RELATIVISTIC CORRECTIONS FOR THE SCATTERING MATRIX FOR SPHERICALLY SYMMETRIC POTENTIALS

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Abstract. We use the framework for the nonrelativistic limit of scattering theory for abstract Dirac operators developed in [5] to prove holomorphy of the scattering matrix at fixed energy with respect to c^{-2} for Dirac operators with spherically symmetric potentials. Relativistic corrections of order c^{-2} to the nonrelativistic limit *partial wave* scattering matrix are explicitly determined.

1. Introduction

Historically (see, e.g., [28]), the first rigorous treatment of the nonrelativistic limit of Dirac Hamiltonians goes back to Titchmarsh [30] who proved holomorphy of the Dirac eigenvalues (rest energy subtracted) with respect to c^{-2} for spherically symmetric potentials and obtained explicit formulas for relativistic bound state corrections of order $\mathrm{O}(c^{-2})$, formally derived in [27]. Holomorphy of the Dirac resolvent in three dimensions in c^{-1} for electrostatic interactions were first obtained by Veselic [31] and then extended to electromagnetic interactions by Hunziker [12]. An entirely different approach, based on an abstract set up, has been used in [6] to prove strong convergence of the unitary groups as $c^{-1} \to 0$. Employing this abstract framework, holomorphy of the Dirac resolvent in c^{-1} under general conditions on the electromagnetic interaction potentials has been obtained in [8], [9]. Moreover, this approach led to the first rigorous derivation of explicit formulas for relativistic corrections of order $\mathrm{O}(c^{-2})$ to bound state energies. Relativistic corrections for energy bands and corresponding corrections for impurity bound states for one-dimensional periodic systems were treated in [4]. Convergence of solutions of the Dirac equation based on semi group methods have also been obtained in [26].

Much less activity has been devoted to the nonrelativistic limit of the Dirac scattering theory. The proof of strong convergence of wave and scattering operators as $c^{-1} \to 0$ was given in [32] and [34]. A treatment of the scattering amplitude based on a different approach was given in [10]. The proof of holomorphy of the scattering matrix at fixed energy with respect to c^{-2} for abstract Dirac operators is established in [5] and explicit formulas for the correction term of order c^{-2} of the scattering matrix in terms of nonrelativistic scattering quantities are given.

In Section 2, following the abstract approach of [6] to Dirac operators, we review some of the basic results of [16] on abstract scattering theory. In Section 3 we recall the main results of [5] on the holomorphy of the scattering matrix at fixed energy with respect to c^{-2} for abstract Dirac operators and the explicit formula

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for the correction term of order c^{-2} of the scattering matrix in terms of nonrelativistic scattering quantities. In Section 4 we apply this abstract theory to Dirac operators with spherically symmetric potentials and obtain an explicit formula for the correction term of order c^{-2} of the *partial wave* scattering matrix. Finally in Appendix A we summarize the main results of [8] concerning the holomorphy of the Dirac resolvent operator with respect to c^{-2} near $c^{-2} = 0$.

2. The Abstract Approach

In this section we define the Dirac operator based on the abstract approach of [6]. Then we summarize some of the results on abstract scattering theory obtained by Kuroda [16] which are most relevant to understand the general formula for the scattering matrix in the next sections. For additional material on scattering theory in the present context we refer to [5] and the references therein, e.g., [1], [2], [3], [7], [11], [14], [17], [18], [19], [22], [23], [24], [29], [35].

Let $\mathfrak{H}_j,\ j = 1,2$ be separable, complex Hilbert spaces and introduce self-adjoint operators α, β in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ of the type

$$\alpha = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1}$$

where A is a densely defined, closed operator from $\mathfrak{H}_1$ into $\mathfrak{H}_2$. Next, we introduce the abstract free Dirac operator $H^0(c)$ by

$$H^0(c) = c\alpha + mc^2\beta, \quad \mathcal{D}(H^0(c)) = \mathcal{D}(\alpha), \quad c \in \mathbb{R} \setminus \{0\}, \quad m > 0 \tag{2.2}$$

and the interaction V by

$$V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}, \tag{2.3}$$

where V_j denotes self-adjoint operators in $\mathfrak{H}_j,\ j = 1,2$. Assuming V_1 (respectively V_2) to be bounded with respect to A (respectively A^*), i.e.,

$$\mathcal{D}(A) \subseteq \mathcal{D}(V_1), \qquad \mathcal{D}(A^*) \subseteq \mathcal{D}(V_2),$$

the abstract Dirac operator $H(c)$ reads

$$H(c) = H^0(c) + V, \quad \mathcal{D}(H(c)) = \mathcal{D}(\alpha). \tag{2.4}$$

Obviously $H(c)$ is self-adjoint for $|c|$ large enough. The corresponding self-adjoint (free) Pauli operators in $\mathfrak{H}_j,\ j = 1,2$ are then defined by

$$H_1^0 = \frac{1}{2m}A^*A, \quad H_1 = H_1^0 + V_1, \quad \mathcal{D}(H_1) = \mathcal{D}(A^*A), \tag{2.5}$$

$$H_2^0 = \frac{1}{2m}AA^*, \quad H_2 = H_2^0 + V_2, \quad \mathcal{D}(H_2) = \mathcal{D}(AA^*). \tag{2.6}$$

Following the usual convention we now subtract the rest energy mc^2 from $H^0(c)$ (similarly one could add the rest energy) and define

$$\hat{H}_1 \;=\; H^0(c) - mc^2, \quad \hat{H}_2 \;=\; H(c) - mc^2. \tag{2.7}$$

We introduce the following factorization of V

$$V_j = v_j^{1/2}|v_j|^{1/2}, \quad j = 1,2, \tag{2.8}$$

where

$$v_j^{1/2} = U_j|V_j|^{1/2}, \quad |v_j|^{1/2} = |V_j|^{1/2}, \quad j = 1, 2 \tag{2.9}$$

with $V_j = U_j|V_j|$ the polar decomposition of V_j. Furthermore,

$$Y = B(c)^{-1}\begin{pmatrix} |v_1|^{1/2} & 0 \\ 0 & |v_2|^{1/2} \end{pmatrix} = \begin{pmatrix} |v_1|^{1/2} & 0 \\ 0 & \frac{1}{c}|v_2|^{1/2} \end{pmatrix}, \tag{2.10}$$

$$Z = B(c)\begin{pmatrix} v_1^{1/2} & 0 \\ 0 & v_2^{1/2} \end{pmatrix} = \begin{pmatrix} v_1^{1/2} & 0 \\ 0 & cv_2^{1/2} \end{pmatrix}, \quad B(c) = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, \tag{2.11}$$

$$R_j(z) = (\hat{H}_j - z)^{-1}, \quad z \in \rho(\hat{H}_j) \quad j = 1, 2. \tag{2.12}$$

Remark 2.1. *The operator $B(c)$ was introduced in* [12].

The following assumptions 2.2–2.4 and 2.6–2.9 are basic in the approach of [16].

Assumption 2.2. *Y and Z are closed operators from $\mathfrak{H}$ to another Hilbert space $\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$ with $\mathcal{D}(\hat{H}_1) \subseteq \mathcal{D}(Y)$ and $\mathcal{D}(\hat{H}_1) \subseteq \mathcal{D}(Z)$.*

This implies that $YR_1(z), ZR_1(z) \in \mathrm{B}(\mathfrak{H}, \mathfrak{K})$ (see, e.g., [1], [13]). Here $\mathrm{B}(\mathfrak{H}, \mathfrak{K})$ denotes the set of bounded operators from $\mathfrak{H} \to \mathfrak{K}$ and $\mathrm{B}_\infty(\mathfrak{H}, \mathfrak{K})$ the set of compact operators from $\mathfrak{H} \to \mathfrak{K}$. The set of bounded (compact) operators on a Hilbert space X we denote by $\mathrm{B}(X)$ ($\mathrm{B}_\infty(X)$).

Assumption 2.3. *$ZR_1(z)Y^*$ is closable and the closure of $ZR_1(z)Y^* \in \mathrm{B}(\mathfrak{K})$ for one (or equivalently for all) $z \in \rho(\hat{H}_1)$*

$$Q_1(z, c) \;=\; \left[ZR_1(z)Y^*\right]^{(a)}, \quad G_1(z, c) \;=\; 1 + Q_1(z, c), \tag{2.13}$$

where $^{(a)}$ denotes the closure.

Assumption 2.4. *Let $z \in \rho(\hat{H}_1) \cap \rho(\hat{H}_2)$. Then $G_1(z, c)^{-1} \in \mathrm{B}(\mathfrak{K})$ and*

$$R_2(z) = R_1(z) - \left[R_1(z)Y^*\right]^a G_1(z)^{-1} Z R_1(z). \tag{2.14}$$

Thus propositions 2.6 and 2.7 in [16] hold. Define

$$Q_2(z, c) \;=\; \left[ZR_2(z)Y^*\right]^{(a)}, \quad G_2(z, c) \;=\; 1 - Q_2(z, c), \quad z \in \rho(\hat{H}_2). \tag{2.15}$$

Then

$$G_2(z, c) \;=\; G_1(z, c)^{-1}, \quad z \in \rho(\hat{H}_2). \tag{2.16}$$

Remark 2.5. *From our assumptions on $H^0(c)$ and V we infer that*

(i) $V^{1/2}$ is $\hat{H}^0(c)$ bounded with bound 0 and hence Assumption 2.2 is fulfilled.
(ii) $V^{1/2}$ is $\hat{H}^0(c)^{1/2}$ bounded implying that Assumption 2.3 is fulfilled.
(iii) The second resolvent equation yields

$$\begin{aligned} &(1 + \left[ZR_1(z)Y^*\right]^{(a)})(1 - \left[ZR_2(z)Y^*\right]^{(a)}) = 1, \\ &(1 - \left[ZR_2(z)Y^*\right]^{(a)})(1 + \left[ZR_1(z)Y^*\right]^{(a)}) = 1 \end{aligned} \tag{2.17}$$

(see, e.g., [1]*) and thus Assumption 2.4 is fulfilled.*

Next let E_j denote the spectral measures associated with $\hat{H}_j, \quad j = 1, 2$.

Assumption 2.6. *There exists a Hilbert space $\mathfrak{C}$, a non-empty open set $I \subseteq \mathbb{R}$, and a unitary operator F from $E_1(I)\mathfrak{H}$ onto $L^2(I; \mathfrak{C})$ such that for every Borel set $I' \subseteq I$ one has $F E_1(I') F^{-1} = \chi_{I'}$, where $\chi_{I'}$ denotes the operator of multiplication by the characteristic function of I'.*

Assumption 2.7. *There exist $B(\mathfrak{K}, \mathfrak{C})$-valued functions $T(\lambda, c, Y)$ and $T(\lambda, c, Z)$, $\lambda \in I$, such that*

(i) $T(\cdot, c, Y)$ and $T(\cdot, c, Z)$ are locally Hölder continuous in I with respect to the operator norm.

(ii) There exist dense subsets $D \subseteq \mathcal{D}(Y^*)$ and $D' \subseteq \mathcal{D}(Z^*)$ such that for any $u \in D$ and $v \in D'$ one infers for a. e. $\lambda \in I$

$$T(\lambda, c, Y)u = (F E_1(I) Y^* u)(\lambda), \quad T(\lambda, c, Z)v = (F E_1(I) Z^* v)(\lambda). \tag{2.18}$$

Assumption 2.8. *For one (or equivalently all) $z \in \rho(\hat{H}_1)$ either $Y R_1(z) \in B_\infty(\mathfrak{H}, \mathfrak{K})$ or $Z R_1(z) \in B_\infty(\mathfrak{H}, \mathfrak{K})$.*

Assumption 2.9. *The subspace generated by $\{E_j(I') Y^* u \mid u \in \mathcal{D}(Y^*), I' \subseteq I$ a Borel set$\}$ is dense in $E_j(I)\mathfrak{H}, \quad j = 1, 2$.*

Remark 2.10. [16] *Since $\mathfrak{H}$ is separable, Assumption 2.6 is equivalent to assuming that $\hat{H}_1$ has absolutely continuous spectrum in I with constant multiplicity. Moreover, $\mathfrak{C}$ is determined uniquely up to unitary equivalence and F is uniquely determined up to unitary equivalence with decomposable, unitary operators on $L^2(I; \mathfrak{C})$.*

Since these assumptions are identical with the ones in [16] we have all the results of ([16] §3, §4) at our disposal, e.g., the norm limits

$$G_{1\pm}(\lambda, c) = \mathrm{n} - \lim_{\epsilon \downarrow 0} G_1(\lambda \pm i\epsilon, c), \quad Q_{1\pm}(\lambda, c) = \mathrm{n} - \lim_{\epsilon \downarrow 0} Q_1(\lambda \pm i\epsilon, c) \tag{2.19}$$

exist (see [16] Theorem 3.9) and introducing

$$e_\pm(c) = \{\lambda \in I \mid G_{1\pm}(\lambda, c) \text{ is not one to one }\}, \quad e(c) = e_+(c) \cup e_-(c) \tag{2.20}$$

($e(c)$ is a closed set of Lebesgue measure zero [16]) we get for $\lambda \in I \setminus e_\pm(c)$ the existence of the boundary values

$$G_{2\pm}(\lambda, c) = \mathrm{n} - \lim_{\epsilon \downarrow 0} G_2(\lambda \pm i\epsilon, c) \tag{2.21}$$

and

$$G_{2\pm}(\lambda, c) = G_{1\pm}(\lambda, c)^{-1} \tag{2.22}$$

(see [16] Theorem 3.10).

Also Theorems 3.11–3.13 and 6.3 of [16] are valid. In particular, we obtain for the fibers of the scattering operator

Theorem 2.11. [16] *For $\lambda \in I \setminus e(c)$ the scattering matrix $S(\lambda, c)$ in $\mathfrak{C}$ associated with the pair $(\hat{H}_2, \hat{H}_1)$ is given by*

$$S(\lambda, c) = 1 - 2\pi i T(\lambda, c, Y) G_{2+}(\lambda, c) T(\lambda, c, Z)^*. \tag{2.23}$$

$S(\cdot, c)$ is unitary in $\mathfrak{C}$ and locally Hölder continuous on $I \setminus e(c)$ with respect to the norm in $\mathrm{B}(\mathfrak{C})$.

3. Holomorphy of the scattering matrix in c^{-2} and relativistic corrections

In this section we recall the results obtained in [5] on holomorphy of the abstract scattering matrix with respect to c^{-2}. Moreover, explicitly corrections of the scattering matrix of order c^{-2} in terms of nonrelativistic scattering quantities are given in Theorem 3.3.

Let $I \subseteq \mathbb{R}^+ = (0, \infty)$ and define

$$I_{\pm 0} = \{\lambda \mid \lambda \in I \setminus e_\pm(c^{-2} = 0)\}, \quad I_0 = I_{+0} \cap I_{-0}. \tag{3.1}$$

In addition we strengthen Assumptions 2.3 and 2.7 by introducing

Assumption 3.1. *(i) For $\lambda \in I$, $T(\lambda, c, Y)$ and $T(\lambda, c, Z)$ are holomorphic in c^{-2} around $c^{-2} = 0$ and*

(ii) for $\lambda \in I_{+0}$

$$Q_{1+}(\lambda, c) = \lim_{\epsilon \downarrow 0} Q_{1+}(\lambda + i\epsilon, c) \tag{3.2}$$

is holomorphic in c^{-2} around $c^{-2} = 0$.

Remark 3.2. *For later purposes we note that Assumption 3.1 (ii) implies that*

$$v_1^{1/2}(H_1^0 - \lambda - i0)^{-2}|v_1|^{1/2} = \frac{d}{d\lambda} v_1^{1/2}(H_1^0 - \lambda - i0)^{-1}|v_1|^{1/2}. \tag{3.3}$$

We define

$$\begin{aligned} g_2(z) &= (1 + v_1^{1/2}(H_1^0 - z)^{-1}|v_1|^{1/2})^{-1}, \quad z = \lambda + i\epsilon, \epsilon > 0, \\ g_{2\pm}(\lambda) &= \lim_{\epsilon \downarrow 0} g_2(\lambda \pm i\epsilon). \end{aligned} \tag{3.4}$$

By Assumption 2.6 , α^2 and hence A^*A, AA^* are absolutely continuous in $\tilde{I}^2$ with constant multiplicity.

Now we consider the analogs U_0, M of F and T when A^*A replaces $\hat{H}_1$.

Let U_0 be the unitary operator that diagonalizes A^*A on $\tilde{I}^2$. For $h \in E_0(\tilde{I}^2)\mathfrak{H}_1$ (where $E_0(\cdot)$ denotes the spectral measure for A^*A) U_0 yields

$$U_0 E_0(\tilde{I}^2)\mathfrak{H}_1 \to L^2(\tilde{I}^2, d\mu; \mathfrak{C}), \quad (U_0 A^*Ah)(\mu) = \mu(U_0 h)(\mu), \quad \mu \in \tilde{I}^2. \tag{3.5}$$

In addition we need the operator $M(k, D) : \mathcal{D}(D) \to \mathfrak{C}$, where $D : \mathcal{D}(D) \to \mathfrak{H}_1$, $\mathcal{D}(D) \subseteq \mathfrak{K}_1$ or $\mathfrak{K}_2, D$ closed

$$M(k, D)h = (U_0 E_0(\tilde{I}^2)Dh)(k^2), \quad h \in \mathcal{D}(D), \quad k = \sqrt{\mu}, \quad \text{for a. e. } k \in \tilde{I}. \tag{3.6}$$

In concrete applications the closure of $M(k, D)$ will be a Hilbert-Schmidt operator. This closure is then denoted by $M(k, D)$, too.

We can now state the following result for the fibers of the scattering operator.

Theorem 3.3. *Assume Assumptions 2.2–2.4, 2.6–2.9 and 3.1 to be fulfilled. Then for $\lambda \in I_0$, the scattering matrix $S(\lambda, c)$ associated with the pair $(H(c) - mc^2, H^0(c) - mc^2)$ is holomorphic in c^{-2} around $c^{-2} = 0$ and we get the following expansion*

$$S(\lambda, c) = 1 - 2\pi i T(\lambda, c, Y) G_{2+}(\lambda, c) T(\lambda, c, Z)^* = \sum_{j=0}^{\infty} c^{-2j} S^{(j)}(\lambda). \tag{3.7}$$

with

$$S^{(0)}(\lambda) = 1 - 2\pi i \left(2mM(k^s, |v_1|^{1/2}) g_{2+}(\lambda) M(k^s, v_1^{1/2})^* \right), \quad \lambda \in I_0, \ k^s = \sqrt{2m\lambda} \tag{3.8}$$

the scattering matrix for the associated pair of Pauli operators (H_1, H_1^0) *(illustrating the nonrelativistic limit) and the explicit correction term of order* c^{-2}

$$\begin{aligned} S^{(1)}(\lambda) &= \frac{(k^s)^2}{4m^2}(S^{(0)}(\lambda) - 1) - 2\pi i \left\{ \frac{(k^s)^3}{4m} M'(k^s, |v_1|^{1/2}) g_{2+}(\lambda) M(k^s, v_1^{1/2})^* \right. \\ &- \frac{1}{2m} M(k^s, A^*|v_2|^{1/2}) \Big(v_2^{1/2} A (H_1^0 - \lambda - i0)^{-1} |v_1|^{1/2} \Big) g_{2+}(\lambda) M(k^s, v_1^{1/2})^* \\ &+ \frac{1}{2m} M(k^s, A^*|v_2|^{1/2}) M(k^s, A^* v_2^{1/2})^* + \frac{(k^s)^3}{4m} M(k^s, |v_1|^{1/2}) g_{2+}(\lambda) M'(k^s, v_1^{1/2})^* \\ &- \frac{(k^s)^2}{(2m)^2} M(k^s, |v_1|^{1/2}) g_{2+}(\lambda) \Big(v_1^{1/2} (H_1^0 - \lambda - i0)^{-1} A^* A (H_1^0 - \lambda - i0)^{-1} |v_1|^{1/2} \Big) \\ &\times g_{2+}(\lambda) M(k^s, v_1^{1/2})^* + \frac{1}{2m} M(k^s, |v_1|^{1/2}) g_{2+}(\lambda) \Big(v_1^{1/2} (H_1^0 - \lambda - i0)^{-1} A^* |v_2|^{1/2} \Big) \\ &\times \Big(v_2^{1/2} A (H_1^0 - \lambda - i0)^{-1} |v_1|^{1/2} \Big) g_{2+}(\lambda) M(k^s, v_1^{1/2})^* \\ &\left. - \frac{1}{2m} M(k^s, |v_1|^{1/2}) g_{2+}(\lambda) \Big(v_1^{1/2} (H_1^0 - \lambda - i0)^{-1} A^* |v_2|^{1/2} \Big) M(k^s, A^* v_2^{1/2})^* \right\}, \\ &\qquad \lambda \in I_0, \end{aligned} \tag{3.9}$$

where $^{(\prime)}$ *denotes the derivative with respect to* k^s.

4. The Dirac operator in $L^2(\mathbb{R}^3)^4$ with a spherically symmetric potential

We apply the abstract theory developed in previous chapters now to concrete Dirac operators in $L^2(\mathbb{R}^3)^4$ with spherically symmetric potentials. The free Dirac operator $H^{0,D}(c)$ in $L^2(\mathbb{R}^3)^4$ is defined by

$$H^{0,D}(c) = c\,\boldsymbol{\alpha}\,\boldsymbol{p} + \beta\, mc^2, \quad m, c > 0, \quad \mathcal{D}(H^{0,D}(c)) = H^{2,1}(\mathbb{R}^3)^4, \tag{4.1}$$

where

$$\begin{aligned} &\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ &\alpha_\ell = \begin{pmatrix} 0 & \sigma_\ell \\ \sigma_\ell & 0 \end{pmatrix}, \ \ell = 1,2,3, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ &\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3), \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \quad \boldsymbol{p} = -i\nabla, \quad \mathcal{D}(\boldsymbol{p}) = H^{2,1}(\mathbb{R}^3). \end{aligned} \tag{4.2}$$

Let V be the maximal operator of multiplication with the real-valued function $v = v(r)$, where $r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Assumption 4.1. *Assume that* V *fulfills*

$$\int_0^\infty dr e^{\alpha r} |v(r)| < \infty, \quad \alpha > 0. \tag{4.3}$$

The Dirac operator $H^D(c)$ in $L^2(\mathbb{R}^3)^4$ is now defined as

$$H^D(c) = H^{0,D}(c) + V, \quad \mathcal{D}(H^D(c)) = \mathcal{D}(H^{0,D}(c)). \tag{4.4}$$

Remark 4.2. *Our Assumption 4.1 does not include Coulomb-like singularities since these are strongly singular with respect to the Dirac operator (cf.* [15]*,* [21]*,* [25]*,* [33]*).*

Furthermore we recall the definition of the "angular momentum operators" (cf. [28], p. 8)

$$\begin{aligned} &\boldsymbol{S} = -\frac{i}{4}\boldsymbol{\alpha} \wedge \boldsymbol{\alpha} && \textit{spin} \text{ angular momentum,} \\ &\boldsymbol{L} = \boldsymbol{x} \wedge \boldsymbol{p} && \textit{orbital} \text{ angular momentum,} \\ &\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} && \textit{total} \text{ angular momentum.} \end{aligned}$$

Since the potential V is spherically symmetric the symmetry induced by invariance under rotations allows the so called "partial wave" expansion. This expresses the conservation of total angular momentum $\boldsymbol{J}$. The Hilbert space is decomposed in the following way (cf. [28], p. 122 ff.), where the operators J^2, J_3, and K ($K = \beta(2\boldsymbol{SL}+1)$ is the relativistic analog of the spin-orbit coupling) are diagonal with quantum numbers $j(j+1), m_j$, and $-\kappa_j$. To achieve this goal we first we introduce polar coordinates in $L^2(\mathbb{R}^3)^4$ and then the unitary transformation U

$$\begin{aligned} &U : (Uf)(r) = rf(r), \\ &L^2(\mathbb{R}^3 \setminus \{0\})^4 \to L^2\Big((0,\infty), r^2dr; L^2(S^2)^4\Big) \to L^2\Big((0,\infty), dr; L^2(S^2)^4\Big), \end{aligned} \tag{4.5}$$

i.e., for every Ψ in $L^2(\mathbb{R}^3)$ we write

$$\psi(r,\vartheta,\varphi) = r\,\Psi(x_1(r,\vartheta,\varphi), x_2(r,\vartheta,\varphi), x_3(r,\vartheta,\varphi)). \tag{4.6}$$

Define the vectors $\Psi^{m_j}_{j\pm1/2}$ by

$$\Psi^{m_j}_{j-1/2}(\vartheta,\varphi) = \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j}\; Y^{m_j-1/2}_{j-1/2}(\vartheta,\varphi) \\ \sqrt{j-m_j}\; Y^{m_j+1/2}_{j-1/2}(\vartheta,\varphi) \end{pmatrix}, \tag{4.7}$$

$$\Psi^{m_j}_{j+1/2}(\vartheta,\varphi) = \frac{1}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j+1-m_j}\; Y^{m_j-1/2}_{j+1/2}(\vartheta,\varphi) \\ -\sqrt{j+1+m_j}\; Y^{m_j+1/2}_{j+1/2}(\vartheta,\varphi) \end{pmatrix}, \tag{4.8}$$

where Y^m_l are the usual spherical harmonics. Then the vectors Φ_{j,m_j,κ_j} in $L^2(S^2)^4$ defined by

$$\begin{aligned} &\Phi^+_{j,m_j,\mp(j+1/2)}(\vartheta,\varphi) = \begin{pmatrix} i\Psi^{m_j}_{j\mp1/2}(\vartheta,\varphi) \\ 0 \end{pmatrix}, \\ &\Phi^-_{j,m_j,\mp(j+1/2)}(\vartheta,\varphi) = \begin{pmatrix} 0 \\ \Psi^{m_j}_{j\pm1/2}(\vartheta,\varphi) \end{pmatrix}, \\ &\Phi_{j,m_j,\kappa_j}(\vartheta,\varphi) = c_+\Phi^+_{j,m_j,\kappa_j}(\vartheta,\varphi) + c_-\Phi^-_{j,m_j,\kappa_j}(\vartheta,\varphi), \quad c_+, c_- \in \mathbb{C} \end{aligned} \tag{4.9}$$

are eigenvectors of J^2, J_3, K with eigenvalues $j(j+1), m_j$, and $-\kappa_j$. These vectors form a complete orthonormal set in $L^2(S^2)^4$.

The Hilbert space $L^2(S^2)^4$ is the orthogonal direct sum of the two dimensional Hilbert spaces $\mathfrak{N}_{j,m_j,\kappa_j}$ which are spanned by the vectors $\Phi^{\pm}_{j,m_j,\kappa_j}$

$$L^2(S^2)^4 = \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{\kappa_j=\mp(j+\frac{1}{2})} \mathfrak{N}_{j,m_j,\kappa_j}. \tag{4.10}$$

This decomposition of the angular Hilbert space implies a similar decomposition of the Hilbert space $L^2(\mathbb{R}^3)^4$. Each "partial wave subspace" $L^2((0,\infty),dr)\otimes\mathfrak{N}_{j,m_j,\kappa_j}$ is isomorphic to $L^2((0,\infty),dr)^2$ if we choose the basis $\{\Phi^{+}_{j,m_j,\kappa_j},\Phi^{-}_{j,m_j,\kappa_j}\}$ in $\mathfrak{N}_{j,m_j,\kappa_j}$.

The full free Dirac operator $H^{0,\mathrm{D}}(c)$ in $L^2(\mathbb{R}^3)^4$ is unitarily equivalent to the direct sum of the "partial wave" Dirac operators $h^0_{j,m_j,\kappa_j}(c)$

$$H^{0,\mathrm{D}}(c) \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{\kappa_j=\mp(j+\frac{1}{2})} h^0_{j,m_j,\kappa_j}(c), \tag{4.11}$$

where the free "partial wave" Dirac operator $H^0(c)$ in $L^2((0,\infty),dr)^2$ now reads

$$H^0(c) = h^0_{j,m_j,\kappa_j}(c) = \begin{pmatrix} mc^2 & cA^*_{j,m_j,\kappa_j} \\ cA_{j,m_j,\kappa_j} & -mc^2 \end{pmatrix}. \tag{4.12}$$

Here A_{j,m_j,κ_j} is the closure of $\dot{A}_{j,m_j,\kappa_j}$, where

$$\begin{aligned} \dot{A}_{j,m_j,\kappa_j} &= \frac{d}{dr} + \frac{\kappa_j}{r}, \quad \mathcal{D}(\dot{A}_{j,m_j,\kappa_j}) = C_0^\infty((0,\infty)), \\ \kappa_j &= \mp(j+\frac{1}{2}), \quad 2j = 1,3,5,\dots, \quad m_j = -j,-j+1,\dots,j. \end{aligned} \tag{4.13}$$

The "partial wave" Dirac operator $H(c)$ in $L^2((0,\infty),dr)^2$ is now defined as

$$\begin{aligned} H(c) = H^0(c) + V = h_{j,m_j,\kappa_j}(c) &= \begin{pmatrix} mc^2 + v(r) & cA^*_{j,m_j,\kappa_j} \\ cA_{j,m_j,\kappa_j} & -mc^2 + v(r) \end{pmatrix}, \\ \mathcal{D}(H(c)) &= \mathcal{D}(H^0(c)). \end{aligned} \tag{4.14}$$

Subtracting the rest energy according to (2.7) we therefore identify

$$\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{K}_1 = \mathfrak{K}_2 = L^2((0,\infty),dr), \quad I_{\pm 0} = \mathbb{R}^+ \setminus e_\pm(c^{-2}=0), \quad \mathfrak{C} = \mathbb{C}^1, \tag{4.15}$$

$$V_1 = V_2 = V, \quad V = v^{1/2}|v|^{1/2}, \quad v^{1/2} = |v|^{1/2}\,\mathrm{sgn}(v), \tag{4.16}$$

$$Y = Y^* = B(c)^{-1}|v|^{1/2}, \quad Z = Z^* = v^{1/2}B(c). \tag{4.17}$$

Clearly Assumptions 2.2–2.4, 2.6–2.8 are satisfied. Assumption 2.9 can be dealt with in exactly the same way as in [5] Section 5.1. It remains to verify Assumption 3.1.

(i) Holomorphy of $Q_{1+}(\lambda,c)$, $\lambda \in I_{+0}$. Let $A_\ell = \frac{d}{dr} + \frac{\ell}{r}$. Then the kernel $g_{l,\pm}(\lambda)(r,r')$ of $(A_l^*A_l - \lambda \mp i0)^{-1}$ reads

$$g_{l,\pm}(\lambda)(r,r') = \frac{1}{\sqrt{\lambda}}\hat{j}_l(\sqrt{\lambda}r_<)\hat{h}_l^{\pm}(\sqrt{\lambda}r_>), \quad l = 0,1,2,..., \tag{4.18}$$

where $r_> = \max(r, r'), r_< = \min(r, r')$ and $\hat{j}_l, \hat{h}_l^\pm$ are the Riccati-Bessel and Riccati-Hankel functions (cf. [1], p. 496 ff. and p. 481).

Thus we obtain for the kernel $q(r, r', \lambda, c)$ of

$$Q_{1+}(\lambda, c) = v^{1/2} B(c)(h^0_{j,m_j,\kappa_j}(c) - mc^2 - \lambda - i0)^{-1} B(c)^{-1}|v|^{1/2} \tag{4.19}$$

$$q(r, r', \lambda, c) = v(r)^{1/2} \begin{pmatrix} 1 + \frac{\lambda}{2mc^2} & \frac{1}{2mc^2}\left(-\frac{d}{dr} + \frac{\kappa_j}{r}\right) \\ \frac{1}{2m}\left(\frac{d}{dr} + \frac{\kappa_j}{r}\right) & \frac{\lambda}{2mc^2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{2m}{k^d}\hat{j}_{l(\kappa_j)}(k^d r_<)\hat{h}^+_{l(\kappa_j)}(k^d r_>) & 0 \\ 0 & \frac{2m}{k^d}\hat{j}_{l(\kappa_j-1)}(k^d r_<)\hat{h}^+_{l(\kappa_j-1)}(k^d r_>) \end{pmatrix} |v(r')|^{1/2},$$

$$l(\kappa_j) = |\kappa_j| + \frac{1}{2}(\mathrm{sgn}(\kappa_j) - 1), \tag{4.20}$$

$$\lambda \in I_{+0}, \quad k^d(\lambda, c) = k^s(1 + \frac{\lambda}{2mc^2})^{1/2}, \quad k^s = \sqrt{2m\lambda}.$$

Define the compact set $M \subset \mathbb{C}$

$$M = \left\{ c^{-2} \in \mathbb{C} \;\middle|\; |c^{-2}| \le |c_0^{-2}| < \frac{2m}{\lambda} \text{ and } 2|\mathrm{Im} k^d(\lambda, c)| \le k^s \frac{\lambda}{m|c_0^2|} \le \alpha \right\}. \tag{4.21}$$

Using

$$|k^d| \le k^s(1 + \frac{\lambda}{2m|c_0^2|})^{1/2} \tag{4.22}$$

and a matrix norm in $\mathbb{C}^2$ we get for $c^{-2} \in M$

$$\int ||q(r, r', \lambda, c)||dr \le c_1 < \infty \quad \text{and} \quad \int ||q(r, r', \lambda, c)||dr' \le c_2 < \infty, \quad c_1, c_2 \in \mathbb{R}. \tag{4.23}$$

For $c^{-2} \in M$ and fixed λ we have a family of uniformly bounded operators (using [33], Theorem 6.24 "Folgerung" 4). Since the integral kernel $q(r, r', \lambda, c)$ is a holomorphic function of c^{-2} around $c^{-2} = 0$, we obtain holomorphy of $Q_{1+}(\lambda, c)$.

(ii) Holomorphy of $T(\lambda, c, Y), \lambda > 0$.

The integral kernel $t(r, \lambda, c)$ of $T(\lambda, c, Y) : L^2((0, \infty))^2 \to \mathbb{C}^1$ is given by

$$t(r, \lambda, c) = \sqrt{\frac{1}{ck_0}} \frac{1}{\sqrt{\pi}} (-i)^{l(\kappa_j)} |v(r)|^{1/2}$$
$$\times \begin{pmatrix} \hat{j}_{l(\kappa_j)}(k^d r) & \frac{k_0}{ck^d}\frac{\kappa_j}{r}\hat{j}_{l(\kappa_j)}(k^d r) + \frac{k_0}{c}\hat{j}'_{l(\kappa_j)}(k^d r) \end{pmatrix},$$

$$k^d = \sqrt{2m\lambda(1 + \frac{\lambda}{2mc^2})}, \quad k_0 = \sqrt{\frac{\lambda}{\lambda + 2mc^2}}, \quad \lambda > 0. \tag{4.24}$$

For $\lambda \in I$ we obtain

$$||t(r, \lambda, c)|| \le c_3(\lambda, \alpha)|v(r)|^{1/2} e^{\frac{\alpha}{2} r}, \quad c_3 \in \mathbb{R}. \tag{4.25}$$

For $c^{-2} \in M$ this is a family of uniformly bounded Hilbert Schmidt operators (since the right hand side of (4.25) is in $L^2((0, \infty), dr)$) with integral kernel holomorphic in c^{-2} and therefore $T(\lambda, c, Y)$ is holomorphic in c^{-2} around $c^{-2} = 0$.

The holomorphy of $T(\lambda, c, Z)^*$ follows similarly.

Thus we have shown that all assumptions are fulfilled which guaranty the holomorphy of the scattering matrix in c^{-2}. It remains to calculate the relativistic correction terms.

The operator U_{j,m_j,κ_j} that diagonalizes the radial Pauli operator

$$A^*_{j,m_j,\kappa_j} A_{j,m_j,\kappa_j} = -\frac{d^2}{dr^2} + \frac{\kappa_j(\kappa_j+1)}{r^2} \tag{4.26}$$

is given by $U_{j,m_j,\kappa_j} : L^2((0,\infty), dr) \to L^2((0,\infty), d\mu)$

$$(U_{j,m_j,\kappa_j} f)(\mu) = \frac{\mu^{-1/4}}{\sqrt{\pi}} (-i)^{l(\kappa_j)} \int_0^\infty dr \hat{j}_{l(\kappa_j)}(\sqrt{\mu} r) f(r), \tag{4.27}$$

where $\hat{j}_l(x)$ denote the Riccati-Bessel functions and the integral is to be taken in the sense

$$\int_0^\infty dr \hat{j}_{l(\kappa_j)}(\sqrt{\mu} r) f(r) = \ s - \lim_{R\to\infty} \int_0^R dr \hat{j}_{l(\kappa_j)}(\sqrt{\mu} r) f(r). \tag{4.28}$$

Thus we get

$$\begin{aligned}
&M(k^d, |v|^{1/2}) : \ L^2((0,\infty), dr) \to \mathbb{C}^1, \\
&M(k^d, |v|^{1/2}) f = \frac{1}{\sqrt{\pi}} (k^d)^{-1/2} (-i)^{l(\kappa_j)} \int_0^\infty dr \hat{j}_{l(\kappa_j)}(k^d r) |v(r)|^{1/2} f(r) \\
&= \frac{1}{\sqrt{\pi}} (k^d)^{-1/2} (-i)^{l(\kappa_j)} < |v|^{1/2} \hat{j}_{l(\kappa_j)}(k^d), f >, \quad f \in L^2((0,\infty), dr),
\end{aligned} \tag{4.29}$$

where $< \cdot, \cdot >$ denotes the scalar product in $L^2((0,\infty))$. Similarly we have

$$\begin{aligned}
&M(k^d, A^*|v|^{1/2}) : \ L^2((0,\infty), dr) \to \mathbb{C}^1, \\
&M(k^d, A^*|v|^{1/2}) f \\
&= \frac{1}{\sqrt{\pi}} (k^d)^{-1/2} (-i)^{l(\kappa_j)} < |v|^{1/2} \Big(\frac{\kappa_j}{r} \hat{j}_{l(\kappa_j)}(k^d) + k^d \hat{j}'_{l(\kappa_j)}(k^d) \Big), f >, \\
&\qquad f \in L^2((0,\infty), dr).
\end{aligned} \tag{4.30}$$

For the corresponding adjoint operators we obtain

$$\begin{aligned}
&M(k^d, v^{1/2})^* : \ \mathbb{C}^1 \to L^2((0,\infty), dr), \\
&\Big(M(k^d, v^{1/2})^* h \Big)(r) = \frac{1}{\sqrt{\pi}} (k^d)^{-1/2} (i)^{l(\kappa_j)} v^{1/2}(r) \hat{j}_{l(\kappa_j)}(k^d r) h, \quad h \in \mathbb{C}^1,
\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}
&M(k^d, A^* v^{1/2})^* : \ \mathbb{C}^1 \to L^2((0,\infty), dr), \\
&\Big(M(k^d, A^* v^{1/2})^* h \Big)(r) \\
&= \frac{1}{\sqrt{\pi}} (k^d)^{-1/2} (i)^{l(\kappa_j)} v^{1/2}(r) \Big(\frac{\kappa_j}{r} \hat{j}_{l(\kappa_j)}(k^d r) + k^d \hat{j}'_{l(\kappa_j)}(k^d r) \Big) h, \quad h \in \ \mathbb{C}^1.
\end{aligned} \tag{4.32}$$

The physical solutions $\psi^s_{\kappa_j,\pm}$ of the radial Schrödinger (Pauli) equation are defined by the Fredholm (resp. Lippmann-Schwinger) equation

$$v^{1/2} \psi^s_{\kappa_j,\pm}(k^s) = g_{2\pm}(\lambda) v^{1/2} \hat{j}_{l(\kappa_j)}(k^s), \quad k^s = \sqrt{2m\lambda}, \quad \lambda \in I_{\pm 0}. \tag{4.33}$$

For the nonrelativistic limit $S^{(0)}_{\kappa_j}(\lambda)$ we obtain from (3.8)

$$\begin{aligned} S^{(0)}_{\kappa_j}(\lambda) &= 1 - 2\pi i 2mM(k^s, |v|^{1/2}) g_{2+}(\lambda) M(k^s, v^{1/2})^* \\ &= 1 - \frac{4im}{k^s} < |v|^{1/2}\hat{j}_{l(\kappa_j)}(k^s), v^{1/2}\psi^s_{\kappa_j,+}(k^s) >, \quad \lambda \in I_{+0}. \end{aligned} \tag{4.34}$$

Calculating the remaining terms on the right hand side of (3.9) yields

2^{nd} term

$$\frac{(k^s)^2}{2im} < |v|^{1/2} r\hat{j}_{l(\kappa_j)}(k^s), v^{1/2}\psi^s_{\kappa_j,+}(k^s) > - \frac{k^s}{4im} < |v|^{1/2}\hat{j}_{l(\kappa_j)}(k^s), v^{1/2}\psi^s_{\kappa_j,+}(k^s) >, \tag{4.35}$$

3^{rd} term

$$\begin{aligned} &-\frac{1}{imk^s} \\ &\times < |v|^{1/2}\Big(\frac{\kappa_j}{r}\hat{j}_{l(\kappa_j)}(k^s) - k^s\hat{j}'_{l(\kappa_j)}(k^s)\Big), v^{1/2}A_{j,m_j,\kappa_j}\Big(\hat{j}_{l(\kappa_j)}(k^s) - \psi^s_{\kappa_j,+}(k^s)\Big) >, \end{aligned} \tag{4.36}$$

4^{th} term

$$\frac{1}{imk^s} < |v|^{1/2}\Big(\frac{\kappa_j}{r}\hat{j}_{l(\kappa_j)}(k^s) + k^s\hat{j}'_{l(\kappa_j)}(k^s)\Big), v^{1/2}\Big(\frac{\kappa_j}{r}\hat{j}_{l(\kappa_j)}(k^s) + k^s\hat{j}'_{l(\kappa_j)}(k^s)\Big) >, \tag{4.37}$$

5^{th} term

$$\frac{(k^s)^2}{2im} < |v|^{1/2}\psi^s_{\kappa_j,-}(k^s), v^{1/2} r\hat{j}_{l(\kappa_j)}(k^s) > - \frac{k^s}{4im} < |v|^{1/2}\hat{j}_{l(\kappa_j)}(k^s), v^{1/2}\psi^s_{\kappa_j,+}(k^s) >, \tag{4.38}$$

6^{th} term

$$\begin{aligned} &-\frac{k^s}{im} < |v|^{1/2}\Big(\hat{j}_{l(\kappa_j)}(k^s) - \psi^s_{\kappa_j,-}(k^s)\Big), v^{1/2}\psi^s_{\kappa_j,+}(k^s) > \\ &-\frac{(k^s)^3}{2im^2} < |v|^{1/2}\psi^s_{\kappa_j,-}(k^s), (v^{1/2}(H^0_1 - \lambda - i0)^{-2}|v|^{1/2}) v^{1/2}\psi^s_{\kappa_j,+}(k^s) >, \end{aligned} \tag{4.39}$$

7^{th} term

$$\begin{aligned} &\frac{1}{imk^s} \\ &\times < |v|^{1/2}A_{j,m_j,\kappa_j}\Big(\hat{j}_{l(\kappa_j)}(k^s) - \psi^s_{\kappa_j,-}(k^s)\Big), v^{1/2}A_{j,m_j,\kappa_j}\Big(\hat{j}_{l(\kappa_j)}(k^s) - \psi^s_{\kappa_j,+}(k^s)\Big) >, \end{aligned} \tag{4.40}$$

8^{th} term

$$-\frac{1}{imk^s} < |v|^{1/2}A_{j,m_j,\kappa_j}\Big(\hat{j}_{l(\kappa_j)}(k^s) - \psi^s_{\kappa_j,-}(k^s)\Big), v^{1/2}A_{j,m_j,\kappa_j}\hat{j}_{l(\kappa_j)}(k^s) > . \tag{4.41}$$

Summing up yields for the first order correction term in c^{-2} of the scattering matrix

$$\begin{aligned} S^{(1)}_{\kappa_j}(\lambda) &= \frac{(k^s)^4}{8m^3}\frac{dS^{(0)}_{\kappa_j}(\lambda)}{d\lambda} + \frac{k^s}{im} < |v|^{1/2}\psi^s_{\kappa_j,-}(k^s), v^{1/2}\psi^s_{\kappa_j,+}(k^s) > \\ &+ \frac{1}{imk^s} < |v|^{1/2}A_{j,m_j,\kappa_j}\psi^s_{\kappa_j,-}(k^s), v^{1/2}A_{j,m_j,\kappa_j}\psi^s_{\kappa_j,+}(k^s) >, \quad \lambda \in I_0. \end{aligned} \tag{4.42}$$

We summarize our results in

Theorem 4.3. *Assume Assumptions 4.1 to be fulfilled. Then the* partial wave *scattering matrix* $S_{\kappa_j}(\lambda, c)$ *is holomorphic in* c^{-2} *and*

$$S_{\kappa_j}(\lambda, c) = 1 - 2\pi i T(\lambda, c, Y) G_{2+}(\lambda, c) T(\lambda, c, Z)^* = \sum_{\ell=0}^{\infty} c^{-2\ell} S_{\kappa_j}^{(\ell)}(\lambda) \tag{4.43}$$

with

$$S_{\kappa_j}^{(0)}(\lambda) = 1 - \frac{4im}{k^s} \int_0^\infty dr \hat{j}_{l(\kappa_j)}(k^s r) v(r) \psi_{\kappa_j,+}^s(k^s r), \quad \lambda \in I_{+0} \tag{4.44}$$

the partial wave *scattering matrix for the associated pair of Pauli operators* (H_1, H_1^0) *(illustrating the nonrelativistic limit) and the explicit correction term of order* c^{-2}

$$\begin{aligned} S_{\kappa_j}^{(1)}(\lambda) = {} & \frac{(k^s)^4}{8m^3} \frac{dS_{\kappa_j}^{(0)}(\lambda)}{d\lambda} + \frac{k^s}{im} \int_0^\infty dr\, \psi_{\kappa_j,-}^s(k^s r) v(r) \psi_{\kappa_j,+}^s(k^s r) \\ & + \frac{1}{imk^s} \int_0^\infty dr \left(A_{j,m_j,\kappa_j} \psi_{\kappa_j,-}^s\right)(k^s r) v(r) \left(A_{j,m_j,\kappa_j} \psi_{\kappa_j,+}^s\right)(k^s r), \quad \lambda \in I_0. \end{aligned} \tag{4.45}$$

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Appendix A. Holomorphy in c^{-2} of the Dirac resolvent operator

In this Appendix we recall the main theorem from [8] concerning the holomorphy of the Dirac resolvent operator with respect to c^{-2} near $c^{-2} = 0$.

Theorem A.1. *Let* $H(c)$ *be defined as in Section 2 and fix* $z \in \mathbb{C} \setminus \mathbb{R}$. *Then*
(i) $(H(c) - mc^2 - z)^{-1}$ *is holomorphic with respect to* c^{-1} *around* $c^{-1} = 0$

$$\begin{aligned} & (H(c) - mc^2 - z)^{-1} \\ & = \left(1 + \begin{pmatrix} 0 & (2mc)^{-1}(H_1 - z)^{-1} A^*(V_2 - z) \\ (2mc)^{-1} A (H_1^0 - z)^{-1} V_1 & (2mc^2)^{-1} z (H_2^0 - z)^{-1}(V_2 - z) \end{pmatrix}\right)^{-1} \\ & \times \begin{pmatrix} (H_1 - z)^{-1} & (2mc)^{-1}(H_1 - z)^{-1} A^* \\ (2mc)^{-1} A (H_1^0 - z)^{-1} & (2mc^2)^{-1} z (H_2^0 - z)^{-1} \end{pmatrix}. \end{aligned} \tag{A.1}$$

(ii) $B(c)(H(c) - mc^2 - z)^{-1} B(c)^{-1}$ *is holomorphic with respect to* c^{-2} *around* $c^{-2} = 0$ *and*

$$\begin{aligned} & B(c)(H(c) - mc^2 - z)^{-1} B(c)^{-1} \\ & = \left(1 + \begin{pmatrix} 0 & (2mc^2)^{-1}(H_1 - z)^{-1} A^*(V_2 - z) \\ 0 & (2mc^2)^{-1}\big((2m)^{-1} A (H_1 - z)^{-1} A^* - 1\big)(V_2 - z) \end{pmatrix}\right)^{-1} \\ & \times \begin{pmatrix} (H_1 - z)^{-1} & (2mc^2)^{-1}(H_1 - z)^{-1} A^* \\ (2m)^{-1} A (H_1 - z)^{-1} & (2mc^2)^{-1}\big((2m)^{-1} A (H_1 - z)^{-1} A^* - 1\big) \end{pmatrix}. \end{aligned} \tag{A.2}$$

First order expansions in (A.1) and (A.2) yield

$$(H(c) - mc^2 - z)^{-1} = \begin{pmatrix} (H_1 - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ c^{-1} \begin{pmatrix} 0 & (2m)^{-1}(H_1 - z)^{-1}A^* \\ (2m)^{-1}A(H_1 - z)^{-1} & 0 \end{pmatrix} + O(c^{-2}) \quad \text{(A.3)}$$

(clearly illustrating the nonrelativistic limit $|c| \to \infty$) and

$$B(c)(H(c) - mc^2 - z)^{-1}B(c)^{-1}$$
$$= \begin{pmatrix} (H_1 - z)^{-1} & 0 \\ (2m)^{-1}A(H_1 - z)^{-1} & 0 \end{pmatrix} + c^{-2} \begin{pmatrix} R_{11}(z) & R_{12}(z) \\ R_{21}(z) & R_{22}(z) \end{pmatrix} + O(c^{-4}), \quad \text{(A.4)}$$

where

$$\begin{aligned} R_{11}(z) &= (2m)^{-2}(H_1 - z)^{-1}A^*(z - V_2)A(H_1 - z)^{-1}, \\ R_{12}(z) &= (2m)^{-1}(H_1 - z)^{-1}A^*, \\ R_{21}(z) &= (2m)^{-2}\big((2m)^{-1}A(H_1 - z)^{-1}A^* - 1\big)(z - V_2)A(H_1 - z)^{-1}, \\ R_{22}(z) &= (2m)^{-1}\big((2m)^{-1}A(H_1 - z)^{-1}A^* - 1\big). \end{aligned} \quad \text{(A.5)}$$

Remark A.2. *The holomorphy of the free Dirac resolvent can be easily derived from*

$$\begin{aligned} H^0(c) &= \begin{pmatrix} mc^2 & cA^* \\ cA & -mc^2 \end{pmatrix}, \\ (H^0(c))^2 &= \begin{pmatrix} c^2A^*A + m^2c^4 & 0 \\ 0 & c^2AA^* + m^2c^4 \end{pmatrix}, \end{aligned} \quad \text{(A.6)}$$

and

$$\begin{aligned} (H^0(c) - z)^{-1} &= (H^0(c) + z)(H^0(c) + z)^{-1}(H^0(c) - z)^{-1} \\ &= (H^0(c) + z)((H^0(c))^2 - z^2)^{-1}. \end{aligned}$$

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