (s, y))

$$\frac{d\widetilde{m}}{ds} - [\widetilde{d}_n, \widetilde{m}] = \mathrm{mKdV}_n(\widetilde{\varphi}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad n \in \boldsymbol{N}_0. \tag{4.18}$$

Formally define d_n by $W^{-1}\widetilde{d}_n W$ (see (2.20) for the definition of W) then

$$\frac{dm}{dt} = [d_n, m], \quad n \in \boldsymbol{N}_0 \tag{4.19}$$

yields $p_t = 0$ and the generalized mKdV-hierarchy $\varphi_t = [d_n, m]$. To include the time dependence of p we recall (3.39) and compute with

$$\bar{d}_n = d_n + d, \qquad d = -\frac{1}{2}\left[p\delta\frac{d}{dx} + \frac{d}{dx}\delta p\right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \delta = \delta(p, p_x, p_{xx}, \ldots), \tag{4.20}$$

$$\begin{aligned} \frac{dm}{ds} - [\bar{d}_n, m] &= \frac{dm}{ds} - [d_n, m] - [d, m] \\ &= \begin{pmatrix} 0 & -p_t\frac{d}{dx} - \frac{1}{2}p_{x,t} + \varphi_t \\ p_t\frac{d}{dx} + \frac{1}{2}p_{x,t} + \varphi_t & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} 0 & -p^2\delta_x\frac{d}{dx} - \frac{1}{2}p^2\delta_{xx} - pp_x\delta_x - \delta p\varphi_x \\ p_x^2\frac{d}{dx} + \frac{1}{2}p^2\delta_{xx} + pp_x\delta_x - \delta p\varphi_x & 0 \end{pmatrix} - [d_n, m]. \end{aligned} \tag{4.21}$$

Requiring $dm/dt = [\bar{d}_n, m]$ then yields

$$p_t = p^2\delta_x, \tag{4.22}$$

$$\varphi_t = [d_n, m] - \delta p\varphi_x, \quad n \in \boldsymbol{N}_0. \tag{4.23}$$

Introducing

$$\mathrm{gmKdV}_n(\varphi) = \varphi_t - [d_n, m] + \delta p \varphi_x, \qquad n \in \boldsymbol{N}_0. \tag{4.24}$$

Miura's identity then reads in the special case where $\beta = \delta$ in (3.41) and (4.22)

$$\mathrm{gKdV}_n(\varphi^2 + (-1)^j p\varphi_x) = [2\varphi + (-1)^j p\partial_x]\, \mathrm{gmKdV}_n(\varphi),$$
$$j = 1,2, \qquad n \in \boldsymbol{N}_0, \qquad \beta = \delta \tag{4.25}$$

and we emphasize that for $\beta(t,x) = \delta(t,x)$ the "modified" equation for p in (4.22) is identical to its "unmodified" version (3.41).

5. The HD-equation

Due to its importance we now isolate the Harry Dym (HD)-equation as a special case of Sections 3 and 4. In accordance with our earlier comments on the HD-equation, we shall use Hypothesis (H.5.1) throughout this section:

(H.5.1). Assume Hypotheses (H.2.1), (H.2.2), and (H.2.4) with $r(t,x) = 1$,

$$q(t,x) = 0, \qquad \varphi(t,x) = -\tfrac{1}{2} p_x(t,x) \qquad (\text{i.e., } \phi(t,x) = 0).$$

Introducing $m(t), a(t), a(t)^*, l_j(t), j = 1,2$ in (4.1), (4.3)–(4.5) with $\varphi(t,x) = -\frac{1}{2}p_x(t,x)$ yields the HD-Lax pairs $(l_j, b_{2,j}), j = 1,2$, where

$$l_1 = a^* a = -\frac{d}{dx} p^2 \frac{d}{dx}, \tag{5.1}$$

$$b_{2,1} = -8p^3 \frac{d^3}{dx^3} - 36 p^2 p_x \frac{d^2}{dx^2} - (24 p p_x^2 + 12 p^2 p_{xx}) \frac{d}{dx}, \tag{5.2}$$

$$l_2 = a a^* = -p \frac{d^2}{dx^2} p, \tag{5.3}$$

$$b_{2,2} = -8p^3 \frac{d^3}{dx^3} - 36 p^2 p_x \frac{d^2}{dx^2} - (24 p p_x^2 + 24 p^2 p_{xx}) \frac{d}{dx} - 12 p p_x p_{xx} - 6 p^2 p_{xxx}. \tag{5.4}$$

Since

$$[b_{2,1}, l_1] = \frac{d}{dx} 4 p^4 p_{xxx} \frac{d}{dx} \tag{5.5}$$

and

$$[b_{2,2}, l_2] = 4p^4 p_{xxx} \frac{d^2}{dx^2} + (16 p^3 p_x p_{xxx} + 4 p^4 p_{xxxx}) \frac{d}{dx}$$
$$+ 12 p^2 p_x^2 p_{xxx} + 8 p^3 p_{xx} p_{xxx} + 12 p^3 p_x p_{xxxx} + 2 p^4 p_{xxxxx}, \tag{5.6}$$

$dl_j/dt = [b_{2,j}, l_j]$, $j = 1,2$ are both equivalent to the HD-equation

$$p_t = -2 p^3 p_{xxx}. \tag{5.7}$$

Similarly (see (4.7)–(4.13))

$$d_{2,j} = \delta_{2,j,1} \frac{d}{dx} + \frac{d}{dx} \delta_{2,j,1} - 4p^3 \frac{d^3}{dx^3} - \frac{d^3}{dx^3} 4p^3, \qquad j = 1,2,$$
$$\delta_{2,1,1} = 24 p p_x^2 + 12 p^2 p_{xx}, \qquad \delta_{2,2,1} = 24 p p_x^2 + 6 p^2 p_{xx}, \tag{5.8}$$

$$d_2 = \begin{pmatrix} d_{2,1} & 0 \\ 0 & d_{2,2} \end{pmatrix}, \qquad [d_2, m] = \begin{pmatrix} 0 & d_{2,1}a^* - a^* d_{2,2} \\ d_{2,2}a - a d_{2,1} & 0 \end{pmatrix} \tag{5.9}$$

yield

$$\frac{d}{dt}m - [d_2, m] = \begin{pmatrix} 0 & -(p_t + 2p^3 p_{xxx})\frac{d}{dx} - p_{xt} - 2(p^3 p_{xxx})_x \\ (p_t + 2p^3 p_{xxx})\frac{d}{dx} & 0 \end{pmatrix}. \tag{5.10}$$

Thus $\frac{d}{dt}m = [d_2, m]$ is also equivalent to the HD-equation (5.7) in agreement with our comment following (4.25).

An auto-Bäcklund transformation for the HD-equation (5.7) can be obtained by the following sequence of transformations [21]:

$$p_t = -2p^3 p_{xxx} \tag{5.11}$$

is transformed by

$$\rho = \frac{1}{p}, \qquad s = t, \qquad \xi = \int_{x_0}^{x} \rho(t, x')^2\, dx' + \zeta(t), \qquad \zeta(t) = 4\int^{t} dt' p_{xx}(t', x_0),$$

$$\widehat{\rho}(s, \xi) = \rho(t, x(s, \xi)), \qquad \frac{\partial}{\partial x} = \widehat{\rho}^2 \frac{\partial}{\partial \xi},$$

$$\frac{\partial \xi}{\partial t} = 4p_{xx}(t, x) - 4p_{xx}(t, x_0) + \zeta_t(t) = 4p_{xx}(t, x), \qquad \widehat{\frac{\partial \xi}{\partial t}} = -4\widehat{\rho}^2 \widehat{\rho}_{\xi\xi} \tag{5.12}$$

into

$$\widehat{\rho}_s + \widehat{\rho}_\xi \widehat{\frac{\partial \xi}{\partial t}} = -2\widehat{\rho}(\widehat{\rho}^2 \widehat{\rho}_{\xi\xi})_\xi, \tag{5.13}$$

and finally into

$$\widehat{\rho}_s = -2\widehat{\rho}^3 \widehat{\rho}_{\xi\xi\xi}. \tag{5.14}$$

(This transformation corresponds to the transformation $\widetilde{\varphi} \to -\widetilde{\varphi}$, resp. $\widetilde{p} \to \widetilde{p}^{-1}$ in (5.21), (5.23).)

The following example shows that this transformation also generates singular HD-solutions where p violates (H.5.1).

EXAMPLE 5.2. Let $p(t, x) = \alpha^2 x^2$, $\alpha \in \boldsymbol{R}$ which fulfills the HD-equation. Then

$$\rho = \frac{1}{p} = \frac{1}{\alpha^2 x^2}, \qquad \text{implies} \quad \alpha^4 \xi = -\tfrac{1}{3}x^{-3} + \tfrac{1}{3}x_0^{-3} + \alpha^4 \zeta(t). \tag{5.15}$$

Since $p_{xx}(t, x_0) = 2\alpha^2$ we choose $x_0 = -\infty$ and by $\zeta(t) = 8\alpha^2 t$ get

$$x = (24\alpha^6 s - 3\alpha^4 \xi)^{-1/3} \tag{5.16}$$

and

$$\widehat{\rho}(s,\xi) = (24\alpha^3 s - 3\alpha\xi)^{2/3} \tag{5.17}$$

which fulfills the HD-equation too.

In the following we reconsider the construction of solutions of the HD-equation from solutions of the KdV and mKdV-equation. The link between the HD-equation and (m)KdV-equation has been discussed by a variety of authors [4, 8, 12, 13, 15, 16, 17, 20, 21, 22, 25]. Here we shall recover these results very naturally within our approach.

As is well-known [9, 10], solutions of the KdV-equation

$$\widetilde{v}_s - 12\widetilde{v}\widetilde{v}_y + 2\widetilde{v}_{yyy} = 0 \tag{5.18}$$

yield solutions of the mKdV-equation

$$\widetilde{\varphi}_s - 12\widetilde{\varphi}^2\widetilde{\varphi}_y + 2\widetilde{\varphi}_{yyy} = 0 \tag{5.19}$$

satisfying

$$\widetilde{v}_j = \widetilde{\varphi}^2 + (-1)^j\widetilde{\varphi}_y, \qquad j = 1,2, \tag{5.20}$$

where $\widetilde{\varphi}$ is given by

$$\widetilde{\varphi}(s,y) = \partial_y \ln\widetilde{\psi}(s,y), \tag{5.21}$$

and $\widetilde{\psi}$ is assumed to satisfy

$$\tilde{l}(s)\widetilde{\psi}(s) = 0, \qquad (\partial_s - \widetilde{b}_2(s))\widetilde{\psi}(s) = 0 \tag{5.22}$$

with $\tilde{l}$, $\widetilde{b}_2$ defined in (3.21), (3.26). The ansatz

$$\widetilde{p}_\pm(s,y) = [\widetilde{\psi}(s,y)]^{\pm 2}, \tag{5.23}$$

as suggested by the relation $\widetilde{\varphi} = -\frac{\widetilde{p}_y}{2\widetilde{p}}$ (see (5.32)) and the invariance of the mKdV-equation with respect to $\widetilde{\varphi} \to -\widetilde{\varphi}$, then yields solutions of the transformed Harry Dym (tHD)-equation

$$\widetilde{p}_s - 6\frac{\widetilde{p}_y\widetilde{p}_{yy}}{\widetilde{p}} + 3\frac{\widetilde{p}_y^3}{\widetilde{p}^2} + 2\widetilde{p}_{yyy} = 0. \tag{5.24}$$

Note that if $\widetilde{p}$ solves the tHD-equation, then $\widetilde{p}^{-1}$ and *const* $\cdot\widetilde{p}$ solve the tHD-equation too. A further transformation of the variables

$$t = s, \qquad x = \int_{y_0(s)}^{y} \widetilde{p}_\pm(s,y')dy' + \eta_\pm(t), \qquad p_\pm(t,x) = \widetilde{p}_\pm(s,y(t,x)), \tag{5.25}$$

with the condition

$$\eta'_\pm(s) - y'_0(s)\widetilde{p}_\pm(s,y_0(s)) + 2\widetilde{p}_{\pm,yy}(s,y_0(s)) - 3\frac{\widetilde{p}_{\pm,y}(s,y_0))^2}{\widetilde{p}_\pm(s,y_0(s))} = 0, \tag{5.26}$$

then yields solutions of the HD-equation

$$p_t + 2p^3p_{xxx} = 0. \tag{5.27}$$

The simplest way to satisfy (5.26) is to choose $y_0'(s) = 0$ and take

$$\eta_\pm(s) = \int^s ds' \left(-2\widetilde{p}_{\pm,yy}(s', y_0) + 3\frac{\widetilde{p}_{\pm,y}(s', y_0)^2}{\widetilde{p}_\pm(s', y_0)}\right). \tag{5.28}$$

Conversely, in order to transform the HD-equation (5.27) back to the tHD-equation (5.24) we use the transformation (see (2.3)) of the variables

$$s = t, \qquad y = \int_{x_0(t)}^{x} p(t, x')^{-1} dx' + \eta(s), \qquad \widetilde{p}(s, y) = p(t, x(s, y)) \tag{5.29}$$

with

$$\eta'(t) - x_0'(t)p(t, x_0(t))^{-1} - 2p(t, x_0(t))p_{xx}(t, x_0(t)) + p_x(t, x_0(t))^2 = 0. \tag{5.30}$$

E.g., if $x_0'(t) = 0$ then

$$\eta(t) = \int^t dt'(2p(t', x_0)p_{xx}(t', x_0) - p_x(t', x_0)^2). \tag{5.31}$$

Remark 5.3: The conclusion following (5.10) and the results in [16] as presented above clearly point out that the Dirac-type differential expression

$$m = \begin{pmatrix} 0 & -p\frac{d}{dx} - \frac{1}{2}p_x \\ p\frac{d}{dx} + \frac{1}{2}p_x & 0 \end{pmatrix}, \quad \widetilde{m} = \begin{pmatrix} 0 & -\frac{d}{dy} + \widetilde{\varphi} \\ \frac{d}{dy} + \widetilde{\varphi} & 0 \end{pmatrix}, \quad \widetilde{\varphi} = -\frac{\widetilde{p}_y}{2\widetilde{p}} \tag{5.32}$$

is the natural choice in a Lax pair for the HD-equation.

This approach can most effectively be combined with Hirota's τ-function formalism [14] (see Appendix A) as will be shown below.

Assume that $\widetilde{\psi}_2$ is a solution of

$$\widetilde{l}_2(s)\widetilde{\psi}_2(s) = 0, \quad \text{i.e., } a^*(s)\widetilde{\psi}_2(s) = 0, \quad s \in \boldsymbol{R} \tag{5.33}$$

and

$$(\partial_s - \widetilde{b}_2(s))\widetilde{\psi}_2(s) = 0 \tag{5.34}$$

of the type

$$\widetilde{\psi}_2(s, y) = e^{Dy+Es}\frac{\tau_1(s, y)}{\tau_2(s, y)}, \quad (s, y) \in \boldsymbol{R}^2, \ D, E \in \boldsymbol{R}, \ \tau_j \in C^\infty(\boldsymbol{R}^2), \ j = 1, 2. \tag{5.35}$$

Making the ansatz

$$\widetilde{v}_2(s, y) = C - 2\partial_y^2 \ln \tau_2(s, y), \qquad C \in \boldsymbol{R} \tag{5.36}$$

one infers

$$\widetilde{v}_1(s, y) = C - 2\partial_y^2 \ln \tau_1(s, y), \tag{5.37}$$

$$C - D^2 = 2D\frac{\tau_{1,y}}{\tau_1} - 2D\frac{\tau_{2,y}}{\tau_2} - 2\frac{\tau_{1,y}\tau_{2,y}}{\tau_1\tau_2} + \frac{\tau_{1,yy}}{\tau_1} + \frac{\tau_{2,yy}}{\tau_2}, \tag{5.38}$$

$$\widetilde{\varphi} = \partial_y \ln \widetilde{\psi}_2(s,y) = D + \frac{\tau_{1,y}}{\tau_1} - \frac{\tau_{2,y}}{\tau_2}. \tag{5.39}$$

By the ansatz (5.23) we get

$$\widetilde{p}_\pm(s,y) = [\widetilde{\psi}_2(s,y)]^{\pm 2} = \left[e^{Dy+Es}\left(\frac{\tau_1(s,y)}{\tau_2(s,y)}\right)\right]^{\pm 2} \tag{5.40}$$

for solutions of the tHD-equation (5.24).

A further variable transformation then yields solutions p of the HD-equation as described in (5.25)–(5.28).

We illustrate formula (5.40) with the help of soliton and quasi-periodic finite-gap solutions.

EXAMPLE 5.4. (N-soliton solutions). Let

$$\tau_2^N(s,y) = \det[1 + C_2^N(s,y)], \qquad N \in \boldsymbol{N}, \tag{5.41}$$

$$C_2^N(s,y) = \left[\frac{c_{2,l}c_{2,m}}{\kappa_l + \kappa_m} e^{-(\kappa_l+\kappa_m)(y+12V_\infty s)+8(\kappa_l^3+\kappa_m^3)s}\right]_{l,m=1}^N, \quad c_{2,l} > 0,\ 1 \le l \le N, \tag{5.42}$$

$$0 < \kappa_N < \kappa_{N-1} < \ldots < \kappa_1 \le V_\infty^{1/2} \tag{5.43}$$

describe the N-soliton KdV-solutions $\widetilde{v}_2^N(s,y)$,

$$\widetilde{v}_2^N(s,y) = V_\infty - 2\partial_y^2 \ln \tau_2^N(s,y). \tag{5.44}$$

We distinguish two cases [10].

(i) $V_\infty = \kappa_1^2$ (the critical case in the terminology of [10]). This yields a unique $(N-1)$-soliton KdV-solution $\widetilde{v}_1^{(N-1)}$ given by

$$\widetilde{v}_1^{(N-1)}(s,y) = V_\infty - 2\partial_y^2 \ln \tau_1^{(N-1)}(s,y), \tag{5.45}$$

$$\tau_1^{(N-1)}(s,y) = \det[1 + C_1^{(N-1)}(s,y)], \tag{5.46}$$

$$C_1^{(N-1)}(s,y) = \left[\left(\frac{(\kappa_1+\kappa_l)(\kappa_1+\kappa_m)}{(\kappa_1-\kappa_l)(\kappa_1-\kappa_m)}\right)^{1/2} C_{2,l,m}^N(s,y)\right]_{l,m=2}^N, \qquad N \ge 2, \tag{5.47}$$

$$C_1^{(0)}(s,y) = 0, \qquad N = 1, \tag{5.48}$$

$$C = \kappa_1^2, \qquad D = -\kappa_1, \qquad E = -4\kappa_1^3. \tag{5.49}$$

(ii) $V_\infty > \kappa_1^2$ (the subcritical case in the terminology of [10]). This yields KdV-solutions $\widetilde{v}_{1,\sigma}^N$, $\sigma = \pm 1$

$$\widetilde{v}_{1,\sigma}^N(s,y) = V_\infty - 2\partial_y^2 \ln \tau_{1,\sigma}^N(s,y), \tag{5.50}$$

$$\tau_{1,\sigma}^N(s,y) = \det[1 + C_{1,\sigma}^N(s,y)], \tag{5.51}$$

$$C^N_{1,\sigma}(s,y) = \left[\left(\frac{(\sigma V_\infty^{1/2} + \kappa_l)(\sigma V_\infty^{1/2} + \kappa_m)}{(\sigma V_\infty^{1/2} - \kappa_l)(\sigma V_\infty^{1/2} - \kappa_m)} \right)^{1/2} C^N_{2,l,m}(s,y) \right]^N_{l,m=1}, \tag{5.52}$$

$$C = V_\infty, \qquad D = -\sigma V_\infty^{1/2}, \qquad E = -4\sigma V_\infty^{3/2}, \qquad \sigma = \pm 1. \tag{5.53}$$

In both cases one reads off the corresponding mKdV-solutions $\widetilde{\varphi}_0$, resp. $\widetilde{\varphi}_\pm$ from (5.39) and obtains the associated solution $\widetilde{p}_{0,\pm}$ resp. $\widetilde{p}_{\pm,\sigma}$ of the tHD-equation from (5.40) as follows:

(i) $V_\infty = \kappa_1^2$ (critical)

$$\widetilde{\varphi}_0(s,y) = -\kappa_1 - \partial_y \ln\left(\frac{\det(1 + C_2^N(s,y))}{\det(1 + C_1^{(N-1)}(s,y))} \right). \tag{5.54}$$

Then we get from (5.40)

$$\widetilde{p}_{\pm,0}(s,y) = [\widetilde{\psi}^N_{2,0}(s,y)]^{\pm 2} = \left[e^{-\kappa_1 y - 4\kappa_1^3 s} \left(\frac{\det(1 + C_1^{(N-1)}(s,y))}{\det(1 + C_2^N(s,y))} \right) \right]^{\pm 2}. \tag{5.55}$$

In the special case where $N = 1$, $c_{2,1}^2 = 2\kappa_1$ one obtains

$$C_2^1(s,y) = e^{-2\kappa_1 y - 8\kappa_1^3 s}, \tag{5.56}$$
$$\widetilde{v}_2^1(s,y) = \kappa_1^2 - 2\kappa_1^2[\cosh(\kappa_1 y + 4\kappa_1^3 s)]^{-2}, \tag{5.57}$$
$$\widetilde{\varphi}_0(s,y) = -\kappa_1 \tanh(\kappa_1 y + 4\kappa_1^3 s), \tag{5.58}$$
$$\widetilde{p}_{\pm,0}(s,y) = [2\cosh(\kappa_1 y + 4\kappa_1^3 s]^{\mp 2}. \tag{5.59}$$

For $\widetilde{p}_{+,0}$ we take $y_0 = -\infty$, $\eta_+ = 0$ and get

$$x = \frac{1}{4} \int_{-\infty}^{y} dy' \frac{1}{(\cosh(\kappa_1 y' + 4\kappa_1^3 s))^2} = \frac{1}{4\kappa_1}(\tanh(\kappa_1 y + 4\kappa_1^3 s) + 1), \tag{5.60}$$

$$y = \frac{1}{\kappa_1} \operatorname{arctanh}(4\kappa_1 x - 1) - 4\kappa_1^2 s. \tag{5.61}$$

Hence

$$p_{+,0}(t,x) = \kappa_1 x(2 - 4\kappa_1 x), \qquad x \in \left(0, \frac{1}{2\kappa_1}\right). \tag{5.62}$$

(ii) $V_\infty > \kappa_1^2$ (subcritical)

$$\widetilde{\varphi}_\sigma(s,y) = -\sigma V_\infty^{1/2} - \partial_y \ln\left(\frac{\det(1 + C_2^N(s,y))}{\det(1 + C_{1,\sigma}^N(s,y))} \right), \qquad \sigma = \pm 1, \tag{5.63}$$

$$\widetilde{p}_{\pm,\sigma}(s,y) = [\widetilde{\psi}^N_{2,\sigma}(s,y)]^{\pm 2} = \left[e^{-\sigma V_\infty^{1/2} y - 4\sigma V_\infty^{3/2} s} \left(\frac{\det(1 + C_{1,\sigma}^N(s,y))}{\det(1 + C_2^N(s,y))} \right) \right]^{\pm 2}, \quad \sigma = \pm 1. \tag{5.64}$$

Remark 5.5: The critical and subcritical cases in Example 5.4 exhibit a very different qualitative behavior if $\widetilde{p}(s,y)$ is further transformed into HD-solutions $p(t,x)$. In fact, since

$$\lim_{y\to\pm\infty} \widetilde{\varphi}_0(s,y) = \mp V_\infty^{1/2} = \mp\kappa_1, \tag{5.65}$$

$$\lim_{y\to\pm\infty} \widetilde{\varphi}_\sigma(s,y) = -\sigma V_\infty^{1/2}, \tag{5.66}$$

one infers from (5.54) resp. (5.55) and (5.63) resp. (5.64) that

$$\widetilde{p}_{+,0}(s,y) \overset{y\to\pm\infty}{=} O(e^{\mp 2\kappa_1 y}), \tag{5.67}$$

$$\widetilde{p}_{-,0}(s,y) \overset{y\to\pm\infty}{=} O(e^{\pm 2\kappa_1 y}), \tag{5.68}$$

$$\widetilde{p}_{+,\sigma}(s,y) \overset{y\to\pm\infty}{=} O(e^{-2\sigma V_\infty^{1/2} y}), \tag{5.69}$$

$$\widetilde{p}_{-,\sigma}(s,y) \overset{y\to\pm\infty}{=} O(e^{+2\sigma V_\infty^{1/2} y}) \tag{5.70}$$

and hence

(i) $p_{+,0}(t,x)$ is defined for x on a finite interval I. E.g. if $y_0 = -\infty$, $\eta_+ = 0$ in (5.25) then $I = (0, c_{2,1}^{-2})$ since one can show that

$$\int_{-\infty}^{\infty} \widetilde{p}_{+,0}(s,y)dy = \int_{-\infty}^{\infty} [\widetilde{\psi}_{2,0}^N(s,y)]^2 dy = c_{2,1}^{-2}. \tag{5.71}$$

(This case is further illustrated in Appendix B.)

(ii) $p_{-,0}(t,x)$ is defined for $x \in \boldsymbol{R}$.

(iii) $p_{+,\sigma}(t,x)$ with $y_0 = \sigma\infty$, $\eta_+ = 0$ is defined for $x \in (0, -\sigma\infty)$, $\sigma = \pm 1$.

(iv) $p_{-,\sigma}(t,x)$ with $y_0 = -\sigma\infty$, $\eta_- = 0$ is defined for $x \in (0, \sigma\infty)$, $\sigma = \pm 1$.

Finally we turn to quasi-periodic finite-gap solutions.

EXAMPLE 5.6. Let

$$\tau_2(s,y) = \Theta(\underline{\zeta}_{P_0} - \underline{A}_{P_0}(P_\infty) + \underline{\alpha}_{P_0}(\underline{\mu}(0,0)) + \frac{y}{2\pi}\underline{U}_0 + \frac{12s}{\pi}\underline{U}_2), \tag{5.72}$$

where Θ denotes Riemann's theta function associated with the hyperelliptic curve

$$R_0(z)^{1/2} = \left[\prod_{n=0}^{2g}(E_n - z)\right]^{1/2}, \qquad 0 \le E_0 < E_1 < \ldots < E_{2g},\ g \in \boldsymbol{N} \tag{5.73}$$

and an appropriate homology basis $\{a_j, b_j\}_{j=1}^g$ with intersection matrix $a_j \circ b_l = \delta_{j,l}$. Here $\underline{\zeta}_{P_0}$ is Riemann's vector with base point $P_0 = (E_0, 0)$, P_∞ the point at infinity, $\underline{A}_{P_0}(P)$ denotes the corresponding Abel map, $\mu(0,0) = (\mu_1(0,0), \ldots, \mu_g(0,0))$ is the Dirichlet divisor at $t = 0$, $x = 0$, $\underline{\alpha}_{P_0}(P_1, \ldots, P_g) = \sum_{j=1}^g \underline{A}_{P_0}(P_j)$ and $\underline{U}_0$, $\underline{U}_2$ are b-periods of normalized differentials of the second kind $\omega_0^{(2)}$, $\omega_2^{(2)}$ with a prescribed pole of order two respectively four at P_∞. The corresponding quasi-periodic finite-gap KdV-solutions are then given by

$$\widetilde{v}_2(s,y) = \Lambda - 2\partial_y^2 \ln \tau_2(s,y), \tag{5.74}$$

where Λ is a constant only depending on the underlying hyperelliptic curve. (See e.g. [11] for a complete discussion of such quasi-periodic finite-gap solutions.) Next we introduce

$$\tau_{1,\pm 1}(\lambda,s,y) = \Theta(\underline{\zeta}_{P_0} \mp \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\underline{\mu}(0,0)) + \frac{y}{2\pi}\underline{U}_0 + \frac{12s}{\pi}\underline{U}_2), \tag{5.75}$$

$$P = (\lambda, \lim_{\epsilon\downarrow 0} R_0(\lambda + i\epsilon)^{1/2}), \qquad \lambda \in \boldsymbol{R},$$

$$\tau_{1,\pm 1}(s,y) = \tau_{1,\pm 1}(0,s,y), \tag{5.76}$$

$$\widetilde{\psi}_{2,\pm 1}(s,y) = e^{\mp iy\int_{P_0}^P \omega_0^{(2)} \mp 24s \int_{P_0}^P \omega_2^{(2)}} \frac{\tau_{1,\pm 1}(s,y)}{\tau_2(s,y)} \tag{5.77}$$

and the quasi-periodic finite-gap KdV solutions

$$\widetilde{v}_{1,\pm 1}(s,y) = \Lambda - 2\partial_y^2 \ln \tau_{1,\pm 1}(s,y). \tag{5.78}$$

Again we distinguish two cases [11].

(i) $E_0 = 0$ (the critical case). Then

$$\widetilde{\psi}_{2,+1}(s,y) = \widetilde{\psi}_{2,-1}(s,y) \equiv \widetilde{\psi}_{2,0}(s,y), \qquad \widetilde{v}_{1,+1}(s,y) = \widetilde{v}_{1,-1}(s,y) \equiv \widetilde{v}_{1,0}(s,y), \tag{5.79}$$

and therefore

$$\widetilde{p}_{\pm,0}(s,y) = [\widetilde{\psi}_{2,0}(s,y)]^{\pm 2} \tag{5.80}$$

satisfies the tHD-equation (5.24). Since in this case $\widetilde{\psi}_{2,0}$ is periodic in y, a further transformation to $p_{\pm,0}(t,x)$ as in (5.25) shows that in the critical case, x varies on the whole real line $\boldsymbol{R}$.

(ii) $E_0 > 0$ (the subcritical case). Then again

$$\widetilde{p}_{\pm,\sigma}(s,y) = [\widetilde{\psi}_{2,\sigma}(s,y)]^{\pm 2}, \qquad \sigma = \pm 1 \tag{5.81}$$

satisfy the tHD-equation (5.24). Since in this case $\widetilde{\psi}_{2,\pm 1}(s) \in L^2((R,\pm\infty);dy)$,$[\widetilde{\psi}_{2,\pm 1}(s)]^{-1} \in L^2((R,\mp\infty);dy)$ for all $R \in \boldsymbol{R}$, a further transformation to $p_{\pm,\sigma}(t,x)$ as in (5.25) shows that in the subcritical case, x varies on half-lines.

Remark 5.7. What we called the transformed Harry Dym (tHD)-equation in (5.24) is the special case $\lambda = 0$ of the following equation

$$\widetilde{p}_s - 6\frac{\widetilde{p}_y \widetilde{p}_{yy}}{\widetilde{p}} + 3\frac{\widetilde{p}_y^3}{\widetilde{p}^2} + 2\widetilde{p}_{yyy} + 3\lambda \widetilde{p}_y = 0, \qquad \lambda \in \boldsymbol{R} \tag{5.82}$$

studied in [7, 8, 13, 25] and called the "interacting soliton equation" in [3]. Equation (5.82) (like (5.24)) has the property that if $\widetilde{p}$ is a solution, so is $\widetilde{p}^{-1}$ and *const* $\cdot\widetilde{p}$. Applying the variable transformation (5.25), (5.26) yields

$$p_t + 2p^3 p_{xxx} + 3\lambda p p_x = 0 \tag{5.83}$$

generalizing the HD-equation (5.27). However, a simple Galilei transformation

$$(s,y) \to (s, z = y - 3\lambda s)$$

reduces equation (5.82) to the case $\lambda = 0$ due to the identity

$$\widetilde{p}_s - 6\frac{\widetilde{p}_y\widetilde{p}_{yy}}{\widetilde{p}} + 3\frac{\widetilde{p}_y^3}{\widetilde{p}^2} + 2\widetilde{p}_{yyy} + 3\lambda\widetilde{p}_y = P_s - 6\frac{P_zP_{zz}}{P} + 3\frac{P_z^3}{P^2} + 2P_{zzz}, \qquad \widetilde{p}(s,y) = P(s,z). \tag{5.84}$$

Consequently, our methods immediately extend to equation (5.83).

Acknowledgements Bulla

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Appendix A: τ-functions and commutation methods

Since the explicit change of the variables in (2.3), (5.25) is possible only in special cases we found it useful to develop the τ-function method for the gKdV-equation directly.

Suppose that

$$0 < p, \tau_j \in C^\infty(\boldsymbol{R}^2), \qquad j = 1,2 \tag{A.1}$$

and introduce

$$l_2(t) = -p(t,x)\frac{d^2}{dx^2}p(t,x) + v_2(t,x) + \frac{1}{2}p(t,x)p_{xx}(t,x) - \frac{1}{4}p_x(t,x)^2, \qquad (t,x) \in \boldsymbol{R}^2, \tag{A.2}$$

where v_2 is of the type

$$v_2(t,x) = C - 2p(t,x)\partial_x[p(t,x)\partial_x \ln \tau_2(t,x)], \qquad C \in \boldsymbol{C}. \tag{A.3}$$

Moreover, assume ψ_2 to be a solution of

$$l_2(t)\psi_2(t) = 0, \qquad (\partial_t - b_2(t))\psi_2(t) = 0 \tag{A.4}$$

of the type

$$\psi_2(t,x) = p(t,x)^{-1/2}e^{D\int_{x_0}^x dx' p(t,x')^{-1} + Et}\frac{\tau_1(t,x)}{\tau_2(t,x)}, \qquad D, E \in \boldsymbol{C}. \tag{A.5}$$

Define

$$\varphi(t,x) = p(t,x)\partial_x \ln \psi_2(t,x) + \frac{1}{2}p_x(t,x) = D + p\frac{\tau_{1,x}}{\tau_1} - p\frac{\tau_{2,x}}{\tau_2}, \tag{A.6}$$

and

$$a(t) = p(t,x)\frac{d}{dx} + \varphi(t,x) + \frac{1}{2}p_x(t,x), \tag{A.7}$$

$$a(t)^+ = -p(t,x)\frac{d}{dx} + \varphi(t,x) - \frac{1}{2}p_x(t,x). \tag{A.8}$$

Then

$$l_2(t) = a(t)a(t)^+. \tag{A.9}$$

Next consider

$$l_1(t) = a(t)^+ a(t), \tag{A.10}$$

then

$$l_1(t) = -\frac{d}{dx}p(t,x)^2\frac{d}{dx} + v_1(t,x) - \frac{1}{2}p(t,x)p_{xx}(t,x) - \frac{1}{4}p_x(t,x)^2, \tag{A.11}$$

where

$$v_j = \varphi^2 + (-1)^j p\varphi_x, \qquad j = 1,2. \tag{A.12}$$

Moreover,

$$\begin{aligned} v_2 &= \varphi^2 + p\partial_x\varphi = D^2 + pp_x\left(\frac{\tau_{1,x}}{\tau_1} - \frac{\tau_{2,x}}{\tau_2}\right) + 2Dp\left(\frac{\tau_{1,x}}{\tau_1} - \frac{\tau_{2,x}}{\tau_2}\right) \\ &\quad + p^2\left(-\frac{2\tau_{1,x}\tau_{2,x}}{\tau_1\tau_2} + \frac{2\tau_{2,x}^2}{\tau_2^2} + \frac{\tau_{1,xx}}{\tau_1} - \frac{\tau_{2,xx}}{\tau_2}\right) \\ &= C - 2p\partial_x[p\partial_x \ln\tau_2] = C - 2pp_x\frac{\tau_{2,x}}{\tau_2} + 2p^2\left(\frac{\tau_{2,x}^2}{\tau_2^2} - \frac{\tau_{2,xx}}{\tau_2}\right), \end{aligned} \tag{A.13}$$

$$\begin{aligned} v_1 &= \varphi^2 - p\partial_x\varphi = D^2 + pp_x\left(-\frac{\tau_{1,x}}{\tau_1} + \frac{\tau_{2,x}}{\tau_2}\right) + 2Dp\left(\frac{\tau_{1,x}}{\tau_1} - \frac{\tau_{2,x}}{\tau_2}\right) \\ &\quad + p^2\left(\frac{2\tau_{1,x}^2}{\tau_1^2} - \frac{2\tau_{1,x}\tau_{2,x}}{\tau_1\tau_2} - \frac{\tau_{1,xx}}{\tau_1} + \frac{\tau_{2,xx}}{\tau_2}\right), \end{aligned} \tag{A.14}$$

$$\begin{aligned} v_2 - v_1 &= 2pp_x\left(\frac{\tau_{1,x}}{\tau_1} - \frac{\tau_{2,x}}{\tau_2}\right) + 2p^2\left(-\frac{\tau_{1,x}^2}{\tau_1^2} + \frac{\tau_{2,x}^2}{\tau_2^2} + \frac{\tau_{1,xx}}{\tau_1} - \frac{\tau_{2,xx}}{\tau_2}\right) \\ &= 2p\partial_x[p\partial_x \ln\tau_1] - 2p\partial_x[p\partial_x\ln\tau_2]. \end{aligned} \tag{A.15}$$

Thus

$$v_1(x,t) = C - 2p\partial_x[p\partial_x\ln\tau_1] \tag{A.16}$$

and

$$C - D^2 = pp_x\left(\frac{\tau_{1,x}}{\tau_1} + \frac{\tau_{2,x}}{\tau_2}\right) + 2Dp\left(\frac{\tau_{1,x}}{\tau_1} - \frac{\tau_{2,x}}{\tau_2}\right) + p^2\left(\frac{\tau_{1,xx}}{\tau_1} + \frac{\tau_{2,xx}}{\tau_2} - 2\frac{\tau_{1,x}\tau_{2,x}}{\tau_1\tau_2}\right). \tag{A.17}$$

Appendix B: A self-adjoint operator on a finite interval having non trivial absolutely continuous spectrum

In this appendix we further illustrate Remark 5.5 and generate a simple nontrivial example of a self-adjoint operator on a finite interval with a nonempty absolutely continuous component in its spectrum as follows: Consider the one-soliton operator $\widetilde{L}$ in $L^2(\boldsymbol{R}; dy)$

$$\widetilde{L}f = \widetilde{l}f, \qquad f \in \mathcal{D}(\widetilde{L}) = H^2(\boldsymbol{R}), \tag{B.1}$$

where

$$\tilde{l} = -\frac{d^2}{dy^2} + \kappa_1^2 - 2\kappa_1^2[\cosh(\kappa_1 y)]^{-2}, \qquad y \in \boldsymbol{R}. \tag{B.2}$$

(This corresponds to (5.57) at $s = 0$.) Then the spectrum of $\widetilde{L}$ is given by

$$\sigma(\widetilde{L}) = \{0\} \cup [\kappa_1^2, \infty), \tag{B.3}$$

$$\sigma_{ess}(\widetilde{L}) = \sigma_{ac}(\widetilde{L}) = [\kappa_1^2, \infty). \tag{B.4}$$

The (generalized) eigenfunctions of $\widetilde{L}$ are given by

$$\psi_0(y) = \sqrt{\frac{\kappa_1}{2}}\frac{1}{\cosh \kappa_1 y}, \qquad \psi_0 \in H^2(\boldsymbol{R}), \qquad ||\psi_0||_2 = 1, \tag{B.5}$$

$$\psi_\lambda(y) = c_1 e^{i\sqrt{\lambda-\kappa_1^2}y}(\kappa_1 \tanh \kappa_1 y - i\sqrt{\lambda - \kappa_1^2}) + c_2 e^{-i\sqrt{\lambda-\kappa_1^2}y}(\kappa_1 \tanh \kappa_1 y + i\sqrt{\lambda - \kappa_1^2}), \tag{B.6}$$

$$(\tilde{l} - \lambda)\psi_\lambda = 0, \qquad \psi_\lambda \notin L^2(\boldsymbol{R}; dy), \qquad \psi_\lambda \in L^\infty(\boldsymbol{R}), \qquad \lambda \geq \kappa_1^2. \tag{B.7}$$

Transforming with U^{-1}, $p(x) = 2\kappa_1 x(1 - 2\kappa_1 x)$

$$U^{-1} : L^2(\boldsymbol{R}; dy) \to L^2\left(\left(0, \frac{1}{2\kappa_1}\right); dx\right),$$

$$(U^{-1}f)(x) = \frac{1}{\sqrt{p(x)}}f(y(x)) = \frac{1}{\sqrt{2\kappa_1 x(1 - 2\kappa_1 x)}}f\left(\frac{1}{2\kappa_1}\ln\left(\frac{2\kappa_1 x}{1 - 2\kappa_1 x}\right)\right), \tag{B.8}$$

we get the Sturm Liouville operator in $L^2\left(\left(0, \frac{1}{2\kappa_1}\right); dx\right)$

$$Lf = lf, \qquad f \in \mathcal{D}(L) = \left\{g \in L^2\left(\left(0, \frac{1}{2\kappa_1}\right); dx\right) | g, g' \in \mathrm{AC}_{\mathrm{loc}}\left(\left(0, \frac{1}{2\kappa_1}\right)\right); \; lg \in L^2\left(\left(0, \frac{1}{2\kappa_1}\right); dx\right)\right\}, \tag{B.9}$$

where

$$l = -\frac{d}{dx}4\kappa_1^2 x^2(1 - 2\kappa_1 x)^2\frac{d}{dx}, \qquad x \in \left(0, \frac{1}{2\kappa_1}\right). \tag{B.10}$$

The transformed eigenvector $w_0 = U^{-1}\psi_0$ then becomes

$$w_0(x) = \sqrt{2\kappa_1}, \qquad x \in \left(0, \frac{1}{2\kappa_1}\right) \tag{B.11}$$

and the continuum solutions $w_\lambda = U^{-1}\psi_\lambda$ turn into

$$w_\lambda(x) = c_1(1 - 2\kappa_1 x)^{-\frac{1}{2}(\frac{i}{\kappa_1}\sqrt{\lambda-\kappa_1^2}+1)}(2\kappa_1 x)^{\frac{1}{2}(\frac{i}{\kappa_1}\sqrt{\lambda-\kappa_1^2}-1)}(4\kappa_1^2 x - \kappa_1 - i\sqrt{\lambda - \kappa_1^2}) + c_2(1 - 2\kappa_1 x)^{\frac{1}{2}(\frac{i}{\kappa_1}\sqrt{\lambda-\kappa_1^2}-1)}(2\kappa_1 x)^{-\frac{1}{2}(\frac{i}{\kappa_1}\sqrt{\lambda-\kappa_1^2}+1)}(4\kappa_1^2 x - \kappa_1 + i\sqrt{\lambda - \kappa_1^2}),$$

$$\lambda \geq \kappa_1^2, \qquad x \in \left(0, \frac{1}{2\kappa_1}\right). \tag{B.12}$$

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