Isospectral deformations for Sturm-Liouville and Dirac-type operators and associated nonlinear evolution equations[1]

F. Gesztesy[1] and K. Unterkofler[1,2,3]

[1] Department of Mathematics, University of Missouri, Columbia, MO 65211, USA.
[2] Institut für Theoretische Physik, TU Graz, A-8010 Graz, Austria.
[3] Supported by Fonds zur Förderung der wissenschaftlichen Forschung in Österreich by an E. Schrödinger Fellowship and by Project No. P7425.

Abstract We give a systematic account of isospectral deformations for Sturm-Liouville and Dirac-type operators and associated hierarchies of nonlinear evolution equations. In particular, we study generalized KdV and modified KdV-hierarchies and their reduction to the standard (m)KdV-hierarchy. As an example we study the Harry Dym equation in some detail and relate its solutions to KdV-solutions and to Hirota's τ-functions.

1. Introduction

In this note we attempt to give a systematic treatment of certain isospectral deformations for Sturm-Liouville and Dirac-type operators and nonlinear evolution equations associated with them. The differential expressions we are most interested in are of the type

$$l(t) = -\frac{d}{dx}p(t,x)^2\frac{d}{dx} + q(t,x), \tag{1.1}$$

and

$$m(t) = \begin{pmatrix} 0 & -p(t,x)\frac{d}{dx} - p_x(t,x) + \phi(t,x) \\ p(t,x)\frac{d}{dx} + \phi(t,x) & 0 \end{pmatrix}, \tag{1.2}$$

where $t \in \mathbb{R}$, x varies on a (finite or infinite) interval (a,b), and p, q, φ satisfies appropriate conditions.

[1] *Rep. Math. Phys.* **31** (1992), 113-137.

In Section 2 we recall the Liouville transformation which transforms (1.1) into

$$\tilde{l}(s) = -\frac{d^2}{dy^2} + \tilde{v}(s,y), \quad (s,y) \in \mathbb{R}^2 \tag{1.3}$$

and (1.2) into

$$\tilde{m}(s) = \begin{pmatrix} 0 & -\frac{d}{dy} + \tilde{\varphi}(s,y) \\ \frac{d}{dy} + \tilde{\varphi}(s,y) & 0 \end{pmatrix}, \tag{1.4}$$

for appropriate coefficients $\tilde{v}$ and $\tilde{\varphi}$. In Section 3 we study the differential expression $l(t)$ in (1.1) and a hierarchy of Lax differential expressions $b_n(t), n \in \mathbb{N}_0$. The Lax equations

$$\frac{dl}{dt} = [b_n, l], \qquad n \in \mathbb{N}_0 \tag{1.5}$$

then yield a hierarchy of coupled nonlinear evolution equations (3.11), (3.12). In the remainder of Section 3 we then show how to reduce these generalized hierarchies to the standard Korteweg-de Vries (KdV)-hierarchy by means of the Liouville transformation of Section 2. As special cases of these generalized hierarchies we isolate various examples, most notably the Harry Dym (HD)-hierarchy. In Section 4 we study the modified versions of the hierarchies introduced in Section 3 and the analog of Miura-type transformations that link solutions of the (generalized) KdV and (generalized) modified Korteweg-de Vries (mKdV)-hierarchy. This modified hierarchy is defined in terms of the Lax equations

$$\frac{dm}{dt} = [d_n, m], \qquad n \in \mathbb{N}_0 \tag{1.6}$$

where m is the Dirac-type differential expression (1.2) and d_n are appropriate (matrix-valued) Lax differential expressions. Section 5 finally gives a systematic treatment of the Harry Dym equation within our approach. In particular, we provide a detailed discussion of how to generate solutions of the HD-equation with the help of solutions of the KdV-equation extending various earlier results on this subject [4], [8], [12], [13], [15], [17], [18], [21], [22], [23], [27] (see also the references therein). As shown by several illustrations involving solitons and quasi-periodic finite-gap solutions of the KdV-equation, our approach to the HD-equation is most effectively combined with Hirota's τ-function methods. We conclude with two appendices summarizing Hirota's τ-functions as needed in Section 5 and the construction of a typical example of a differential operator on a finite interval with a nontrivial absolutely continuous component in its spectrum. (Such spectral properties, although perhaps unexpected at first sight, turn out to be quite typical in connection with the HD-equation.)

2. Liouville-type transformations for Schrödinger and Dirac operators

In this section we briefly recall the well known Liouville transformation for one-dimensional Schrödinger and Dirac operators needed later on.

Assuming hypothesis

(H.2.1). $p, r > 0, p, r, q \in C^\infty(\mathbb{R} \times (a,b)), \frac{r}{p} \notin L^1((x_0,b); dx), \frac{r}{p} \notin L^1((a,x_0); dx)$ for some $x_0 \in (a,b)$

we introduce on (a,b) ($a = -\infty$ and/or $b = +\infty$ included) the differential expression

$$l(t) = r(t,x)^{-2}\left[-\frac{d}{dx}p(t,x)^2\frac{d}{dx} + q(t,x)\right], \qquad t \in \mathbb{R},\ x \in (a,b) \tag{2.1}$$

and the associated maximal Sturm-Liouville operator in $L^2((a,b); r(t,x)^2dx)$

$$\begin{aligned} &L(t)f = l(t)f, \\ &\quad f \in \mathcal{D}(L(t)) = \{g \in L^2((a,b); r(t,x)^2dx) \,\big|\, g, g' \in \mathrm{AC}_{\mathrm{loc}}((a,b)); \\ &\qquad\qquad\qquad\qquad l(t)g \in L^2((a,b); r(t,x)^2dx)\}, \quad t \in \mathbb{R}. \end{aligned} \tag{2.2}$$

(Here $\mathrm{AC}_{\mathrm{loc}}(\Omega)$ denotes the set of locally absolutely continuous functions on $\Omega \subseteq \mathbb{R}$ open.) $L(t)$ is well known to be a densly defined and closed operator. In addition we require

(H.2.2). $l(t)$ is in the limit point case at a and b (i.e., $L(t)$ is self-adjoint) for all $t \in \mathbb{R}$.

Next we recall the Liouville transformation from the variables (t,x) to (s,y), (see e.g. [6] page 1500), where

$$s = t, \quad y = y(t,x) = \int_{x_0}^{x} \frac{r(t,x')}{p(t,x')}dx' + \eta(t), \qquad x_0 \in (a,b), \quad \eta \in C^\infty(\mathbb{R}). \tag{2.3}$$

Since y is strictly monotone in x, the inverse function $x = x(s,y)$ exists and one introduces

$$\begin{aligned} &\tilde{r}(s,y) = r(t,x(s,y)), \quad \tilde{p}(s,y) = p(t,x(s,y)), \quad \tilde{q}(s,y) = q(t,x(s,y)), \\ &\tilde{v}(s,y) = v(t,x(s,y)), \quad v(t,x) = \frac{q}{r^2} + \frac{1}{4r^2}\left(p_x^2 + 2pp_{xx} + 2pp_x\frac{r_x}{r} + 2p^2\frac{r_{xx}}{r} - 3p^2\frac{r_x^2}{r^2}\right) \end{aligned} \tag{2.4}$$

and the family of unitary operators

$$U(s) : L^2((a,b); r^2dx) \to L^2(\mathbb{R}; dy), \quad (U(s)f)(y) = \sqrt{\tilde{r}(s,y)\tilde{p}(s,y)}\ f(x(s,y)). \tag{2.5}$$

A straightforward computation then yields for the differential expression $\tilde{l}(s)$ associated with $\tilde{L}(s) = U(s)L(s)U^{-1}(s)$ in $L^2(\mathbb{R}; dy)$

$$\tilde{l}(s) = -\frac{d^2}{dy^2} + \tilde{v}(s,y), \qquad \tilde{v} = \frac{\tilde{q}}{\tilde{r}^2} + \frac{1}{(\tilde{r}\tilde{p})^2}\left(\frac{1}{2}\tilde{r}\tilde{p}(\tilde{r}\tilde{p})_{yy} - \frac{1}{4}(\tilde{r}\tilde{p})_y^2\right). \tag{2.6}$$

Thus the nonlinear evolution equations that leave the spectrum of $\tilde{L}(s)$ (and hence of $L(t)$) invariant are given by the KdV-hierarchy for $\tilde{v}$. Since $\tilde{v} = \tilde{v}(\tilde{r}, \tilde{p}, \tilde{q})$ two of the three functions can be chosen freely.

Example 2.3. (i). If $r = p = \sqrt{k},\ a = -\infty, b = \infty,\ x_0 = 0,\ \eta = 0,\ q = \hat{q}r^2$ then $x = y$ and

$$l = -\frac{1}{k}\frac{d}{dx}k\frac{d}{dx} + \hat{q}, \qquad \tilde{l} = -\frac{d^2}{dy^2} + v, \quad v = \hat{q} + \frac{1}{2}\frac{k_{yy}}{k} - \frac{1}{4}\frac{k_y^2}{k^2} = \hat{q} + \frac{p_{yy}}{p}. \tag{2.7}$$

l turns out to be the differential expression of the impedance equation [5].
(ii). If $r^{-1} = p = \sqrt{k}$ then

$$l = k\left(-\frac{d}{dx}k\frac{d}{dx} + q\right), \qquad \tilde{l} = -\frac{d^2}{dy^2} + \tilde{v}, \qquad \tilde{v} = \tilde{q}\tilde{k}. \tag{2.8}$$

In particular, if $q = 0$ then $\tilde{v} = 0$.
(iii). If $r = 1, p = s^2, q = 0$ then

$$l = -\frac{d}{dx}s^4\frac{d}{dx}, \qquad \tilde{l} = -\frac{d^2}{dy^2} + \frac{\tilde{s}_{yy}}{\tilde{s}}. \tag{2.9}$$

Next we turn to certain Dirac-type operators. Assuming

(H.2.4). $\phi \in C^\infty((\mathbb{R} \times (a,b))$ real-valued

in addition to (H.2.1) we define the minimal operator

$$\hat{A}(t) = r(t,\cdot)^{-2}\Big[r(t,\cdot)p(t,\cdot)\frac{d}{dx} + \phi(t,\cdot)\Big], \tag{2.10}$$

$$\mathcal{D}(\hat{A}(t)) = \{g \in L^2((a,b); r(t,x)^2dx)\Big|\ g \in \mathrm{AC_{loc}}((a,b)), \mathrm{supp}(g) \subset (a,b) \text{ compact}\}, \quad t \in \mathbb{R}$$

and let $A(t)$ be the closure of $\hat{A}(t), t \in \mathbb{R}$. Then introducing the self-adjoint Dirac-type operator in $\Big[L^2((a,b); r(t,x)^2dx)\Big]^2$

$$M(t) = \begin{pmatrix} 0 & A(t)^* \\ A(t) & 0 \end{pmatrix}, \quad \mathcal{D}(M(t)) = \mathcal{D}(A(t)) \oplus \mathcal{D}(A(t)^*), \tag{2.11}$$

one infers

$$M(t)^2 = \begin{pmatrix} L_1(t) & 0 \\ 0 & L_2(t) \end{pmatrix}, \qquad t \in \mathbb{R}, \tag{2.12}$$

where

$$L_1(t) = A(t)^* A(t), \quad L_2(t) = A(t) A(t)^*. \tag{2.13}$$

Here $L_1(t)$ and $L_2(t)$ are generated by differential expressions of the type

$$l_1(t) = r(t,x)^{-2} \left\{ -\frac{d}{dx} p(t,x)^2 \frac{d}{dx} - \Big[r(t,x)^{-1} p(t,x)\phi(t,x) \Big]_x + r(t,x)^{-2}\phi(t,x)^2 \right\}, \tag{2.14}$$

$$\begin{aligned} l_2(t) = r(t,x)^{-2} \Big\{ & -\frac{d}{dx} p(t,x)^2 \frac{d}{dx} + r(t,x)^{-1} p(t,x)\phi_x(t,x) - r(t,x)^{-1} p_x(t,x)\phi(t,x) \\ & -3r(t,x)^{-2} r_x(t,x) p(t,x)\phi(t,x) - p(t,x)p_{xx}(t,x) - r(t,x)^{-1} r_{xx}(t,x) p(t,x)^2 \\ & +2r(t,x)^{-2} r_x(t,x)^2 p(t,x)^2 + r(t,x)^{-2}\phi(t,x)^2 \Big\} \end{aligned} \tag{2.15}$$

and $M(t)$ is generated by the matrix differential expression

$$m(t) = \begin{pmatrix} 0 & a(t)^* \\ a(t) & 0 \end{pmatrix}, \tag{2.16}$$

$$a(t) = r(t,x)^{-2} \Big[r(t,x)p(t,x)\frac{d}{dx} + \phi(t,x) \Big], \tag{2.17}$$

$$a(t)^* = r(t,x)^{-2} \Big[-r(t,x)p(t,x)\frac{d}{dx} - (r(t,x)p(t,x))_x + \phi(t,x) \Big]. \tag{2.18}$$

Given (2.3) and (2.4) and

$$\tilde\phi(s,y) = \phi(t, x(s,y)), \quad \tilde\varphi(s,y) = \varphi(t,x(s,y)), \ \varphi(t,x) = \frac{\phi(t,x)}{r(t,x)^2} - \frac{1}{2}(r(t,x)p(t,x))_x \tag{2.19}$$

we introduce the following family of unitary operators

$$\begin{aligned} & W(s) : L^2((a,b);dx)^2 \to L^2(\mathbb{R};dy)^2, \quad W(s) = U(s)\cdot 1_2 \\ & (W(s)f)(y)_j = \sqrt{\tilde r(s,y)\tilde p(s,y)}\ f(x(s,y))_j, \quad j=1,2. \end{aligned} \tag{2.20}$$

A computation analogous to (2.6) then yields

$$\tilde M(s) = W(s)M(s)W^{-1}(s) = \begin{pmatrix} 0 & \tilde A(s)^* \\ \tilde A(s) & 0 \end{pmatrix}, \tag{2.21}$$

$$\tilde A(s) = U(s)A(s)U(s)^{-1}, \quad \tilde A(s)^* = U(s)A(s)^*U(s)^{-1}, \tag{2.22}$$

where $\tilde A(s)$ and $A(s)^*$ are generated by the differential expressions

$$a(s) = \frac{d}{dy} + \tilde\varphi(s,y), \qquad a(s)^* = -\frac{d}{dy} + \tilde\varphi(s,y), \tag{2.23}$$

$$\tilde\varphi = \frac{\tilde\phi}{\tilde r^2} - \frac{1}{2\tilde r\tilde p}(\tilde r\tilde p)_y \tag{2.24}$$

and hence $\tilde{A}(s)^*\tilde{A}(s)$, $\tilde{A}(s)A(s)^*$ and $\tilde{M}(s)$ are generated by

$$\tilde{l}_1(s) = \tilde{a}(s)^*\tilde{a}(s) = -\frac{d^2}{dy^2} + \tilde{v}_1(s,y), \quad \tilde{l}_2(s) = \tilde{a}(s)\tilde{a}(s)^* = -\frac{d^2}{dy^2} + \tilde{v}_2(s,y), \quad (2.25)$$

$$\tilde{v}_j(s,y) = \tilde{\varphi}(s,y)^2 + (-1)^j \tilde{\varphi}_y(s,y), \quad j = 1,2, \quad (2.26)$$

$$\tilde{m}(s) = \begin{pmatrix} 0 & -\frac{d}{dy} + \tilde{\varphi}(s,y) \\ \frac{d}{dy} + \tilde{\varphi}(s,y) & 0 \end{pmatrix}. \quad (2.27)$$

Example 2.5. (i). If $p = 1, r = 1, a = -\infty, b = \infty, x_0 = 0, \eta = 0$, then $q = v,\ \phi = \varphi,\ x = y$ and

$$q_j = \varphi^2 + (-1)^j \varphi_x, \quad j = 1,2 \quad (2.28)$$

is the well known Miura transformation for the KdV-hierarchy.

(ii). If $r = 1, \phi = 0$ we get

$$\tilde{v}_1 = \frac{1}{2}\frac{\tilde{p}_{yy}}{\tilde{p}} - \frac{1}{4}\frac{\tilde{p}_y^2}{\tilde{p}^2}, \qquad \tilde{v}_2 = -\frac{1}{2}\frac{\tilde{p}_{yy}}{\tilde{p}} + \frac{3}{4}\frac{\tilde{p}_y^2}{\tilde{p}^2}. \quad (2.29)$$

By the transformation $\tilde{p} \to \frac{1}{\tilde{\rho}},\ \tilde{v}_j,\ j = 1,2$ transform into

$$\tilde{v}_1 \to -\frac{1}{2}\frac{\tilde{\rho}_{yy}}{\tilde{\rho}} + \frac{3}{4}\frac{\tilde{\rho}_y^2}{\tilde{\rho}^2}, \qquad \tilde{v}_2 \to \frac{1}{2}\frac{\tilde{\rho}_{yy}}{\tilde{\rho}} - \frac{1}{4}\frac{\tilde{\rho}_y^2}{\tilde{\rho}^2}. \quad (2.30)$$

3. A generalized KdV-hierarchy for the case $r(t,x) = 1$

In this section we will concentrate on the special case $r(t,x) = 1$ and study a hierarchy of nonlinear evolution equations associated with L in (2.2). (The case $r \neq 1, p = 1$ is discussed in detail in [1] using the inverse scattering method (see also [20], [24], [26], [28]).)

At the end of the section we illustrate a reduction of this hierarchy to the KdV-hierarchy by means of the Liouville transformation of Section 2.

Throughout this section we shall use hypothesis

(H.3.1). Assume Hypotheses (H.2.1) and (H.2.2) with $r(t,x) = 1$.

Introducing v by

$$v = q + \frac{p_x^2}{4} + \frac{pp_{xx}}{2} \quad (3.1)$$

we can rewrite $l(t)$ in the form

$$l = -\frac{d}{dx}p^2\frac{d}{dx} + v - \frac{{p_x}^2}{4} - \frac{pp_{xx}}{2}, \qquad t \in \mathbb{R},\ x \in (a,b). \quad (3.2)$$

Then

$$\frac{d}{dt}l = -\frac{d}{dx}2pp_t\frac{d}{dx} - \frac{p_xp_{xt}}{2} - \frac{p_tp_{xx}}{2} - \frac{pp_{xxt}}{2} + v_t. \tag{3.3}$$

For the Lax differential expressions $b_n(t)$ we make the usual Ansatz

$$\begin{aligned} b_n(t) &= \sum_{l=1}^{n}\Big(\beta_{2l-1}(t,x)\frac{d^{2l-1}}{dx^{2l-1}} + \frac{d^{2l-1}}{dx^{2l-1}}\beta_{2l-1}(t,x)\Big), \quad b_0(t) = \beta_0(t),\ t \in \mathbb{R},\ x \in (a,b), \\ &\quad \beta_m \in C^\infty(\mathbb{R} \times (a,b)),\ m \in \mathbb{N}_0. \end{aligned} \tag{3.4}$$

In order to illustrate some of the nonlinear equations covered by this ansatz we present a few special examples:

Example 3.2. (i). $\beta_1 = -\frac{1}{2}p(\beta - 2)$ yields

$$b_1 = -\frac{1}{2}\left(p(\beta-2)\frac{d}{dx} + \frac{d}{dx}(\beta-2)p\right), \tag{3.5}$$

$$[b_1, l] = -\frac{d}{dx}2p^3\beta_x\frac{d}{dx} - \left(2pp_x{}^2\beta_x + \frac{3}{2}p^2p_{xx}\beta_x + \frac{5}{2}p^2p_x\beta_{xx} + \frac{1}{2}p^3\beta_{xxx} + p(\beta-2)v_x\right) \tag{3.6}$$

The requirement $\frac{dl}{dt} = [b_1, l]$ then gives the evolution equations

$$p_t = p^2\beta_x, \tag{3.7}$$

$$v_t = 2pv_x - \beta pv_x, \tag{3.8}$$

where the smooth function $\beta = \beta(p, p_x, p_{xx}, \ldots)$ can be chosen freely.

(ii). $\beta_1 = -\frac{1}{2}\beta p + 6pv + 23pp_x{}^2 + 8p^2p_{xx},\ \beta_3 = -4p^3$ yields

$$\begin{aligned} b_2 &= -8p^3\frac{d^3}{dx^3} - 36p^2p_x\frac{d^2}{dx^2} + \left(-\beta p + 12pv - 26pp_x{}^2 - 20p^2p_{xx}\right)\frac{d}{dx} \\ &\quad -\frac{1}{2}p\beta_x - \frac{1}{2}\beta p_x + 6vp_x - p_x{}^3 + 6pv_x - 10pp_xp_{xx} - 4p^2p_{xxx}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} [b_2, l] &= -\frac{d}{dx}2p^3\beta_x\frac{d}{dx} - p\beta v_x - 2pp_x{}^2\beta_x + 12pvv_x - 2pp_x{}^2v_x - \frac{3}{2}p^2p_{xx}\beta_x \\ &\quad -\frac{5}{2}p^2p_x\beta_{xx} - 2p^2p_{xx}v_x - 6p^2p_xv_{xx} - \frac{1}{2}p^3\beta_{xxx} - 2p^3v_{xxx}. \end{aligned} \tag{3.10}$$

$\frac{dl}{dt} = [b_2, l]$ then yields

$$p_t = p^2\beta_x, \tag{3.11}$$

$$v_t = 12pvv_x - (2p^3v_{xx})_x - (2p_x{}^2 + 2pp_{xx} + \beta)pv_x, \tag{3.12}$$

where again the smooth function $\beta = \beta(p, p_x, p_{xx}, \ldots)$ can be chosen freely.

Consequently we define the generalized Korteweg-de Vries (gKdV)-equation by

$$\mathrm{gKdV}(v) = v_t - 12pvv_x + (2p^3 v_{xx})_x + (2{p_x}^2 + 2pp_{xx} + \beta)pv_x = 0. \tag{3.13}$$

Remark 3.3. The freedom in the choice of the function β just expresses the fact that we have two functions p, v and one can be chosen freely.

Remark 3.4. In the special case where $v(t,x) = 0$ (and hence $\tilde{v}(s,y) = 0$ in (2.6)), any smooth solution $p(t,x)$ of (3.11) leaves the spectrum of $L(t)$ invariant. Actually, one infers quite generally that in this case (independently of (3.11))

$$\sigma(L(t)) = \sigma_{ac}(L(t)) = [0, \infty) \tag{3.14}$$

since

$$f_\pm(\lambda, t, x) = p(t,x)^{-1/2} e^{\pm i\sqrt{\lambda}\int_{x_0}^{x} p(t,x')^{-1}dx'}, \qquad \lambda \geq 0 \tag{3.15}$$

are the generalized eigenfunctions of $L(t)$. (Here $\sigma(\cdot), \sigma_{ac}(\cdot)$ denote the spectrum and the absolutely continuous spectrum respectively.)

Remark 3.5. Imposing conditions on v (or q) fixes the choice of β. E.g. $q = 0$ is equivalent to $v = \frac{1}{4}p_x^2 + \frac{1}{2}pp_{xx}$ which implies $\beta = -2pp_{xx} + p_x^2$ and p must now fulfill the Harry Dym (HD)-equation

$$p_t = -2p^3 p_{xxx}. \tag{3.16}$$

Also mixed types are possible, giving other forms of evolution equations:

Example 3.6. (i). Setting $v = p$ in (3.8) we get $(\beta - 2) = -p^{-1}$ and hence

$$p_t = p_x. \tag{3.17}$$

(ii). Setting $v = p$ in (3.12) we get $\beta = 6p - 2pp_{xx} - 2p_x^2$ and hence

$$p_t = 6p^2 p_x - 6p^2 p_x p_{xx} - 2p^3 p_{xxx}. \tag{3.18}$$

(This equation is called "modified" magma equation in [25], page 219.) By (2.3) and (3.55) this equation is also transformed into the KdV-equation.

(iii). Setting $v = p^2$ in (3.12) yields $b = 4p^2 - 2pp_{xx} - 8p_x^2 + 6p^{-1}\int_{x_0}^{x} pp_{x'}p_{x'x'}dx'$ and hence

$$p_t = 8p^3 p_x - 12p^2 p_x p_{xx} - 2p^3 p_{xxx} - 6p_x \int_{x_0}^{x} pp_{x'}p_{x'x'}dx'. \tag{3.19}$$

Next we shall describe a hierarchy of nonlinear evolution equations associated with (3.2) and (3.4) in two different ways.

The first way is to construct the Lax pairs (l, b_n) from the corresponding Lax pairs $(\tilde{l}, \tilde{b}_n)$ of the Korteweg-de Vries (KdV)-equation. Consider

$$\frac{d\tilde{l}}{ds} - [\tilde{b}_n, \tilde{l}], \tag{3.20}$$

$$\tilde{l} = -\frac{d^2}{dy^2} + \tilde{v}, \tag{3.21}$$

where $(\tilde{l}, \tilde{b}_{,n})$ are the Lax pairs of the KdV-hierarchy, (see e.g. [19])

$$\tilde{b}_n = \sum_{m=1}^{n} \left(2\frac{\delta F_{m-1}}{\delta \tilde{v}} \partial_y - X_{m-1}(\tilde{v}) \right) (4\tilde{l})^{n-m}, \quad n \in \mathbb{N}, \quad \tilde{b}_0(t) = \beta_0(t), \tag{3.22}$$

with the sequence $\frac{\delta F_n}{\delta \tilde{v}}$ defined by

$$\partial_y \frac{\delta F_n}{\delta \tilde{v}} = (4\tilde{v}\partial_y + 2\tilde{v}_y - \partial_y^3)\frac{\delta F_{n-1}}{\delta \tilde{v}}, \qquad \frac{\delta F_0}{\delta \tilde{v}} = 1, \tag{3.23}$$

$$X_n(\tilde{v}) = \partial_y \frac{\delta F_n}{\delta \tilde{v}} = (4\tilde{v} + 2\tilde{v}_y \partial_y^{-1} - \partial_y^2) X_{n-1}(\tilde{v}), \tag{3.24}$$

$$\frac{d\tilde{l}}{ds} - [\tilde{b}_n, \tilde{l}] = \tilde{v}_s - \partial_y \frac{\delta F_n}{\delta \tilde{v}}. \tag{3.25}$$

Hence we get

$$\frac{\delta F_1}{\delta \tilde{v}} = 2\tilde{v}, \qquad \frac{\delta F_2}{\delta \tilde{v}} = 6\tilde{v}^2 - 2\tilde{v}_{yy}, \quad X_0 = 0, \quad X_1 = 2\tilde{v}_y, \quad X_2 = 12\tilde{v}\tilde{v}_y - 2\tilde{v}_{yyy},$$
$$\tilde{b}_1 = 2\partial_y, \quad \tilde{b}_2 = -\partial_y^3 + 12\tilde{v}\partial_y + 6\tilde{v}_y. \tag{3.26}$$

Considering first the special case where $p_t = 0$, we formally transform by U in (2.5) and get

$$U^{-1} \left(\frac{d\tilde{l}}{dt} - [\tilde{b}_n, \tilde{l}] \right) U = \frac{dl}{dt} - [b_n, l], \tag{3.27}$$

where

$$b_n = U^{-1} \tilde{b}_n U. \tag{3.28}$$

The b_n are the transformed Lax differential expressions of the KdV-hierarchy. We have

$$\frac{dl}{dt} - [b_n, l] = -\frac{d}{dx} 2pp_t \frac{d}{dx} - \frac{p_x p_{xt}}{2} - \frac{p_t p_{xx}}{2} - \frac{pp_{xxt}}{2} + v_t - [b_n, l]. \tag{3.29}$$

Now

$$\frac{dl}{dt} - [b_n, l] = 0 \tag{3.30}$$

implies (the commutator is still a multiplication operator!)

$$p_t = 0, \tag{3.31}$$
$$v_t = [b_n, l], \qquad n \in \mathbb{N}_0. \tag{3.32}$$

The second way to obtain the Lax differential expressions b_n is essentially due to [2]: According to our conventions we define

$$A = -\left(p^3 \frac{d^3}{dx^3} + 3p^2 p_x \frac{d^2}{dx^2} + \left(-4pv + p{p_x}^2 + p^2 p_{xx}\right)\frac{d}{dx} - 2pv_x\right), \tag{3.33}$$
$$J = p\frac{d}{dx}, \tag{3.34}$$
$$G_0 = 1, \quad JG_{n+1} = AG_n, \quad n \in \mathbb{N}_0. \tag{3.35}$$

Then this sequence is well defined [2] and the evolution equations are given by

$$p_t = 0, \tag{3.36}$$
$$v_t = JG_n, \qquad n \in \mathbb{N}_0. \tag{3.37}$$

This yields the same b_n as in (3.28) by

$$[b_n, l] = JG_n, \qquad n \in \mathbb{N}_0. \tag{3.38}$$

In order to include the time dependence of $p, p_t \neq 0$, we extend the formalism of [2] by setting

$$\bar{b}_n = b_n + b,$$
$$b = -\frac{1}{2}\left(p\beta\frac{d}{dx} + \frac{d}{dx}\beta p\right), \qquad \beta = \beta(p, p_x, p_{xx}, \ldots) \tag{3.39}$$

and therefore get (since $[\bar{b}_n, l] = [b_n, l] + [b, l] = JG_n + [b, l]$),

$$\frac{dl}{dt} - [\bar{b}_n, l] = -\frac{d}{dx}2pp_t\frac{d}{dx} - \frac{p_x p_{xt}}{2} - \frac{p_t p_{xx}}{2} - \frac{pp_{xxt}}{2} + v_t$$
$$+\frac{d}{dx}2p^3\beta_x\frac{d}{dx} + \left(2p{p_x}^2\beta_x + \frac{3}{2}p^2 p_{xx}\beta_x + \frac{5}{2}p^2 p_x\beta_{xx} + \frac{1}{2}p^3\beta_{xxx} + p\beta v_x\right) - JG_n \tag{3.40}$$

Requiring $\frac{dl}{dt} = [\bar{b}_n, l]$ then yields the pair of equations

$$p_t = p^2\beta_x, \tag{3.41}$$
$$v_t = JG_n - p\beta v_x, \qquad n \in \mathbb{N}_0. \tag{3.42}$$

Thus we define the generalized KdV-hierarchy by

$$\mathrm{gKdV}_n(v) = v_t - JG_n + p\beta v_x, \qquad n \in \mathbb{N}_0. \tag{3.43}$$

The first few equations of the sequence $\mathrm{gKdV}_n(v) = 0$ are given by

$$G_0 = 1, \quad G_1 = 2v, \quad G_2 = 6v^2 - 2p^2 v_{xx} - 2pp_x v_x, \tag{3.44}$$

$$\begin{aligned} n = 0 &: \ v_t = -pv_x\beta, \\ n = 1 &: \ v_t = 2pv_x - pv_x\beta, \\ n = 2 &: \ v_t = 12pvv_x - (2p^3 v_{xx})_x - (2{p_x}^2 + 2pp_{xx} + \beta)pv_x. \end{aligned} \tag{3.45}$$

Choosing p in (3.41) which fixes β, the hierarchy for v is then determined by equation (3.42). On the other hand, choosing a relation between p and v fixes β in (3.42) and one gets a hierarchy for p by (3.41). This is well illustrated e.g. in

Example 3.7. Let $q = 0$, i.e. $v = \frac{p_x^2}{4} + \frac{pp_{xx}}{2}$ and define $m = n - 1$. Taking $\beta = -2H_m$, where

$$H_{m+1,x} = -p(pH_m)_{xxx}, \qquad H_0 = -1, \qquad H_1 = pp_{xx} - \frac{p_x^2}{2}, \quad G_1 = pp_{xx} + \frac{p_x^2}{2}, \tag{3.46}$$

(3.41) yields the HD-hierarchy for p

$$p_t = -2p^2 H_{m,x}, \tag{3.47}$$

$$m = 0 \ : \ p_t = 0, \tag{3.48}$$

$$m = 1 \ : \ p_t = -2p^3 p_{xxx}. \tag{3.49}$$

In this case (3.42) becomes the identity

$$p^2 H_{m,xxx} + 5pp_x H_{m,xx} + (4p_x^2 + 3pp_{xx})H_{m,x} + (2p_x p_{xx} + pp_{xxx})H_m = -G_{m+1,x} \tag{3.50}$$

as can be shown by a straightforward induction argument.

Another example illustrating (3.41), (3.42) is given by

Example 3.8. Taking $v = p$ and $\beta = p^{-1}G_n$ we get from (3.41) and (3.42)

$$p_t = p^2 \left(p^{-1}G_n\right)_x = -p_x G_n + pG_{n,x}, \tag{3.51}$$

$$n = 0 \ : \ p_t = -p_x, \tag{3.52}$$

$$n = 1 \ : \ p_t = 0, \tag{3.53}$$

$$n = 2 \ : \ p_t = 6p^2 p_x - 2p^3 p_{xxx} - 6p^2 p_x p_{xx}. \tag{3.54}$$

Having introduced the hierarchy (3.41), (3.42) with the help of the KdV-hierarchy (3.21), (3.22) we now briefly consider the converse approach, i.e. given the hierarchy (3.41), (3.42) we shall reduce it to the KdV-hierarchy. Consider the Liouville transformation (2.3) where η is defined in terms of β by

$$\eta(t) = -\int^t dt' \beta(t', x_0) \tag{3.55}$$

implying

$$\frac{\partial}{\partial x} = \frac{1}{\tilde{p}}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial s} + \widetilde{\frac{\partial y}{\partial t}}\frac{\partial}{\partial y}, \quad \frac{\partial y}{\partial t} = -\beta \tag{3.56}$$

by (2.3) and (3.41) where

$$\begin{aligned} &\tilde{p}(s,y) = p(t,x(s,y)), \quad \tilde{v}(s,y) = v(t,x(s,y)), \quad \tilde{\beta}(s,y) = \beta(t,x(s,y)),\\ &\dot{y} = \frac{\partial y}{\partial t}, \quad \tilde{\dot{y}}(s,y) = \dot{y}(t,x(s,y)), \quad \Rightarrow \quad \tilde{\dot{y}}(s,y)_y = -\frac{\tilde{p}_s + \tilde{p}_y\tilde{\dot{y}}}{\tilde{p}} \end{aligned} \tag{3.57}$$

Now we get for the transformed gKdV-equation (3.13) the ordinary KdV-equation

$$\mathrm{KdV}(\tilde{v}) = \tilde{v}_s - 12\tilde{v}\tilde{v}_y + 2\tilde{v}_{yyy} = 0. \tag{3.58}$$

To transform the entire hierarchy we describe again two possibilities.
First we observe that

$$\begin{aligned} G_0 &= 1, \qquad \tilde{G}_n(\tilde{v}(s,y)) = G_n(v(t,x(s,y))),\\ J &= p\frac{d}{dx} = \frac{d}{dy} = \tilde{J},\\ A &= -\left(p^3\frac{d^3}{dx^3} + 3p^2p_x\frac{d^2}{dx^2} + \left(-4pv + pp_x{}^2 + p^2p_{xx}\right)\frac{d}{dx} - 2pv_x\right)\\ &= -\frac{d^3}{dy^3} + 2(\tilde{v}\frac{d}{dy} + \frac{d}{dy}\tilde{v}) = \tilde{A}. \end{aligned} \tag{3.59}$$

Now $v_t = JG_n - p\beta v_x$ implies $\tilde{v}_s + \tilde{v}_y\widetilde{\frac{\partial y}{\partial t}} = \tilde{J}\tilde{G}_n - \tilde{\beta}\tilde{v}_y$ which in turn implies

$$\tilde{v}_s = \tilde{J}\tilde{G}_n. \tag{3.60}$$

Thus we have reduced this problem to the KdV-hierarchy, e.g. if $\tilde{v}(s,y)$ is a solution of the n-th KdV-equation then $v(t,x) = \tilde{v}(s,y(t,x))$ solves the n-th gKdV-equation.
A second way is to transform the Lax-equation

$$U\left(\frac{dl}{dt} - [\bar{b}_n, l]\right)U^{-1} = \frac{d\tilde{l}}{ds} - [\tilde{b}_n, \tilde{l}] - [\tilde{b} + \tilde{e}, \tilde{l}], \tag{3.61}$$

where

$$b = -\frac{1}{2}\left(p\beta\frac{d}{dx} + \frac{d}{dx}\beta p\right), \tag{3.62}$$

$$\tilde{b} = U^{-1}bU = -\frac{1}{2}\left(\tilde{\beta}\frac{d}{dy} + \frac{d}{dy}\tilde{\beta}\right), \tag{3.63}$$

$$\tilde{e} = -\frac{1}{2}\left(\tilde{\dot{y}}\frac{d}{dy} + \frac{d}{dy}\tilde{\dot{y}}\right). \tag{3.64}$$

Requiring $\frac{dl}{dt} = [\bar{b}_n, l]$, which implies $p_t = p^2\beta_x$, we infer $-\beta = \dot{y}$, $-\tilde{\beta} = \tilde{\dot{y}}$ and hence $\tilde{b} + \tilde{e} = 0$.
We conclude this section with the simple example of a one-soliton solution.

Example 3.9. Suppose p satisfies (3.41) and η is defined as in (3.55). Then

$$\mathrm{gKdV}(v_{sol}) = 0, \tag{3.65}$$

$$v_{sol}(t,x) = -2\kappa^2 \left(\mathrm{Cosh}\kappa \left(D + \eta(t,x_0) - 8\kappa^2 t + \int_{x_0}^{x} dx' \, \frac{1}{p(t,x')} \right)\right)^{-2}, \ \kappa, D \in \mathbb{R}. \tag{3.66}$$

Other solutions of the KdV-equation transform in an analogous way.

4. The modified gKdV-hierarchy for $r(t,x) = 1$

In this section we derive the modified version of the generalized KdV-hierarchy of Section 3 by invoking Miura's transformation. Throughout this section we shall use hypothesis

(H.4.1). Assume Hypotheses (H.2.1), (H.2.2) and (H.2.4) with $r(t,x) = 1$.

Consider the matrix differential expression

$$m(t) = \begin{pmatrix} 0 & a(t)^* \\ a(t) & 0 \end{pmatrix} \tag{4.1}$$

with (see (2.14), (2.15), (2.17), (2.18))

$$\varphi(t,x) = \phi(t,x) - \frac{1}{2} p_x(t,x), \tag{4.2}$$

$$a = p\frac{d}{dx} + \frac{p_x}{2} + \varphi, \quad a^* = -p\frac{d}{dx} - \frac{p_x}{2} + \varphi, \quad a_t = p_t \frac{d}{dx} + \frac{1}{2} p_{x,t} + \varphi_t, \tag{4.3}$$

$$l_1 = a^* a = -\frac{d}{dx} p^2 \frac{d}{dx} - \frac{1}{4} p_x^2 - \frac{1}{2} p p_{xx} + v_1, \tag{4.4}$$

$$l_2 = a a^* = -\frac{d}{dx} p^2 \frac{d}{dx} - \frac{1}{4} p_x^2 - \frac{1}{2} p p_{xx} + v_2. \tag{4.5}$$

Then Miura's transformation reads

$$v_j = \varphi + (-1)^j p \varphi_x, \qquad j = 1, 2. \tag{4.6}$$

Introducing

$$d_{2,l} = \delta_{2,l,1} \frac{d}{dx} + \frac{d}{dx} \delta_{2,l,1} - 4p^3 \frac{d^3}{dx^3} - \frac{d^3}{dx^3} 4p^3, \qquad l = 1, 2 \tag{4.7}$$

$$\delta_{2,1,1} \; = \; -\frac{1}{2}\delta p + 6p\varphi^2 - 6p^2\varphi_x + 23p{p_x}^2 + 8p^2 p_{xx}, \tag{4.8}$$

$$\delta_{2,2,1} \; = \; \delta_{2,1,1} + 12p^2\varphi_x \tag{4.9}$$

and

$$d_2 = \begin{pmatrix} d_{2,1} & 0 \\ 0 & d_{2,2} \end{pmatrix} \tag{4.10}$$

we get

$$[d_2, m] = \begin{pmatrix} 0 & d_{2,1}a^* - a^* d_{2,2} \\ d_{2,2}a - a d_{2,1} & 0 \end{pmatrix}, \tag{4.11}$$

$$d_{2,2}a - a d_{2,1} = p^2 \delta_x \frac{d}{dx} - \delta p \varphi_x + 12 p \varphi^2 \varphi_x + \delta_x p p_x$$
$$-2p{p_x}^2 \varphi_x + \frac{1}{2} p^2 \delta_{xx} - 6p^2 p_x \varphi_{xx} - 2p^2 p_{xx} \varphi_x - 2p^3 \varphi_{xxx}, \tag{4.12}$$
$$d_{2,1}a^* - a^* d_{2,2} = -p^2 \delta_x \frac{d}{dx} - \delta p \varphi_x + 12 p \varphi^2 \varphi_x - \delta_x p p_x$$
$$-2p{p_x}^2 \varphi_x - \frac{1}{2} p^2 \delta_{xx} - 6p^2 p_x \varphi_{xx} - 2p^2 p_{xx} \varphi_x - 2p^3 \varphi_{xxx}. \tag{4.13}$$

The modified nonlinear evolution equations determined by $\frac{d}{dt} m = [d_2, m]$ then read

$$p_t = p^2 \delta_x, \tag{4.14}$$
$$\varphi_t = 12 p \varphi^2 \varphi_x - (2p^3 \varphi_{xx})_x - (2{p_x}^2 + 2p p_{xx} + \delta) p \varphi_x. \tag{4.15}$$

Introducing the generalized modified Korteweg-de Vries functional by

$$\mathrm{gmKdV}(\varphi) = \varphi_t - 12 p \varphi^2 \varphi_x + (2p^3 \varphi_{xx})_x + (2{p_x}^2 + 2p p_{xx} + \delta) p \varphi_x \tag{4.16}$$

we obtain Miura's identity in the special case where $\beta = \delta$ in (3.11) and (4.14)

$$\mathrm{gKdV}(\varphi^2 + (-1)^j p \varphi_x) = \left[2\varphi + (-1)^j p \partial_x\right] \mathrm{gmKdV}(\varphi), \quad j = 1, 2, \quad \beta = \delta. \tag{4.17}$$

In order to derive the hierarchy we proceed as before. Let $\tilde{d}_n$ be the Lax differential expressions for the mKdV-hierarchy (in the variables (s, y))

$$\frac{d\tilde{m}}{ds} - [\tilde{d}_n, \tilde{m}] = \mathrm{mKdV}_n(\tilde{\varphi}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad n \in \mathbb{N}_0. \tag{4.18}$$

Formally define d_n by $W^{-1} \tilde{d}_n W$ (see (2.20) for the definition of W) then

$$\frac{dm}{dt} = [d_n, m], \qquad n \in \mathbb{N}_0 \tag{4.19}$$

yields $p_t = 0$ and the generalized mKdV-hierarchy $\varphi_t = [d_n, m]$. To include the time dependence of p we recall (3.39) and compute with

$$\bar{d}_n = d_n + d,$$
$$d = -\frac{1}{2}\left[p \delta \frac{d}{dx} + \frac{d}{dx} \delta p\right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \delta = \delta(p, p_x, p_{xx}, \ldots), \tag{4.20}$$

$$
\begin{aligned}
&\frac{dm}{ds} - [\bar{d}_n, m] = \frac{dm}{ds} - [d_n, m] - [d, m] \\
&= \begin{pmatrix} 0 & -p_t \frac{d}{dx} - \frac{1}{2} p_{x,t} + \varphi_t \\ p_t \frac{d}{dx} + \frac{1}{2} p_{x,t} + \varphi_t & 0 \end{pmatrix} \\
&- \begin{pmatrix} 0 & -p^2 \delta_x \frac{d}{dx} - \frac{1}{2} p^2 \delta_{xx} - p p_x \delta_x - \delta p \varphi_x \\ p_x^2 \frac{d}{dx} + \frac{1}{2} p^2 \delta_{xx} + p p_x \delta_x - \delta p \varphi_x & 0 \end{pmatrix} - [d_n, m].
\end{aligned} \tag{4.21}
$$

Requiring $\frac{dm}{dt} = [\bar{d}_n, m]$ then yields

$$
p_t = p^2 \delta_x, \tag{4.22}
$$

$$
\varphi_t = [d_n, m] - \delta p \varphi_x, \qquad n \in \mathbb{N}_0. \tag{4.23}
$$

Introducing

$$
\mathrm{gmKdV}_n(\varphi) = \varphi_t - [d_n, m] + \delta p \varphi_x, \qquad n \in \mathbb{N}_0 \tag{4.24}
$$

Miura's identity then reads in the special case where $\beta = \delta$ in (3.41) and (4.22)

$$
\mathrm{gKdV}_n(\varphi^2 + (-1)^j p \varphi_x) = \left[2\varphi + (-1)^j p \partial_x\right] \mathrm{gmKdV}_n(\varphi), \quad j = 1, 2, \ n \in \mathbb{N}_0, \ \beta = \delta \tag{4.25}
$$

and we emphasize that for $\beta(t,x) = \delta(t,x)$ the "modified" equation for p in (4.22) is identical to its "unmodified" version (3.41).

5. The HD-equation

Due to its importance we now isolate the Harry Dym (HD)-equation[2] as a special case of Sections 3 and 4. In accordance with our earlier comments on the HD-equation, we shall use Hypothesis (H.5.1) throughout this section:

(H.5.1). Assume Hypotheses (H.2.1), (H.2.2), and (H.2.4) with $r(t,x) = 1$, $q(t,x) = 0$, $\varphi(t,x) = -\frac{1}{2} p_x(t,x)$ (*i.e.*, $\phi(t,x) = 0$).

Introducing $m(t), a(t), a(t)^*, l_j(t), j = 1, 2$ in (4.1), (4.3)-(4.5) with $\varphi(t,x) = -\frac{1}{2} p_x(t,x)$ yields the HD-Lax pairs $(l_j, b_{2,j}), j = 1, 2$, where

$$
l_1 = a^* a = -\frac{d}{dx} p^2 \frac{d}{dx}, \tag{5.1}
$$

$$
b_{2,1} = -8p^3 \frac{d^3}{dx^3} - 36 p^2 p_x \frac{d^2}{dx^2} - \left(24 p {p_x}^2 + 12 p^2 p_{xx}\right) \frac{d}{dx}, \tag{5.2}
$$

$$
l_2 = a a^* = -p \frac{d^2}{dx^2} p, \tag{5.3}
$$

$$
b_{2,2} = -8p^3 \frac{d^3}{dx^3} - 36 p^2 p_x \frac{d^2}{dx^2} - \left(24 p {p_x}^2 + 24 p^2 p_{xx}\right) \frac{d}{dx} - 12 p p_x p_{xx} - 6 p^2 p_{xxx}. \tag{5.4}
$$

[2] See the recently discovered close connection [16] between the HD-equation and the Saffman-Taylor Problem.

Since

$$[b_{2,1}, l_1] = \frac{d}{dx} 4p^4 p_{xxx} \frac{d}{dx} \tag{5.5}$$

and

$$\begin{aligned}[b_{2,2}, l_2] &= 4p^4 p_{xxx} \frac{d^2}{dx^2} + \left(16p^3 p_x p_{xxx} + 4p^4 p_{xxxx}\right) \frac{d}{dx} \\ &\quad +12p^2 {p_x}^2 p_{xxx} + 8p^3 p_{xx} p_{xxx} + 12p^3 p_x p_{xxxx} + 2p^4 p_{xxxxx} \end{aligned} \tag{5.6}$$

$\frac{dl_j}{dt} = [b_{2,j}, l_j], j = 1, 2$ are both equivalent to the HD-equation

$$p_t = -2p^3 p_{xxx}. \tag{5.7}$$

Similarly (see (4.7)-(4.13))

$$d_{2,j} = \delta_{2,j,1} \frac{d}{dx} + \frac{d}{dx} \delta_{2,j,1} - 4p^3 \frac{d^3}{dx^3} - \frac{d^3}{dx^3} 4p^3, \qquad j = 1, 2,$$

$$\delta_{2,1,1} = 24p{p_x}^2 + 12p^2 p_{xx}, \qquad \delta_{2,2,1} = 24p{p_x}^2 + 6p^2 p_{xx}, \tag{5.8}$$

$$d_2 = \begin{pmatrix} d_{2,1} & 0 \\ 0 & d_{2,2} \end{pmatrix}, \quad [d_2, m] = \begin{pmatrix} 0 & d_{2,1}a^* - a^* d_{2,2} \\ d_{2,2}a - a d_{2,1} & 0 \end{pmatrix} \tag{5.9}$$

yield

$$\frac{d}{dt} m - [d_2, m] = \begin{pmatrix} 0 & -(p_t + 2p^3 p_{xxx}) \frac{d}{dx} - p_{xt} - 2\left(p^3 p_{xxx}\right)_x \\ (p_t + 2p^3 p_{xxx}) \frac{d}{dx} & 0 \end{pmatrix}. \tag{5.10}$$

Thus $\frac{d}{dt} m = [d_2, m]$ is also equivalent to the HD-equation (5.7) in agreement with our comment following (4.25).

An auto-Bäcklund transformation for the HD-equation (5.7) can be obtained by the following sequence of transformations [22]:

$$p_t = -2p^3 p_{xxx} \tag{5.11}$$

is transformed by

$$\begin{aligned} &\rho = \frac{1}{p}, \quad s = t, \quad \xi = \int_{x_0}^{x} \rho(t, x')^2 dx' + \zeta(t), \quad \zeta(t) = 4 \int^{t} dt' p_{xx}(t', x_0), \\ &\hat{\rho}(s, \xi) = \rho(t, x(s, \xi)), \qquad \frac{\partial}{\partial x} = \hat{\rho}^2 \frac{\partial}{\partial \xi}, \\ &\frac{\partial \xi}{\partial t} = 4p_{xx}(t, x) - 4p_{xx}(t, x_0) + \zeta_t(t) = 4p_{xx}(t, x), \quad \widehat{\frac{\partial \xi}{\partial t}} = -4\hat{\rho}^2 \hat{\rho}_{\xi\xi} \end{aligned} \tag{5.12}$$

into

$$\hat{\rho}_s + \hat{\rho}_\xi \widehat{\frac{\partial \xi}{\partial t}} = -2\hat{\rho}\left(\hat{\rho}^2 \hat{\rho}_{\xi\xi}\right)_\xi , \tag{5.13}$$

and finally into

$$\hat{\rho}_s = -2\hat{\rho}^3 \hat{\rho}_{\xi\xi\xi}. \tag{5.14}$$

(This transformation corresponds to the transformation $\tilde{\varphi} \to -\tilde{\varphi}$, resp. $\tilde{p} \to \tilde{p}^{-1}$ in (5.21), (5.23).)

The following example shows that this transformation also generates singular HD-solutions where p violates (H.5.1).

Example 5.2. Let $p(t,x) = \alpha^2 x^2,\ \alpha \in \mathbb{R}$ which fulfills the HD-equation. Then

$$\rho = \frac{1}{p} = \frac{1}{\alpha^2 x^2}, \qquad \text{implies} \qquad \alpha^4 \xi = -\frac{1}{3}x^{-3} + \frac{1}{3}x_0^{-3} + \alpha^4 \zeta(t). \tag{5.15}$$

Since $p_{xx}(t,x_0) = 2\alpha^2$ we choose $x_0 = -\infty$ and by $\zeta(t) = 8\alpha^2 t$ get

$$x = (24\alpha^6 s - 3\alpha^4 \xi)^{-1/3} \tag{5.16}$$

and

$$\hat{\rho}(s,\xi) = (24\alpha^3 s - 3\alpha\xi)^{2/3} \tag{5.17}$$

which fulfills the HD-equation too.

In the following we reconsider the construction of solutions of the HD-equation from solutions of the KdV and mKdV-equation. The link between the HD-equation and (m)KdV-equation has been discussed by a variety of authors [4], [8], [12], [13], [15], [17], [18], [21], [22], [23], [27]. Here we shall recover these results very naturally within our approach.

As is well known [9], [10], solutions of the KdV-equation

$$\tilde{v}_s - 12\tilde{v}\tilde{v}_y + 2\tilde{v}_{yyy} = 0 \tag{5.18}$$

yield solutions of the mKdV-equation

$$\tilde{\varphi}_s - 12\tilde{\varphi}^2\tilde{\varphi}_y + 2\tilde{\varphi}_{yyy} = 0 \tag{5.19}$$

satisfying

$$\tilde{v}_j = \tilde{\varphi}^2 + (-1)^j \tilde{\varphi}_y, \quad j = 1,2, \tag{5.20}$$

where $\tilde{\varphi}$ is given by

$$\tilde{\varphi}(s,y) = \partial_y \ln \tilde{\psi}(s,y), \tag{5.21}$$

and $\tilde{\psi}$ is assumed to satisfy

$$\tilde{l}(s)\tilde{\psi}(s) = 0, \qquad (\partial_s - \tilde{b}_2(s))\tilde{\psi}(s) = 0 \tag{5.22}$$

with $\tilde{l}, \tilde{b}_2$ defined in (3.21), (3.26). The ansatz

$$\tilde{p}_\pm(s,y) = \left[\tilde{\psi}(s,y)\right]^{\pm 2}, \tag{5.23}$$

as suggested by the relation $\tilde{\varphi} = -\frac{\tilde{p}_y}{2\tilde{p}}$ (see (5.32)) and the invariance of the mKdV-equation with respect to $\tilde{\varphi} \to -\tilde{\varphi}$, then yields solutions of the transformed Harry Dym (tHD)-equation

$$\tilde{p}_s - 6\frac{\tilde{p}_y\tilde{p}_{yy}}{\tilde{p}} + 3\frac{\tilde{p}_y^3}{\tilde{p}^2} + 2\tilde{p}_{yyy} = 0. \tag{5.24}$$

Note that if $\tilde{p}$ solves the tHD-equation, then $\tilde{p}^{-1}$ and $const \cdot \tilde{p}$ solve the tHD-equation too. A further transformation of the variables

$$t = s, \qquad x = \int_{y_0(s)}^{y} \tilde{p}_\pm(s,y')dy' + \eta_\pm(t), \qquad p_\pm(t,x) = \tilde{p}_\pm(s, y(t,x)), \tag{5.25}$$

with the condition

$$\eta'_\pm(s) - y'_0(s)\tilde{p}_\pm(s,y_0(s)) + 2\tilde{p}_{\pm,yy}(s,y_0(s)) - 3\frac{\tilde{p}_{\pm,y}(s,y_0(s))^2}{\tilde{p}_\pm(s,y_0(s))} = 0, \tag{5.26}$$

then yields solutions of the HD-equation

$$p_t + 2p^3 p_{xxx} = 0. \tag{5.27}$$

The simplest way to satisfy (5.26) is to choose $y'_0(s) = 0$ and take

$$\eta_\pm(s) = \int^s ds' \left(-2\tilde{p}_{\pm,yy}(s',y_0) + 3\frac{\tilde{p}_{\pm,y}(s',y_0)^2}{\tilde{p}_\pm(s',y_0)}\right). \tag{5.28}$$

Conversly, in order to transform the HD-equation (5.27) back to the tHD-equation (5.24) we use the transformation (see (2.3)) of the variables

$$s = t, \qquad y = \int_{x_0(t)}^{x} p(t,x')^{-1}dx' + \eta(s), \qquad \tilde{p}(s,y) = p(t, x(s,y)) \tag{5.29}$$

with

$$\eta'(t) - x'_0(t)p(t,x_0(t))^{-1} - 2p(t,x_0(t))p_{xx}(t,x_0(t)) + p_x(t,x_0(t))^2 = 0. \tag{5.30}$$

E.g., if $x'_0(t) = 0$ then

$$\eta(t) = \int^t dt' \left(2p(t',x_0)p_{xx}(t',x_0) - p_x(t',x_0)^2\right). \tag{5.31}$$

Remark 5.3. The conclusion following (5.10) and the results in [17] as presented above clearly point out that the Dirac-type differential expression

$$m = \begin{pmatrix} 0 & -p\frac{d}{dx} - \frac{1}{2}p_x \\ p\frac{d}{dx} + \frac{1}{2}p_x & 0 \end{pmatrix}, \quad \tilde{m} = \begin{pmatrix} 0 & -\frac{d}{dy} + \tilde{\varphi} \\ \frac{d}{dy} + \tilde{\varphi} & 0 \end{pmatrix}, \quad \tilde{\varphi} = -\frac{\tilde{p}_y}{2\tilde{p}} \tag{5.32}$$

is the natural choice in a Lax pair for the HD-equation.

This approach can most effectively be combined with Hirota's τ-function formalism [14] (see Appendix A) as will be shown below.

Assume that $\tilde{\psi}_2$ is a solution of

$$\tilde{l}_2(s)\tilde{\psi}_2(s) = 0, \quad \text{i. e., } a^*(s)\tilde{\psi}_2(s) = 0, \ s \in \mathbb{R} \tag{5.33}$$

and

$$(\partial_s - \tilde{b}_2(s))\tilde{\psi}_2(s) = 0 \tag{5.34}$$

of the type

$$\tilde{\psi}_2(s,y) = e^{Dy+Es}\,\frac{\tau_1(s,y)}{\tau_2(s,y)}, \quad (s,y) \in \mathbb{R}^2, \quad D, E \in \mathbb{R}, \quad \tau_j \in C^\infty(\mathbb{R}^2), \quad j = 1,2. \tag{5.35}$$

Making the ansatz

$$\tilde{v}_2(s,y) = C - 2\partial_y^2 \ln \tau_2(s,y), \qquad C \in \mathbb{R} \tag{5.36}$$

one infers

$$\tilde{v}_1(s,y) = C - 2\partial_y^2 \ln \tau_1(s,y), \tag{5.37}$$

$$C - D^2 = 2D\frac{\tau_{1,y}}{\tau_1} - 2D\frac{\tau_{2,y}}{\tau_2} - 2\frac{\tau_{1,y}\tau_{2,y}}{\tau_1\tau_2} + \frac{\tau_{1,yy}}{\tau_1} + \frac{\tau_{2,yy}}{\tau_2}, \tag{5.38}$$

$$\tilde{\varphi} = \partial_y \ln \tilde{\psi}_2(s,y) = D + \frac{\tau_{1,y}}{\tau_1} - \frac{\tau_{2,y}}{\tau_2}. \tag{5.39}$$

By the ansatz (5.23) we get

$$\tilde{p}_\pm(s,y) = \left[\tilde{\psi}_2(s,y)\right]^{\pm 2} = \left[e^{Dy+Es}\left(\frac{\tau_1(s,y)}{\tau_2(s,y)}\right)\right]^{\pm 2} \tag{5.40}$$

for solutions of the tHD-equation (5.24).

A further variable transformation then yields solutions p of the HD-equation as described in (5.25)-(5.28).

We illustrate formula (5.40) with the help of soliton and quasi-periodic finite-gap solutions.

Example 5.4. (N-soliton solutions)
Let

$$\tau_2^N(s,y) = \det\left[1 + C_2^N(s,y)\right], \quad N \in \mathbb{N}, \tag{5.41}$$

$$C_2^N(s,y) = \left[\frac{c_{2,l}c_{2,m}}{\kappa_l + \kappa_m} e^{-(\kappa_l+\kappa_m)(y+12V_\infty s)+8(\kappa_l^3+\kappa_m^3)s}\right]_{l,m=1}^N, \quad c_{2,l} > 0, \quad 1 \le l \le N, \tag{5.42}$$

$$0 < \kappa_N < \kappa_{N-1} < \ldots < \kappa_1 \le V_\infty^{1/2} \tag{5.43}$$

describe the N-soliton KdV-solutions $\tilde{v}_2^N(s,y)$,

$$\tilde{v}_2^N(s,y) = V_\infty - 2\partial_y^2 \ln \tau_2^N(s,y). \tag{5.44}$$

We distinguish two cases [10].
(i). $V_\infty = \kappa_1^2$ (the critical case in the terminology of [10]). This yields a unique (N-1)-soliton KdV-solution $\tilde{v}_1^{(N-1)}$ given by

$$\tilde{v}_1^{(N-1)}(s,y) = V_\infty - 2\partial_y^2 \ln \tau_1^{(N-1)}(s,y), \tag{5.45}$$

$$\tau_1^{(N-1)}(s,y) = \det\left[1 + C_1^{(N-1)}(s,y)\right], \tag{5.46}$$

$$C_1^{(N-1)}(s,y) = \left[\left(\frac{(\kappa_1+\kappa_l)(\kappa_1+\kappa_m)}{(\kappa_1-\kappa_l)(\kappa_1-\kappa_m)}\right)^{1/2} C_{2,l,m}^N(s,y)\right]_{l,m=2}^N, \qquad N \ge 2, \tag{5.47}$$

$$C_1^{(0)}(s,y) = 0, \quad N = 1, \tag{5.48}$$

$$C = \kappa_1^2, \quad D = -\kappa_1, \quad E = -4\kappa_1^3. \tag{5.49}$$

(ii). $V_\infty > \kappa_1^2$ (the subcritical case in the terminology of [10]). This yields KdV-solutions $\tilde{v}_{1,\sigma}^N$, $\sigma = \pm 1$

$$\tilde{v}_{1,\sigma}^N(s,y) = V_\infty - 2\partial_y^2 \ln \tau_{1,\sigma}^N(s,y), \tag{5.50}$$

$$\tau_{1,\sigma}^N(s,y) = \det\left[1 + C_{1,\sigma}^N(s,y)\right], \tag{5.51}$$

$$C_{1,\sigma}^N(s,y) = \left[\left(\frac{(\sigma V_\infty^{1/2}+\kappa_l)(\sigma V_\infty^{1/2}+\kappa_m)}{(\sigma V_\infty^{1/2}-\kappa_l)(\sigma V_\infty^{1/2}-\kappa_m)}\right)^{1/2} C_{2,l,m}^N(s,y)\right]_{l,m=1}^N, \tag{5.52}$$

$$C = V_\infty, \quad D = -\sigma V_\infty^{1/2}, \quad E = -4\sigma V_\infty^{3/2}, \quad \sigma = \pm 1. \tag{5.53}$$

In both cases one reads off the corresponding mKdV-solutions $\tilde{\varphi}_0$, resp. $\tilde{\varphi}_\pm$ from (5.39) and obtains the associated solution $\tilde{p}_{0,\pm}$ resp. $\tilde{p}_{\pm,\sigma}$ of the tHD-equation from (5.40) as follows:

(i). $V_\infty = \kappa_1^2$ (critical)

$$\tilde{\varphi}_0(s,y) = -\kappa_1 - \partial_y \ln\left(\frac{\det(1+C_2^N(s,y))}{\det(1+C_1^{(N-1)}(s,y))}\right). \tag{5.54}$$

Then we get from (5.40)

$$\tilde{p}_{\pm,0}(s,y) = \left[\tilde{\psi}_{2,0}^N(s,y)\right]^{\pm 2} = \left[e^{-\kappa_1 y - 4\kappa_1^3 s}\left(\frac{\det(1+C_1^{(N-1)}(s,y))}{\det(1+C_2^N(s,y))}\right)\right]^{\pm 2}. \tag{5.55}$$

In the special case where $N=1$, $c_{2,1}^2 = 2\kappa_1$ one obtains

$$C_2^1(s,y) = e^{-2\kappa_1 y - 8\kappa_1^3 s}, \tag{5.56}$$

$$\tilde{v}_2^1(s,y) = \kappa_1^2 - 2\kappa_1^2\left[\cosh(\kappa_1 y + 4\kappa_1^3 s)\right]^{-2}, \tag{5.57}$$

$$\tilde{\varphi}_0(s,y) = -\kappa_1 \tanh(\kappa_1 y + 4\kappa_1^3 s), \tag{5.58}$$

$$\tilde{p}_{\pm,0}(s,y) = \left[2\cosh(\kappa_1 y + 4\kappa_1^3 s)\right]^{\mp 2}. \tag{5.59}$$

For $\tilde{p}_{+,0}$ we take $y_0 = -\infty, \eta_+ = 0$ and get

$$x = \frac{1}{4}\int_{-\infty}^{y} dy' \frac{1}{(\cosh(\kappa_1 y' + 4\kappa_1^3 s))^2} = \frac{1}{4\kappa_1}\left(\tanh(\kappa_1 y + 4\kappa_1^3 s) + 1\right), \tag{5.60}$$

$$y = \frac{1}{\kappa_1}\operatorname{arctanh}(4\kappa_1 x - 1) - 4\kappa_1^2 s. \tag{5.61}$$

Hence

$$p_{+,0}(t,x) = \kappa_1 x(2 - 4\kappa_1 x), \qquad x \in \left(0, \frac{1}{2\kappa_1}\right). \tag{5.62}$$

(ii). $V_\infty > \kappa_1^2$ (subcritical)

$$\tilde{\varphi}_\sigma(s,y) = -\sigma V_\infty^{1/2} - \partial_y \ln\left(\frac{\det(1+C_2^N(s,y))}{\det(1+C_{1,\sigma}^N(s,y))}\right), \qquad \sigma = \pm 1, \tag{5.63}$$

$$\tilde{p}_{\pm,\sigma}(s,y) = \left[\tilde{\psi}_{2,\sigma}^N(s,y)\right]^{\pm 2} = \left[e^{-\sigma V_\infty^{1/2} y - 4\sigma V_\infty^{3/2} s}\left(\frac{\det(1+C_{1,\sigma}^N(s,y))}{\det(1+C_2^N(s,y))}\right)\right]^{\pm 2}, \ \sigma = \pm 1. \tag{5.64}$$

Remark 5.5. The critical and subcritical cases in Example 5.4 exhibit a very different qualitative behavior if $\tilde{p}(s,y)$ is further transformed into HD-solutions $p(t,x)$. In fact, since

$$\lim_{y\to\pm\infty} \tilde{\varphi}_0(s,y) = \mp V_\infty^{1/2} = \mp\kappa_1, \tag{5.65}$$

$$\lim_{y\to\pm\infty} \tilde{\varphi}_\sigma(s,y) = -\sigma V_\infty^{1/2}, \tag{5.66}$$

one infers from (5.54) resp. (5.55) and (5.63) resp. (5.64) that

$$\tilde{p}_{+,0}(s,y) \overset{y\to\pm\infty}{=} O(e^{\mp 2\kappa_1 y}), \tag{5.67}$$
$$\tilde{p}_{-,0}(s,y) \overset{y\to\pm\infty}{=} O(e^{\pm 2\kappa_1 y}), \tag{5.68}$$
$$\tilde{p}_{+,\sigma}(s,y) \overset{y\to\pm\infty}{=} O(e^{-2\sigma V_\infty^{1/2} y}), \tag{5.69}$$
$$\tilde{p}_{-,\sigma}(s,y) \overset{y\to\pm\infty}{=} O(e^{+2\sigma V_\infty^{1/2} y}) \tag{5.70}$$

and hence
(i). $p_{+,0}(t,x)$ is defined for x on a finite interval I. E. g. if $y_0 = -\infty, \eta_+ = 0$ in (5.25) then $I = (0, c_{2,1}^{-2})$ since one can show that

$$\int_{-\infty}^{\infty} \tilde{p}_{+,0}(s,y)dy = \int_{-\infty}^{\infty} \left[\tilde{\psi}_{2,0}^{N}(s,y)\right]^2 dy = c_{2,1}^{-2}. \tag{5.71}$$

(This case is further illustrated in Appendix B.)
(ii). $p_{-,0}(t,x)$ is defined for $x \in \mathbb{R}$.
(iii). $p_{+,\sigma}(t,x)$ with $y_0 = \sigma\infty,\ \eta_+ = 0$ is defined for $x \in (0, -\sigma\infty),\ \sigma = \pm 1$.
(iv). $p_{-,\sigma}(t,x)$ with $y_0 = -\sigma\infty,\ \eta_- = 0$ is defined for $x \in (0, \sigma\infty),\ \sigma = \pm 1$.

Finally we turn to quasi-periodic finite-gap solutions.

Example 5.6. Let

$$\tau_2(s,y) = \Theta\left(\underline{\xi}_{P_0} - \underline{A}_{P_0}(P_\infty) + \underline{\alpha}_{P_0}(\underline{\mu}(0,0)) + \frac{y}{2\pi}\underline{U}_0 + \frac{12s}{\pi}\underline{U}_2\right), \tag{5.72}$$

where Θ denotes Riemann's theta function associated with the hyperelliptic curve

$$R_0(z)^{1/2} = \left[\prod_{n=0}^{2g}(E_n - z)\right]^{1/2}, \quad 0 \le E_0 < E_1 < \ldots < E_{2g}, \quad g \in \mathbb{N} \tag{5.73}$$

and an appropriate homology basis $\{a_j, b_j\}_{j=1}^{g}$ with intersection matrix $a_j \circ b_l = \delta_{j,l}$. Here $\underline{\xi}_{P_0}$ is Riemann's vector with base point $P_0 = (E_0, 0), P_\infty = (\infty, \infty)$ the point at infinity, $\underline{A}_{P_0}(P)$ denotes the corresponding Abel map, $\underline{\mu}(0,0) = (\mu_1(0,0), \ldots, \mu_g(0,0))$ is the Dirichlet divisor at $t = 0, x = 0, \underline{\alpha}_{P_0}(P_1, \ldots, P_g) = \sum_{j=1}^{g} \underline{A}_{P_0}(P_j)$ and $\underline{U}_0,\ \underline{U}_2$ are b-periods of normalized differentials of the second kind $\omega_0^{(2)}, \omega_2^{(2)}$ with a prescribed pole of order two respectively four at P_∞. The corresponding quasi-periodic finite-gap KdV-solutions are then given by

$$\tilde{v}_2(s,y) = \Lambda - 2\partial_y^2 \ln \tau_2(s,y), \tag{5.74}$$

where Λ is a constant only depending on the underlying hyperelliptic curve. (See e.g. [11] for a complete discussion of such quasi-periodic finite-gap solutions.) Next we introduce

$$\begin{aligned} \tau_{1,\pm 1}(\lambda, s, y) &= \Theta\left(\underline{\xi}_{P_0} \mp \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\underline{\mu}(0,0)) + \frac{y}{2\pi}\underline{U}_0 + \frac{12s}{\pi}\underline{U}_2\right), \\ & \qquad P = \left(\lambda, \lim_{\epsilon\downarrow 0} R_0(\lambda + i\epsilon)^{1/2}\right), \quad \lambda \in \mathbb{R}, && (5.75) \\ \tau_{1,\pm 1}(s, y) &= \tau_{1,\pm 1}(0, s, y), && (5.76) \\ \tilde{\psi}_{2,\pm 1}(s, y) &= e^{\mp iy \int_{P_0}^{P} \omega_0^{(2)} \mp 24s \int_{P_0}^{P} \omega_2^{(2)}} \frac{\tau_{1,\pm 1}(s, y)}{\tau_2(s, y)} && (5.77) \end{aligned}$$

and the quasi-periodic finite-gap KdV solutions

$$\tilde{v}_{1,\pm 1}(s, y) = \Lambda - 2\partial_y^2 \ln \tau_{1,\pm 1}(s, y). \tag{5.78}$$

Again we distinguish two cases [11].

(i). $E_0 = 0$ (the critical case). Then

$$\tilde{\psi}_{2,+1}(s, y) = \tilde{\psi}_{2,-1}(s, y) \equiv \tilde{\psi}_{2,0}(s, y), \quad \tilde{v}_{1,+1}(s, y) = \tilde{v}_{1,-1}(s, y) \equiv \tilde{v}_{1,0}(s, y), \tag{5.79}$$

and therefore

$$\tilde{p}_{\pm,0}(s, y) = \left[\tilde{\psi}_{2,0}(s, y)\right]^{\pm 2} \tag{5.80}$$

satisfies the tHD-equation (5.24). Since in this case $\tilde{\psi}_{2,0}$ is periodic in y, a further transformation to $p_{\pm,0}(t, x)$ as in (5.25) shows that in the critical case, x varies on the whole real line $\mathbb{R}$.

(ii). $E_0 > 0$ (the subcritical case). Then again

$$\tilde{p}_{\pm,\sigma}(s, y) = \left[\tilde{\psi}_{2,\sigma}(s, y)\right]^{\pm 2}, \qquad \sigma = \pm 1 \tag{5.81}$$

satisfy the tHD-equation (5.24). Since in this case $\tilde{\psi}_{2,\pm 1}(s) \in L^2((R, \pm\infty); dy)$, $\left[\tilde{\psi}_{2,\pm 1}(s)\right]^{-1} \in L^2((R, \mp\infty); dy)$ for all $R \in \mathbb{R}$, a further transformation to $p_{\pm,\sigma}(t, x))$ as in (5.25) shows that in the subcritical case, x varies on half-lines.

Remark 5.7. What we called the transformed Harry Dym (tHD)-equation in (5.24) is the special case $\lambda = 0$ of the following equation

$$\tilde{p}_s - 6\frac{\tilde{p}_y \tilde{p}_{yy}}{\tilde{p}} + 3\frac{\tilde{p}_y^3}{\tilde{p}^2} + 2\tilde{p}_{yyy} + 3\lambda \tilde{p}_y = 0, \qquad \lambda \in \mathbb{R} \tag{5.82}$$

studied in [7], [8], [13], [27] and called the "interacting soliton equation" in [3]. Equation (5.82) (like (5.24)) has the property that if $\tilde{p}$ is a solution, so is $\tilde{p}^{-1}$ and $const \cdot \tilde{p}$. Applying the variable transformation (5.25), (5.26) yields

$$p_t + 2p^3 p_{xxx} + 3\lambda p p_x = 0 \tag{5.83}$$

generalizing the HD-equation (5.27). However, a simple Galilei transformation

$$(s, y) \to (s, z = y - 3\lambda s)$$

reduces equation (5.82) to the case $\lambda = 0$ due to the identity

$$\tilde{p}_s - 6\frac{\tilde{p}_y \tilde{p}_{yy}}{\tilde{p}} + 3\frac{\tilde{p}_y^3}{\tilde{p}^2} + 2\tilde{p}_{yyy} + 3\lambda\tilde{p}_y = P_s - 6\frac{P_z P_{zz}}{P} + 3\frac{P_z^3}{P^2} + 2P_{zzz}, \quad \tilde{p}(s, y) = P(s, z). \tag{5.84}$$

Consequently, our methods immediately extend to equation (5.83).

Acknowledgements.
K. Unterkofler would like to thank the Department of Mathematics of the University of Missouri, Columbia for the hospitality extended to him. He also gratefully acknowledges financial support by the Fonds zur Förderung der wissenschaftlichen Forschung in Österreich by an E. Schrödinger Fellowship and by Project No. P7425 and additional support by the Bundeswirschaftskammer, Österreich.

Appendix A: τ-functions and commutation methods

Since the explicit change of the variables in (2.3), (5.25) is possible only in special cases we found it useful to develop the τ-function method for the gKdV-equation directly.

Suppose that

$$0 < p, \tau_j \in C^\infty(\mathbb{R}^2), \quad j = 1, 2 \tag{A.1}$$

and introduce

$$l_2(t) = -p(t, x)\frac{d^2}{dx^2}p(t, x) + v_2(t, x) + \frac{1}{2}p(t, x)p_{xx}(t, x) - \frac{1}{4}p_x(t, x)^2, \quad (t, x) \in \mathbb{R}^2, \tag{A.2}$$

where v_2 is of the type

$$v_2(t, x) = C - 2p(t, x)\partial_x\Big[p(t, x)\partial_x \ln \tau_2(t, x)\Big], \qquad C \in \mathbb{C}. \tag{A.3}$$

Moreover, assume ψ_2 to be a solution of

$$l_2(t)\psi_2(t) = 0, \quad (\partial_t - b_2(t))\psi_2(t) = 0 \tag{A.4}$$

of the type

$$\psi_2(t, x) = p(t, x)^{-1/2} e^{D\int_{x_0}^x dx'\, p(t, x')^{-1} + Et}\, \frac{\tau_1(t, x)}{\tau_2(t, x)}, \quad D, E \in \mathbb{C}. \tag{A.5}$$

Define

$$\varphi(t,x) \quad = \quad p(t,x)\partial_x \ln \psi_2(t,x) + \frac{1}{2}p_x(t,x) = D + p\frac{\tau_{1,x}}{\tau_1} - p\frac{\tau_{2,x}}{\tau_2}. \tag{A.6}$$

and

$$a(t) = p(t,x)\frac{d}{dx} + \varphi(t,x) + \frac{1}{2}p_x(t,x), \tag{A.7}$$

$$a(t)^{+} = -p(t,x)\frac{d}{dx} + \varphi(t,x) - \frac{1}{2}p_x(t,x). \tag{A.8}$$

Then

$$l_2(t) = a(t)a(t)^{+}. \tag{A.9}$$

Next consider

$$l_1(t) = a(t)^{+}a(t), \tag{A.10}$$

then

$$l_1(t) = -\frac{d}{dx}p(t,x)^2\frac{d}{dx} + v_1(t,x) - \frac{1}{2}p(t,x)p_{xx}(t,x) - \frac{1}{4}p_x(t,x)^2, \tag{A.11}$$

where

$$v_j = \varphi^2 + (-1)^j p\varphi_x, \quad j = 1,2. \tag{A.12}$$

Moreover,

$$\begin{aligned} v_2 \quad &= \quad \varphi^2 + p\partial_x\varphi = D^2 + pp_x\left(\frac{\tau_{1,x}}{\tau_1} - \frac{\tau_{2,x}}{\tau_2}\right) + 2Dp\left(\frac{\tau_{1,x}}{\tau_1} - \frac{\tau_{2,x}}{\tau_2}\right) \\ &\quad +p^2\left(-\frac{2\tau_{1,x}\tau_{2,x}}{\tau_1\tau_2} + \frac{2{\tau_{2,x}}^2}{{\tau_2}^2} + \frac{\tau_{1,xx}}{\tau_1} - \frac{\tau_{2,xx}}{\tau_2}\right) \\ &\quad = C - 2p\partial_x\Big[p\partial_x \ln \tau_2\Big] = C - 2pp_x\frac{\tau_{2,x}}{\tau_2} + 2p^2\left(\frac{{\tau_{2,x}}^2}{{\tau_2}^2} - \frac{\tau_{2,xx}}{\tau_2}\right), \end{aligned} \tag{A.13}$$

$$\begin{aligned} v_1 \quad &= \quad \varphi^2 - p\partial_x\varphi = D^2 + pp_x\left(-\frac{\tau_{1,x}}{\tau_1} + \frac{\tau_{2,x}}{\tau_2}\right) + 2Dp\left(\frac{\tau_{1,x}}{\tau_1} - \frac{\tau_{2,x}}{\tau_2}\right) \\ &\quad +p^2\left(\frac{2{\tau_{1,x}}^2}{{\tau_1}^2} - \frac{2\tau_{1,x}\tau_{2,x}}{\tau_1\tau_2} - \frac{\tau_{1,xx}}{\tau_1} + \frac{\tau_{2,xx}}{\tau_2}\right), \end{aligned} \tag{A.14}$$

$$\begin{aligned} &v_2 - v_1 = 2pp_x\left(\frac{\tau_{1,x}}{\tau_1} - \frac{\tau_{2,x}}{\tau_2}\right) + 2p^2\left(-\frac{{\tau_{1,x}}^2}{{\tau_1}^2} + \frac{{\tau_{2,x}}^2}{{\tau_2}^2} + \frac{\tau_{1,xx}}{\tau_1} - \frac{\tau_{2,xx}}{\tau_2}\right) \\ &= 2p\partial_x\Big[p\partial_x \ln \tau_1\Big] - 2p\partial_x\Big[p\partial_x \ln \tau_2\Big]. \end{aligned} \tag{A.15}$$

Thus

$$v_1(x,t) \quad = \quad C - 2p\partial_x\Big[p\partial_x \ln \tau_1\Big] \tag{A.16}$$

and

$$C - D^2 = pp_x\left(\frac{\tau_{1,x}}{\tau_1} + \frac{\tau_{2,x}}{\tau_2}\right) + 2Dp\left(\frac{\tau_{1,x}}{\tau_1} - \frac{\tau_{2,x}}{\tau_2}\right) + p^2\left(\frac{\tau_{1,xx}}{\tau_1} + \frac{\tau_{2,xx}}{\tau_2} - 2\frac{\tau_{1,x}\tau_{2,x}}{\tau_1\tau_2}\right). \tag{A.17}$$

Appendix B: A self-adjoint operator on a finite interval having nontrivial absolutely continuous spectrum

In this appendix we further illustrate Remark 5.5 and generate a simple nontrivial example of a self-adjoint operator on a finite interval with a nonempty absolutely continuous component in its spectrum as follows: Consider the one-soliton operator $\tilde{L}$ in $L^2(\mathbb{R}; dy)$

$$\tilde{L}f = \tilde{l}f, \quad f \in \mathcal{D}(\tilde{L}) = H^2(\mathbb{R}), \tag{B.1}$$

where

$$\tilde{l} = -\frac{d^2}{dy^2} + \kappa_1^2 - 2\kappa_1^2[\cosh(\kappa_1 y)]^{-2}, \quad y \in \mathbb{R}. \tag{B.2}$$

(This corresponds to (5.57) at $s = 0$.) Then the spectrum of $\tilde{L}$ is given by

$$\sigma(\tilde{L}) = \{0\} \cup [\kappa_1^2, \infty), \tag{B.3}$$

$$\sigma_{ess}(\tilde{L}) = \sigma_{ac}(\tilde{L}) = [\kappa_1^2, \infty). \tag{B.4}$$

The (generalized) eigenfunctions of $\tilde{L}$ are given by

$$\psi_0(y) = \sqrt{\frac{\kappa_1}{2}}\,\frac{1}{\cosh\,\kappa_1 y}, \qquad \psi_0 \in H^2(\mathbb{R}), \quad \|\psi_0\|_2 = 1, \tag{B.5}$$

$$\psi_\lambda(y) = c_1 e^{i\sqrt{\lambda-\kappa_1^2}\,y}\left(\kappa_1 \tanh \kappa_1 y - i\sqrt{\lambda - \kappa_1^2}\right) + c_2 e^{-i\sqrt{\lambda-\kappa_1^2}\,y}\left(\kappa_1 \tanh \kappa_1 y + i\sqrt{\lambda - \kappa_1^2}\right), \tag{B.6}$$

$$(\tilde{l} - \lambda)\psi_\lambda = 0, \quad \psi_\lambda \neq L^2(\mathbb{R}; dy), \quad \psi_\lambda \in L^\infty(\mathbb{R}), \quad \lambda \geq \kappa_1^2. \tag{B.7}$$

Transforming with U^{-1}, $p(x) = 2\kappa_1 x(1 - 2\kappa_1 x)$

$$U^{-1} : \ L^2(\mathbb{R}; dy) \to L^2((0, \frac{1}{2\kappa_1}); dx),$$

$$(U^{-1}f)(x) = \frac{1}{\sqrt{p(x)}} f(y(x)) = \frac{1}{\sqrt{2\kappa_1 x(1-2\kappa_1 x)}}\, f\left(\frac{1}{2\kappa_1}\ln\left(\frac{2\kappa_1 x}{1-2\kappa_1 x}\right)\right), \tag{B.8}$$

we get the Sturm Liouville operator in $L^2((0,\frac{1}{2\kappa_1});dx)$

$$Lf = lf,\ \ f \in \mathcal{D}(L) = \{g \in L^2((0,\frac{1}{2\kappa_1});dx)\ |\ g, g' \in \mathrm{AC}_{\mathrm{loc}}((0,\frac{1}{2\kappa_1})); lg \in L^2((0,\frac{1}{2\kappa_1});dx)\}, \tag{B.9}$$

where

$$l = -\frac{d}{dx} 4\kappa_1^2 x^2 (1-2\kappa_1 x)^2 \frac{d}{dx}, \quad x \in (0,\frac{1}{2\kappa_1}). \tag{B.10}$$

The transformed eigenvector $w_0 = U^{-1}\psi_0$ then becomes

$$w_0(x) = \sqrt{2\kappa_1}, \quad x \in (0,\frac{1}{2\kappa_1}) \tag{B.11}$$

and the continuum solutions $w_\lambda = U^{-1}\psi_\lambda$ turn into

$$\begin{aligned} w_\lambda(x) &= c_1\,(1-2\kappa_1 x)^{-\frac{1}{2}(\frac{i}{\kappa_1}\sqrt{\lambda-\kappa_1^2}+1)}\,(2\kappa_1 x)^{\frac{1}{2}(\frac{i}{\kappa_1}\sqrt{\lambda-\kappa_1^2}-1)}\left(4\kappa_1^2 x - \kappa_1 - i\sqrt{\lambda-\kappa_1^2}\right) \\ &+ c_2\,(1-2\kappa_1 x)^{\frac{1}{2}(\frac{i}{\kappa_1}\sqrt{\lambda-\kappa_1^2}-1)}\,(2\kappa_1 x)^{-\frac{1}{2}(\frac{i}{\kappa_1}\sqrt{\lambda-\kappa_1^2}+1)}\left(4\kappa_1^2 x - \kappa_1 + i\sqrt{\lambda-\kappa_1^2}\right), \\ &\quad \lambda \geq \kappa_1^2,\ x \in (0,\frac{1}{2\kappa_1}). \end{aligned} \tag{B.12}$$

Bibliography

[1] Calogero F., Degasperis A.: "Spectral Transform and Solitons I". North-Holland, Amsterdam, 1982.

[2] Cao, Cewen: An isospectral class for a generalized Hill's equation, in "Applied differential equations", Xiao Shutie, Pu Fuquan (eds.), World Scientific, Singapore, 1986, p. 199-209.

[3] Carillo S., Fuchssteiner B.: The Action-Angle Transformation for the Korteweg-de Vries Equation, in " Nonlinear Evolution Equations and Dynamical Systems," Carillo S., Ragnisco O. (eds.), Springer, Berlin, 1990, p. 127-130.

[4] Clarkson P. A., Fokas A. S., Ablowitz M. J.: Hodograph transformations of linearizeable partial differential equations. SIAM J. Appl. Math. **49**, 1188-1209 (1989).

[5] Degasperis A., Sabatier P. C.: Extension of the one-dimensional scattering theory, and ambiguities. Inverse Problems **3**, 73-109 (1987).

[6] Dunford N., Schwartz J. T.: "Linear operators II: Spectral Theory, Self Adjoint Operators in Hilbert Space". Interscience Publishers, New York, 1988.

[7] Fuchssteiner B.: Solitons in Interaction. Prog. Theor. Phys. **78**, 1022-1050 (1987).

[8] Fuchssteiner B., Carillo S.: Soliton structure versus singularity analysis: Third-order completly integrable nonlinear differential equations in 1+1-dimensions. Physica **A 154**, 467-510 (1989).

[9] Gesztesy F., Simon B.: Constructing Solutions of the mKdV-Equation. J. Funct. Anal. **89**, 53-60 (1990).

[10] Gesztesy F., Schweiger W., Simon B.: Commutation Methods Applied to the mKdV-Equation. Trans. Amer. Math. Soc. **324**, 465-525 (1991).

[11] Gesztesy F.: Quasi-periodic, finite-gap solutions of the modified Korteweg-de Vries Equation, in "Ideas and Methods in Mathematical Physics. In Memory of Raphael Hoegh-Krøhn", S. Albeverio, J. E. Fenstad, H. Holden, T. Lindstrøm (eds.), Cambrige Univ. Press.

[12] Guo Ben-Yu, Rogers C.: On Harry-Dym equation and its solution. Science in China **A 32**, 283-295 (1989).

[13] Hereman W., Banerjee P.P., Chatterjee M.R.: Derivation and implicit solution of the Harry Dym equation and its connections with the Korteweg-de Vries equation. J. Phys. **A 22**, 241-55 (1989).

[14] Hirota R.: Direct methods in soliton theory, in "Solitons", R. K. Bullough, P. J. Caudrey (eds.), Springer, Berlin, 1980, p. 157- 1176.

[15] Ibragimov N.: Sur l'équivalence des équations d'évolution, qui admettent une algèbre de Lie-Bäcklund infinie. C. R. Acad. Sc. Paris. **293**, 657-660 (1981).

[16] Kadanoff, L. P.: Exact Solutions for the Saffman-Taylor Problem with Surface Tension. Phys. Rev. Lett. **65**, 2986-2988 (1990).

[17] Kawamoto S.: An exact transformation from the Harry Dym equation to the modified KdV equation. J. Phys. Soc. Japan. **54**, 2055-2056 (1985).

[18] Kingston J.G., Rogers C., Woodall D.: Reciprocal auto-Bäcklund transformations. J. Phys. **A 17**, L35-38 (1984)

[19] Mc Kean H. P., Van Moerbeke P.: The spectrum of Hill's equation. Invent. Math. **30**, 217-274 (1975).

[20] Leo M., Leo R.A., Soliani G., Solombrino L.: On the isospectral-eigenvalue problem and the recursion operator of the Harry-Dym equation. Lett. Nuovo Cimento **38**, 45-51 (1983).

[21] Levi D., Ragnisco O., Sym A.: The Bäcklund transformations for nonlinear evolution equations which exhibit exotic solitons. Phys. Lett. **100A**, 7-10 (1984).

[22] Rogers C., Nucci M.: On reciprocal Bäcklund Transformations and the Korteweg-de Vries Hierarchy. Physica Scripta **33**, 289-292 (1985).

[23] Rogers C., Wong P.: On reciprocal Bäcklund transformations of inverse scattering schemes. Physica Scripta **30**, 10-14 (1984).

[24] Sabatier P. C.: On Some Spectral Problems and Isospectral Evolutions Connected with the Classical String Problem II. Lett. Nuovo Cimento **26**, 483-486 (1979).

[25] Takahashi D., Sachs J. R., Satsuma J.: Soliton Phenomena in a Porous Medium, in " Nonlinear Physics ," Gu Chaohao, Li Yishen, Tu Guizhang. (eds.), Springer, Berlin, 1990, p. 214-220.

[26] Waddati M., Ichikawa Y. H., Shimizu T.: Cusp Soliton of a New Integrable Nonlinear Evolution Equation. Prog. Theor. Phys. **64**, 1959-1967 (1980).

[27] Weiss J.: Bäcklund transformation and the Painleve property. J. Math. Phys. **27**, 1293-1305 (1986).

[28] Li Yi-Shen: Evolution Equations Associated with the Eigenvalue Problem Based on the Equation $\varphi_{xx} = [v(x) - k^2\rho^2(x)]\varphi$. Nuovo Cimento **70 B**, 1-12 (1982).