# AN EXPLICIT CHARACTERIZATION OF CALOGERO–MOSER SYSTEMS

FRITZ GESZTESY, KARL UNTERKOFLER, AND RUDI WEIKARD

Abstract. Combining theorems of Halphen, Floquet, and Picard and a Frobenius type analysis, we characterize rational, meromorphic simply periodic, and elliptic KdV potentials. In particular, we explicitly describe the proper extension of the Calogero–Moser locus associated with these three classes of algebro-geometric solutions of the KdV hierarchy with special emphasis on the case of multiple collisions between the poles of solutions. This solves a problem left open since the mid-1970s.

## 1. Introduction

The principal purpose of this paper is to analyze rational, meromorphic simply periodic, and elliptic (algebro-geometric) solutions of the Korteweg-de Vries (KdV) hierarchy of nonlinear evolution equations and the associated Calogero–Moser-type models. In particular, we derive an explicit characterization of the (properly extended) Calogero–Moser locus for stationary rational, periodic and elliptic solutions of the KdV hierarchy. Our techniques rely on a combination of a Frobenius-type analysis with results of Halphen, Floquet, and Picard in the rational, simply periodic, and elliptic case, respectively.

Next we describe this topic in more detail. We freely use the notation introduced in Appendix A in connection with the KdV hierarchy. In particular, we will often call a solution $q$ of some equation of the stationary KdV hierarchy (and hence of infinitely many such equations) a *KdV potential*.

We first consider the case of rational solutions $q$ of the stationary KdV hierarchy. All such (nonconstant) solutions $q$ are known to be necessarily of the form

$$q(z) = q_0 - \sum_{\ell=1}^{M} s_\ell(s_\ell+1)(z-\zeta_\ell)^{-2}, \tag{1.1}$$

where

$$\begin{aligned} &q_0 \in \mathbb{C}, \quad \{\zeta_\ell\}_{1\le\ell\le M} \subset \mathbb{C},\ \zeta_{\ell'} \ne \zeta_\ell \text{ for } \ell' \ne \ell,\ 1 \le \ell, \ell' \le M, \\ &s_\ell \in \mathbb{N},\ 1 \le \ell \le M, \quad \sum_{\ell=1}^{M} s_\ell(s_\ell+1) = g(g+1) \text{ for some } g \in \mathbb{N}. \end{aligned} \tag{1.2}$$

---

*Date*: May 28, 2003.

1991 *Mathematics Subject Classification.* Primary 33E05, 34C25; Secondary 58F07.

*Key words and phrases.* Rational, simply periodic, and elliptic KdV solutions, Calogero–Moser systems, Halphen, Floquet, and Picard theorems, KdV hierarchy.

Based upon work supported by the US National Science Foundation under Grant No. DMS-9970299.

The underlying spectral curve is then of the simple rational type

$$y^2 = (E - q_0)^{2g+1}. \tag{1.3}$$

(To avoid annoying case distinctions we will in almost all circumstances exclude the trivial case $N = g = 0$ in this paper.)

On the other hand, not every $q$ of the type (1.1), (1.2) is an algebro-geometric solution of the KdV hierarchy. In general, the points $\zeta_\ell$ must satisfy a set of intricate constraints. In fact, necessary and sufficient conditions on $\zeta_\ell$ for $q$ in (1.1) to be a rational KdV solution are given by

$$\sum_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{M} \frac{s_{\ell'}(s_{\ell'}+1)}{(\zeta_\ell - \zeta_{\ell'})^{2k+1}} = 0 \text{ for } 1 \leq k \leq s_\ell \text{ and } 1 \leq \ell \leq M. \tag{1.4}$$

This result was first derived by Duistermaat and Grünbaum [24, p. 199] in 1986, as a by-product of their investigations of bispectral pairs of differential operators. An elementary alternative derivation of this result on the basis of Halphen's theorem, describing the structure of fundamental systems of solutions of differential equations with rational coefficients (and a growth restriction at infinity), and an explicit Frobenius-type analysis were recently provided in our paper [39]. For the convenience of the reader we will summarize these results in Section 2.

For a fixed $g \in \mathbb{N}$, (1.2) and (1.4) yield a complete parametrization of all rational KdV potentials belonging to the spectral curve (1.3). In other words, they provide a complete characterization of the isospectral class of KdV potentials corresponding to (1.3). The constraints (1.4) represent the proper generalization of the locus of poles introduced by Airault, McKean, and Moser [8] in the sense that they explicitly describe the situation where poles are permitted to collide (i.e., where some of the $s_\ell > 1$). In this context it seems appropriate to recall the collisionless case associated with the traditional rational Calogero–Moser locus. In that case $q$ is of the form,

$$q(z) = q_0 - 2\sum_{j=1}^{N}(z - z_j)^{-2}, \tag{1.5}$$

where

$$\begin{aligned} &q_0 \in \mathbb{C}, \quad N = g(g+1)/2 \text{ for some } g \in \mathbb{N}, \\ &\{z_j\}_{1 \leq j \leq N} \subset \mathbb{C}, \quad z_j \neq z_{j'} \text{ for } j \neq j',\ 1 \leq j, j' \leq N \end{aligned} \tag{1.6}$$

and the corresponding Calogero–Moser locus is then given by

$$\sum_{j'=1, j' \neq j}^{N} (z_j - z_{j'})^{-3} = 0, \quad 1 \leq j \leq N. \tag{1.7}$$

Equations (1.1), (1.2), and (1.4) are then the proper extensions of the traditional equations (1.5), (1.6), and (1.7) in the presence of collisions, where some of the $z_j$ are permitted to cluster in groups of $s_\ell(s_\ell+1)/2$ mutually distinct points $\zeta_\ell$ with $\sum_{\ell=1}^{M} s_\ell(s_\ell+1) = 2N$, $s_\ell \in \mathbb{N}$, $1 \leq \ell \leq M$.

In the case of elliptic solutions of the stationary KdV hierarchy it is known that all such (nonconstant) solutions $q$ are necessarily of the form

$$q(z) = q_0 - \sum_{\ell=1}^{M} s_\ell(s_\ell+1)\wp(z - \zeta_\ell), \tag{1.8}$$

where

$$\begin{aligned} &q_0 \in \mathbb{C}, \quad \{\zeta_\ell\}_{1\le\ell\le M} \subset \mathbb{C},\ \zeta_{\ell'} \neq \zeta_\ell \text{ for } \ell' \neq \ell,\ 1 \le \ell, \ell' \le M, \\ &s_\ell \in \mathbb{N},\ 1 \le \ell \le M. \end{aligned} \tag{1.9}$$

(Here $\wp$ denotes the Weierstrass elliptic function, cf. [2, Ch. 18] and Appendix B.)

On the other hand, as in the rational context, not every $q$ of the type (1.8), (1.9) is an algebro-geometric solution of the KdV hierarchy. Again, the points $\zeta_\ell$ must satisfy an analogous set of intricate constraints. In fact, combining a Frobenius-type analysis and a theorem of Picard, describing the structure of solutions of differential equations with elliptic coefficients, we derive necessary and sufficient conditions on $\zeta_\ell$ for $q$ in (1.8) to be an elliptic solution. More precisely, our principal result, to be proven in Section 2, reads as follows.

**Theorem 1.1.** *Let $q$ be an elliptic function. Then $q$ is a Picard potential, that is, the differential equation $y'' + qy = Ey$ has a meromorphic fundamental system of solutions (w.r.t. $z$) for each value of the complex spectral parameter $E \in \mathbb{C}$, if and only if there are $M \in \mathbb{N}$, $s_\ell \in \mathbb{N}$, $1 \le \ell \le M$, $q_0 \in \mathbb{C}$, and pairwise distinct $\zeta_\ell \in \mathbb{C}$, $1 \le \ell \le M$, such that*

$$q(z) = q_0 - \sum_{\ell=1}^{M} s_\ell(s_\ell + 1)\wp(z - \zeta_\ell) \tag{1.10}$$

*and*

$$\sum_{\substack{\ell'=1 \\ \ell'\neq\ell}}^{M} s_{\ell'}(s_{\ell'} + 1)\wp^{(2k-1)}(\zeta_\ell - \zeta_{\ell'}) = 0 \text{ for } 1 \le k \le s_\ell \text{ and } 1 \le \ell \le M. \tag{1.11}$$

*Moreover, $q$ is an elliptic KdV potential if and only if $q$ is of the type* (1.10) *and the constraints* (1.11) *hold.*

To the best of our knowledge, a characterization of the elliptic Calogero–Moser locus in the presence of collissions of poles remained an open problem since the mid-1970s in spite of the extensive attention this topic has attracted over the years. Equations (1.11) provide an explicit solution of such a characterization. A discussion of the pertinent literature will be provided in Section 2.

Since $\wp(z)$ converges to $1/z^2$ in the limit as both of its periods tend to infinity, condition (1.4) is the rational analog of (1.11).

Again we briefly comment on the collisionless case associated with the traditional elliptic Calogero–Moser locus. In that case $q$ is of the form,

$$q(z) = q_0 - 2\sum_{j=1}^{N} \wp(z - z_j), \tag{1.12}$$

where

$$q_0 \in \mathbb{C}, \quad \{z_j\}_{1\le j\le N} \subset \mathbb{C}, \quad z_j \neq z_{j'} \text{ for } j \neq j',\ 1 \le j, j' \le N \tag{1.13}$$

and the corresponding Calogero–Moser locus is then given by

$$\sum_{j'=1, j'\neq j}^{N} \wp'(z_j - z_{j'}) = 0, \quad 1 \le j \le N. \tag{1.14}$$

Equations (1.10) and (1.11) are then the proper extensions of the traditional equations (1.12), (1.13), and (1.14) in the presence of collisions, where some of the $z_j$ are permitted to cluster in groups of $s_\ell(s_\ell+1)/2$ mutually distinct points $\zeta_\ell$ with $\sum_{\ell=1}^{M} s_\ell(s_\ell+1)=2N$, $s_\ell\in\mathbb{N}$, $1\leq \ell\leq M$.

We also prove the analog of Theorem 1.1 for the case of simply periodic meromorphic KdV potentials bounded near the ends of the period strip, by combining the same kind of Frobenius-type analysis with a variant of Floquet's theorem, describing the structure of solutions of differential equations with simply periodic meromorphic coefficients.

In Section 2 we provide the necessary background for rational, simply periodic, and elliptic KdV potentials and present our principal result on the extended Calogero–Moser locus in Theorem 2.11. Section 3 provides additional results on the extended Calogero–Moser locus in the rational and simply periodic cases. In particular, in these cases we prove that the extended Calogero–Moser locus is the closure of the traditional Calogero–Moser locus in an appropriate (in fact, canonical) topology. We also provide a detailed discussion of the isospectral manifold of simply periodic meromorphic KdV potentials in Section 3. Our final Section 4 then provides some applications to the time-dependent KdV hierarchy and the dynamics of poles of rational, simply periodic, and elliptic KdV solutions with particular emphasis on collisions of poles. Appendix A reviews basic facts on the KdV hierarchy, Appendix B summarizes essentials of elliptic functions, Appendix C recalls some results on symmetric products of Riemann surfaces, and Appendix D provides the proof of Theorem 2.15.

Although this paper is not directly concerned with the Kadomtsev-Petviashvili (KP) hierarchy and its connection with Calogero–Moser-type systems, it is clear that this connection is responsible for much of the fascination surrounding this circle of ideas. In this context we refer to [6], [20], [55], [59]–[62], [69], [70], [76], [93], [96], [97], [102].

## 2. Rational, simply periodic, and elliptic solutions of the stationary KdV hierarchy

In this section we recall an application of Halphen's theorem to rational solutions of the KdV hierarchy recently presented in [39] and then extend these arguments to simply periodic and elliptic KdV potentials using corresponding theorems by Floquet and Picard. More precisely, we revisit stationary rational KdV potentials bounded near infinity (cf. [1], [3], [7], [8], [19], [25], [48], [63], [65], [67], [68], [79], [95], [98] and the literature cited therein), stationary simply periodic KdV potentials bounded near the ends of the period strip (cf. [8], [83], [98]), and stationary elliptic KdV solutions (cf. [5], [8], [14], [16], [19], [21]–[23], [26]–[31], [35], [53], [61], [77], [78], [84]–[95] and the literature cited therein). In particular, we completely characterize the so called *locus* of Calogero–Moser-type systems employing an elementary Frobenius analysis. The time-dependent case (including a discussion of collisions) will be presented in the next section.

The principal results on the stationary KdV hierarchy, as needed in this section, are summarized in Appendix A, and we freely use these results and the notation established there in what follows.

We start by describing Halphen's original result. Consider the following $n$th-order differential equation

$$q_n(z)y^{(n)}(z) + q_{n-1}(z)y^{(n-1)}(z) + \cdots + q_0(z)y(z) = 0, \tag{2.1}$$

where $q_j$, $0 \le j \le n$, are polynomials, and the order of $q_n$ is at least the order of $q_j$ for all $0 \le j \le (n-1)$, that is,

$$q_m \text{ are polynomials, } 0 \le m \le n, \tag{2.2a}$$

$$q_m/q_n \text{ are bounded near } \infty \text{ for all } 0 \le m \le n-1. \tag{2.2b}$$

Then the following theorem due to Halphen holds.

**Theorem 2.1.** (*Halphen* [49], *Ince* [52, p. 372–375]) *Assume* (2.2) *and suppose the differential equation* (2.1) *has a meromorphic fundamental system of solutions. Then the general solution of* (2.1) *is of the form*

$$y(z) = \sum_{m=1}^{n} c_m r_m(z) e^{\lambda_m z}, \tag{2.3}$$

*where $r_m$ are rational functions, $\lambda_m \in \mathbb{C}$, $1 \le m \le n$, and $c_m$, $1 \le m \le n$ are arbitrary complex constants.*
*Conversely, suppose $r_m$ are rational functions and $\lambda_m, c_m \in \mathbb{C}$, $1 \le m \le n$. If $r_1(z)e^{\lambda_1 z}, \dots, r_n(z)e^{\lambda_n z}$ are linearly independent, then*

$$y(z) = \sum_{m=1}^{n} c_m r_m(z) e^{\lambda_m z} \tag{2.4}$$

*is the general solution of an $n$th-order equation of the type* (2.1)*, whose coefficients satisfy* (2.2).

For an extension of Theorem 2.1 to first-order $n \times n$ systems and the explicit structure of the corresponding fundamental system of solutions we refer to our recent paper [39].

Next, we treat the case of a simply periodic meromorphic potential. Halphen's theorem is then replaced by a variant of Floquet's theorem which we state below as Theorem 2.2.

First, we recall a few basic facts from the theory of meromorphic, simply periodic functions (for more information see, e.g., Markushevich [64, Ch. III.4]): If $f$ is a meromorphic periodic function with period $2\pi$, then $f^*(t) = f(-i\ln(t))$ is meromorphic on $\mathbb{C}\backslash\{0\}$. If $f$ is entire then $f^*$ is analytic on $\mathbb{C}\backslash\{0\}$. We call a function simply periodic if it is periodic but not doubly periodic. A meromorphic simply periodic function $q$ with period $\omega \in \mathbb{C}\backslash\{0\}$ which is bounded as $|\mathrm{Im}(z/\omega)|$ tends to infinity, is of the form

$$q(z) = \frac{a_0 + a_1 e^{2\pi i z/\omega} + \cdots + a_m e^{2\pi i m z/\omega}}{b_0 + b_1 e^{2\pi i z/\omega} + \cdots + b_m e^{2\pi i m z/\omega}}. \tag{2.5}$$

In particular, such functions have only finitely many poles in the period strip

$$\mathcal{S}_\omega = \{z \in \mathbb{C} \,|\, 0 \le \mathrm{Re}(z/\omega) < 1\}. \tag{2.6}$$

We will call such functions *bounded near the ends of the period strip* $\mathcal{S}_\omega$. Note that

$$\lim_{\mathrm{Im}(z/\omega)\to\infty} q(z) = \frac{a_0}{b_0} = q^*(0) \tag{2.7}$$

and

$$\lim_{\mathrm{Im}(z/\omega)\to -\infty} q(z) = \frac{a_m}{b_m} = q^*(\infty). \tag{2.8}$$

Next, consider the $n$th-order differential equation

$$y^{(n)}(z) + q_{n-1}(z)y^{(n-1)}(z) + \cdots + q_0(z)y(z) = 0, \tag{2.9}$$

where $q_j$, $0 \le j \le n-1$, are meromorphic, simply periodic functions with a common period $\omega \in \mathbb{C}\backslash\{0\}$, bounded near the ends of the period strip $\mathcal{S}_\omega$. Then the following variant of Floquet's theorem holds.

**Theorem 2.2.** (*Weikard* [99]) *Suppose the differential equation* (2.9) *has a meromorphic fundamental system of solutions. Then there exists a solution* $y_1$ *of the differential equation* (2.9) *of the form*

$$y_1(z) = R(\mathrm{e}^{2\pi i z/\omega}) \exp(i\lambda z), \tag{2.10}$$

*where* $R$ *is a rational function and* $\lambda$ *satisfies*

$$(i\lambda)^n + q^*_{n-1}(0)(i\lambda)^{n-1} + \cdots + q^*_0(0) = 0. \tag{2.11}$$

**Remark 2.3.** (*i*) This version of Floquet's theorem differs from the standard one by imposing considerably stronger hypotheses on the coefficients $q_j$ and the nature of all solutions of (2.9). In return it provides a considerably stronger conclusion with regard to the explicit form of the solution $y_1$. An extension of Theorem 2.2 to the case of first-order systems is discussed in [100].
(*ii*) The conditions in Theorem 2.2 apply to stationary soliton solutions of the Gelfand–Dickey hierarchy which are periodic with a purely imaginary period.

Finally, we turn to elliptic KdV solutions and start by describing Picard's original result. Consider the following $n$th-order differential equation

$$y^{(n)}(z) + q_{n-1}(z)y^{(n-1)}(z) + \cdots + q_0(z)y(z) = 0, \tag{2.12}$$

where $q_j$, $0 \le j \le n-1$, are elliptic functions associated with the same period lattice generated by the fundamental half-periods $\omega_1$, $\omega_3$ with $\mathrm{Im}(\omega_3/\omega_1) > 0$.

Assuming the fundamental system of solutions of (2.1) to be meromorphic, the following theorem due to Picard holds.

**Theorem 2.4.** (*Picard* [71]–[73], *Ince* [52, p. 372–375]) *Suppose the differential equation* (2.12) *has a meromorphic fundamental system of solutions. Then there exists a solution* $y_1$ *of* (2.12) *which is elliptic of the second kind, that is,* $y_1$ *is meromorphic and there exist constants* $\rho_j \in \mathbb{C}$, $j = 1, 2$, *such that*

$$y_1(z + 2\omega_j) = \rho_j y_1(z), \quad j = 1, 3, \ z \in \mathbb{C}. \tag{2.13}$$

*If in addition, the characteristic equation corresponding to the translation* $z \to z + 2\omega_1$ *or* $z \to z + 2\omega_3$ (*see* [52, p. 358, 376]) *has distinct roots, then there exists a fundamental system of solutions of* (2.12) *which are elliptic functions of the second kind.*

The characteristic equation associated with the substitution $z \mapsto z + 2\omega_j$, $j = 1, 2$, alluded to in Theorem 2.4, is given by

$$\det[A - \rho I] = 0, \tag{2.14}$$

where

$$\phi_\ell(z + 2\omega_j) = \sum_{k=1}^{n} a_{\ell,k}\phi_k(z), \quad A = (a_{\ell,k})_{1\le \ell,k\le n} \tag{2.15}$$

and $\phi_1, \dots, \phi_n$ is any fundamental system of solutions of (2.12).

What we call Picard's theorem following the usual convention in [4, p. 182–185], [15, p. 338–343], [50, p. 536–539], [58, p. 181–189], appears, however, to have a much longer history. In fact, Picard's investigations [71]–[73] were inspired by earlier work of Hermite in the special case of Lamé's equation [51, p. 118–122, 266–418, 475–478] (see also [9, Sect. 3.6.4], and [103, p. 570–576]). Further contributions were made by Mittag-Leffler [66], and Floquet [32]–[34]. Detailed accounts of Picard's differential equation can be found in [50, p. 532–574], [58, p. 198–288].

For an extension of Theorem 2.4 to first-order $n \times n$ systems and the explicit structure of the corresponding fundamental system of solutions we refer to [40].

We continue by quoting a number of known results on stationary rational, simply periodic, and elliptic KdV potentials. To simplify notations in what follows we introduce the unifying notation $\mathcal{P}$ to denote

$$\mathcal{P}(z) = \begin{cases} z^{-2} & \text{in the rational case,} \\ \frac{\pi^2}{\omega^2}\Big([\sin(\pi z/\omega)]^{-2} - \frac{1}{3}\Big) & \text{in the simply periodic case,} \\ \wp(z) & \text{in the elliptic case.} \end{cases} \tag{2.16}$$

We note for later purposes that the three cases depicted in (2.16) can be viewed as specializations of the elliptic case in the following sense: we recall the invariants $g_2$ and $g_3$ associated with $\wp(\cdot) = \wp(\cdot\,|g_2, g_3)$ as introduced in (B.2). Then (cf. [2, p. 652]),

$$\mathcal{P}(z) = \begin{cases} \wp(z|0,0) & \text{in the rational case,} \\ \wp\big(z|[2\pi^2/\omega^2]^2/3, [2\pi^2/\omega^2]^3/27\big) & \text{in the simply periodic case,} \\ \wp(z|g_2, g_3) & \text{in the elliptic case.} \end{cases} \tag{2.17}$$

Here and in the following, the rational case always refers to rational potentials bounded near infinity, and similarly the simply periodic case always refers to meromorphic simply periodic potentials bounded near the ends of the period strip.

**Theorem 2.5.** *Let $N \in \mathbb{N}$, $\{z_j\}_{1\leq j\leq N} \subset \mathbb{C}$ and define $\mathcal{P}$ as in* (2.16)*.*
(*i*) (*Airault, McKean, and Moser* [8]*, Gesztesy and Weikard* [43]) *Any rational, simply periodic* (*bounded near the ends of the period strip*)*, or elliptic solution $q$ of some equation* (*and hence infinitely many equations*) *of the stationary KdV hierarchy, or equivalently, any rational, simply-periodic* (*bounded near the ends of the period strip*)*, or elliptic algebro-geometric KdV potential $q$, is necessarily of the form*

$$q(z) = q_0 - 2\sum_{j=1}^{N} \mathcal{P}(z - z_j), \tag{2.18}$$

*for some $q_0 \in \mathbb{C}$ and $N \in \mathbb{N}$. In the rational case, $N$ is of the special type*[1] *$N = g(g+1)/2$ for some $g \in \mathbb{N}$.*
(*ii*) (*Airault, McKean, and Moser* [8]*, Gesztesy and Weikard* [43]*, Weikard* [98]) *If one allows for "collisions" between the $z_j$, that is, if the set $\{z_j\}_{1\leq j\leq N}$ clusters into groups of points, then the corresponding algebro-geometric potential $q$ is necessarily*

[1] $N \in \mathbb{N}$ is called triangular if there is a $g \in \mathbb{N}$ such that $N = g(g+1)/2$.

*of the form*

$$q(z) = q_0 - \sum_{\ell=1}^{M} s_\ell(s_\ell+1)\mathcal{P}(z-\zeta_\ell), \tag{2.19}$$

$$= q_0 - 2\sum_{j=1}^{N} \mathcal{P}(z-z_j), \tag{2.20}$$

*where*

$$\begin{aligned} &\{z_j\}_{1\le j\le N} = \{\zeta_\ell\}_{1\le \ell\le M} \subset \mathbb{C} \text{ with } \zeta_\ell \text{ pairwise distinct,} \\ &s_\ell \in \mathbb{N},\ 1 \le \ell \le M, \quad \sum_{\ell=1}^{M} s_\ell(s_\ell+1) = 2N. \end{aligned} \tag{2.21}$$

$(iii)$ *The extreme case of all $z_j$ colliding into one point, say $\zeta_1$, that is, $\{z_j\}_{1\le j\le N} = \{\zeta_1\} \subset \mathbb{C}$ yields an algebro-geometric KdV potential* (*also called Lamé potential in the elliptic case, cf.* [41], [47] *and the extensive literature therein*) *of the form*

$$q(z) = q_0 - g(g+1)\mathcal{P}(z-\zeta_1), \quad g \in \mathbb{N} \tag{2.22}$$

*and no additional constraints on $\zeta_1 \in \mathbb{C}$.*
$(iv)$ *If $q$ is a KdV potential, the underlying hyperelliptic curve $\mathcal{K}_g$ is of the form*

$$\mathcal{K}_g \colon y^2 = \prod_{m=0}^{2g} (E - E_m) \text{ for some } \{E_m\}_{0\le m\le 2g} \subset \mathbb{C}. \tag{2.23}$$

*If $q$ is a simply periodic meromorphic KdV potential of period $\omega$, bounded near the ends of the period strip $\mathcal{S}_\omega$, one infers* (*Weikard* [99])

$$E_0 = q^*(0) = e_0, \quad E_{2p-1} = E_{2p} = e_p,\ 1 \le p \le g \text{ for some } \{e_m\}_{0\le m\le g} \subset \mathbb{C},$$

$$e_m \ne e_{m'} \text{ for } m \ne m',\ 0 \le m, m' \le g, \tag{2.24}$$

$$q^*(e^{2\pi i z/\omega}) = q(z), \tag{2.25}$$

*and the corresponding simply periodic* (*singular*) *hyperelliptic curve $\mathcal{K}_g$ in* (2.23) *reduces to the special form*

$$\mathcal{K}_g \colon y^2 = (E - e_0)\prod_{p=1}^{g} (E - e_p)^2. \tag{2.26}$$

*In the special case where $q$ is a rational KdV potential, one obtains*

$$E_0 = \cdots = E_{2g} = q_0 \tag{2.27}$$

*and hence* (2.23) *reduces to the especially simple form of a rational curve*

$$\mathcal{K}_g \colon y^2 = (E - q_0)^{2g+1}. \tag{2.28}$$

*In particular, the KdV potentials* (2.18), (2.19), *and* (2.22) *are all isospectral.*
$(v)$ (*Gesztesy and Weikard* [45], *Weikard* [98]) *Suppose $q$ is a rational function, or a meromorphic simply periodic function bounded near the ends of the period strip, or an elliptic function. Then $q$ is a KdV potential if and only if $\psi'' + (q-E)\psi = 0$ has a meromorphic fundamental system of solutions* (*w.r.t. $z$*) *for all values of the spectral parameter $E \in \mathbb{C}$.*
$(vi)$ (*Gesztesy, Unterkofler, and Weikard* [39]) *If $q$ is a rational KdV potential*

*of the form* (2.19)*, then* $y'' + qy = Ey$ *has linearly independent solutions of the Baker–Akhiezer-type*

$$\psi_\pm(E,z) = \big(\pm E^{1/2}\big)^{-g} \Bigg( \prod_{p=1}^{g} \big[\pm E^{1/2} - \nu_p(z)\big] \Bigg) e^{\pm E^{1/2} z}, \qquad E \in \mathbb{C}\backslash\{q_0\},\ z \in \mathbb{C} \tag{2.29}$$

*with* $\mu_p(z) = \nu_p(z)^2$*,* $1 \le p \le g$*, the zeros of* $F_g(z,x)$ *as defined in* (A.13) *and the elementary symmetric functions of* $\nu_p$*,* $1 \le p \le g$*, are rational functions.*
(*vii*) (*Weikard* [99]) *If* $q$ *is a simply periodic meromorphic KdV potential of period* $\omega$*, bounded near the ends of the period strip* $\mathcal{S}_\omega$*, of the form* (2.19)*, then* $y'' + qy = Ey$ *has linearly independent solutions of the Baker–Akhiezer-type*

$$\psi_\pm(E,z) = \Bigg( \sum_{m=0}^{g} r_m\big(e^{2\pi i z/\omega}\big)(\pm\lambda)^m \Bigg) e^{\pm\lambda z}, \quad r_g\big(e^{2\pi i z/\omega}\big) = 1, \tag{2.30}$$
$$E \in \mathbb{C}\backslash\{e_m\}_{0\le m\le g},\ e_0 = q^*(0),\ E = \lambda^2 + q^*(0),\ z \in \mathbb{C},$$

*where* $r_m$*,* $0 \le m \le g-1$*, are rational functions.*

**Remark 2.6.** (*i*) In connection with Theorem 2.5 (v) we note that the "only if" part follows from the explicit theta function representation of the Baker–Akhiezer function due to Its and Matveev [54] in the special case where $\mathcal{K}_g$ is nonsingular and from the loop group and $\tau$-function approach of Segal and Wilson [75] in the general case of possibly singular hyperelliptic curves.
(*ii*) Strictly speaking, the version of Theorem 2.5 (v) in the rational case proven in [98] assumes in addition to $q$ being rational, that $q$ is bounded near infinity. However, a simple inductive argument using (A.1) proves that a rational function $q$ unbounded near infinity cannot satisfy any of the stationary KdV equations (cf. [39]).
(*iii*) While (2.26) is not explicitly recorded in [99], it immediately follows from (2.30) by noting that the curve is of the form

$$\mathcal{K}_g\colon y^2 = W(\psi_+(\lambda,\cdot),\psi_-(\lambda,\cdot))^2 = (E - q^*(0)) \prod_{p=1}^{g} (E - e_p)^2 \tag{2.31}$$

for some $e_p \in \mathbb{C}$, $1 \le p \le g$.

**Remark 2.7.** Combining the explicit form of the rational and simply periodic hyperelliptic curves (2.28) and (2.26) with [36, Theorem 2.3] shows that all rational and meromorphic simply periodic KdV potentials (bounded near the ends of the period strip) satisfying s-KdV$_g(q) = 0$ (cf. (A.16)), can be generated from the genus zero case $q(x) = q_0$, respectively, $q(x) = q^*(0)$, by precisely $g$ Darboux transformations. This is in sharp contrast to the elliptic case and will play an important role in Section 3.

**Remark 2.8.** For future purposes we note the following $\tau$ function representation of the function $q$ in (2.18). In accordance with the three cases discussed in (2.16) we now define

$$\nu(z)=\begin{cases}\sigma(z|0,0)=z & \text{in the rational case,}\\ \sigma\big(z|[2\pi^2/\omega^2]^2/3,[2\pi^2/\omega^2]^3/27\big) & \\ =(\omega/\pi)\sin(\pi z/\omega)\exp[\pi^2z^2/(6\omega)^2] & \text{in the simply periodic case,}\\ \sigma(z|g_2,g_3) & \text{in the elliptic case}\end{cases} \tag{2.32}$$

with $\sigma(\cdot)=\sigma(\cdot\,|g_2,g_3)$ the Weierstrass $\sigma$-function in the elliptic case associated with the invariants $g_2$ and $g_3$ (cf. [2, Sect. 18.1]), and

$$\tau(z;z_1,\dots,z_N)=\prod_{j=1}^{N}\nu(z-z_j). \tag{2.33}$$

Then obviously,

$$\begin{aligned} q(z)&=q_0-2\sum_{j=1}^{N}\mathcal{P}(z-z_j),\\ &=q_0+2[\ln(\tau(z;z_1,\dots,z_N))]''. \end{aligned} \tag{2.34}$$

Theorems 2.1–2.4 motivate the following definition.

**Definition 2.9.** (*i*) Let $q$ be a rational function. Then $q$ is called a *Halphen potential* if it is bounded near infinity and if $y''+qy=Ey$ has a meromorphic fundamental system of solutions (w.r.t. $z$) for each value of the complex spectral parameter $E\in\mathbb{C}$.
(*ii*) Let $q$ be a simply periodic meromorphic function. Then $q$ is called a *Floquet potential* if it is bounded near the ends of the period strip and if $y''+qy=Ey$ has a meromorphic fundamental system of solutions (w.r.t. $z$) for each value of the complex spectral parameter $E\in\mathbb{C}$.
(*iii*) Let $q$ be an elliptic function. Then $q$ is called a *Picard potential* if $y''+qy=Ey$ has a meromorphic fundamental system of solutions (w.r.t. $z$) for each value of the complex spectral parameter $E\in\mathbb{C}$.

By Theorem 2.5 (v), $q$ is a Halphen (respectively, Floquet or Picard) potential if and only if $q$ is a rational (respectively, simply periodic meromorphic (bounded near the ends of the period strip) or elliptic) KdV potential, or equivalently, if and only if it satisfies one and hence infinitely many of the equations of the stationary KdV hierarchy (cf. Definition A.1).

Next, we turn to the principal aim of this paper, the precise restrictions on the set of poles $\{z_j\}_{1\le j\le N}=\{\zeta_\ell\}_{1\le \ell\le M}$ of $q$ in (2.18) to be a KdV potential. We start with the following known fact.

**Lemma 2.10.** *Suppose $q$ is meromorphic in a neighborhood of $z_0\in\mathbb{C}$ with a Laurent expansion about the point $z_0$ of the type*

$$q(z)=\sum_{j=0}^{\infty}q_j(z-z_0)^{j-2}, \tag{2.35}$$

*where $q_0=-s(s+1)$ and, without loss of generality,* $\mathrm{Re}(2s+1)\ge 0$. *Define for $\sigma\in\mathbb{C}$,*

$$f_0(\sigma)=-\sigma(\sigma-1)-q_0=(s+\sigma)(s+1-\sigma), \tag{2.36}$$

$$c_0(\sigma) = \begin{cases} 1 & \textit{if } 2s+1 \notin \mathbb{N}, \\ \prod_{j=1}^{2s+1} f_0(\sigma+j) & \textit{if } 2s+1 \in \mathbb{N}, \end{cases} \tag{2.37}$$

$$c_j(\sigma) = \frac{\sum_{m=0}^{j-1} q_{j-m} c_m(\sigma)}{f_0(\sigma+j)}, \ \ j \in \mathbb{N}, \tag{2.38}$$

$$w(\sigma, z) = \sum_{j=0}^{\infty} c_j(\sigma)(z-z_0)^{\sigma+j}, \tag{2.39}$$

$$v(\sigma, z) = \frac{\partial w}{\partial \sigma}(\sigma, z) = \sum_{j=0}^{\infty} \big(c_j'(\sigma) + c_j(\sigma)\ln(z-z_0)\big)(z-z_0)^{\sigma+j} \ \textit{if } (2s+1) \in \mathbb{N}_0. \tag{2.40}$$

*If* $(2s+1) \notin \mathbb{N}_0$, *then* $y'' + qy = 0$ *has the linearly independent solutions* $y_1 = w(s+1, \cdot)$ *and* $y_2 = w(-s, \cdot)$. *If* $(2s+1) \in \mathbb{N}_0$, *then* $y'' + qy = 0$ *has the linearly independent solutions* $y_1 = w(s+1, \cdot)$ *and* $y_2 = v(-s, \cdot)$.

*Moreover,* $y'' + qy = 0$ *has a meromorphic fundamental system of solutions near* $z_0$ *if and only if* $s \in \mathbb{N}_0$ *and* $c_{2s+1}(-s) = 0$.

This is a classical result in ordinary differential equations (cf., e.g., [52, Chs. XV, XVI]). A recent proof adapted to the present context can be found in Section 3 of [98]. We note that $q$ is not assumed to be rational, simply periodic, or elliptic in Lemma 2.10.

Our principal new result on simply periodic and elliptic solutions of the stationary KdV hierarchy then reads as follows (we recall our notational convention (2.16) to unify the rational, simply periodic, and elliptic cases by the symbol $\mathcal{P}$).

**Theorem 2.11.** *Let* $q$ *be a rational function bounded near infinity, or a simply periodic function bounded near the ends of the period strip, or an elliptic function. Then* $q$ *is a Halphen, or a Floquet, or a Picard potential if and only if there are* $M \in \mathbb{N}$, $s_\ell \in \mathbb{N}$, $1 \le \ell \le M$, $q_0 \in \mathbb{C}$, *and pairwise distinct* $\zeta_\ell \in \mathbb{C}$, $1 \le \ell \le M$, *such that*

$$q(z) = q_0 - \sum_{\ell=1}^{M} s_\ell(s_\ell+1)\mathcal{P}(z-\zeta_\ell), \tag{2.41}$$

*and*

$$\sum_{\substack{\ell'=1 \\ \ell' \ne \ell}}^{M} s_{\ell'}(s_{\ell'}+1)\mathcal{P}^{(2k-1)}(\zeta_\ell - \zeta_{\ell'}) = 0 \ \textit{for } 1 \le k \le s_\ell \textit{ and } 1 \le \ell \le M. \tag{2.42}$$

*Moreover,* $q$ *is a rational, simply periodic* (*bounded near the ends of the period strip*), *or elliptic KdV potential if and only if* $q$ *is of the type* (2.41) *and the constraints* (2.42) *hold.*
*In the particular rational case, for fixed* $g \in \mathbb{N}$, *the constraints* (2.42) *characterize the isospectral class of all rational KdV potentials associated with the curve* $y^2 = (E-q_0)^{2g+1}$, *where* $g(g+1) = \sum_{\ell=1}^{M} s_\ell(s_\ell+1)$.

*Proof.* The proof of the current theorem is analogous to the one presented in the rational case in [39]. However, we use this opportunity to improve the presentation of the proof and to remove some inaccuracies. As pointed out at the end of the proof, it is sufficient to focus on the elliptic case.

By Theorem 2.5 (v), it suffices to prove the characterization of Picard potentials. Suppose that $q$ is a nonconstant Picard potential. Then a pole $z_0$ of $q$ is a regular singular point of $y'' + qy = Ey$ and hence

$$q(z) - E = \sum_{j=0}^{\infty} Q_j (z - z_0)^{j-2} \tag{2.43}$$

in a sufficiently small neighborhood of $z_0$, where $Q_2$ is a polynomial of first degree in $E$, while $Q_j$ for $j \neq 2$ are independent of $E$. The indices associated with $z_0$, defined as the roots of $\sigma(\sigma-1)+Q_0 = 0$ (hence they are $E$- independent), must be distinct integers whose sum equals one. We denote them by $-s$ and $s+1$, where $s \in \mathbb{N}$, and note that $Q_0 = -s(s+1)$. We intend to prove that $Q_{2j+1} = 0$ whenever $j \in \{0, \dots, s\}$ by applying Lemma 2.10. Proceeding by way of contradiction, we thus assume that for some nonnegative integer $k \in \{0, \dots, s\}$, $Q_{2k+1} \neq 0$ and $k$ is the smallest such integer.

We note that $f_0(\cdot + j)$ are positive in $(-s-1, -s+1)$ for $j = 1, \dots, 2s$, whereas $f_0(\cdot + 2s + 1)$ has a simple zero at $-s$ and its derivative is negative at $-s$. Next one defines

$$\gamma_0(\sigma) = \prod_{j=1}^{2s+1} f_0(\sigma + j). \tag{2.44}$$

Note that $\gamma_0$ has a simple zero at $-s$ and that $\gamma_0'(-s)$ is negative.

The functions $c_0 = \gamma_0$ and $c_1 = Q_1\gamma_0/f_0(\cdot + 1)$ are polynomials with respect to $E$. Actually, $c_0$ has degree zero in $E$ and $c_1$ has degree at most zero ($c_1$ might be equal to zero). Hence the relations (2.45), (2.46), (2.47), and (2.48) below are satisfied for $j = 1$ if we let $\gamma_1(\sigma) = \gamma_0(\sigma)/f_0(\sigma + 1)$. Next let $\ell$ be some integer in $\{1, \dots, s\}$. Assume that there are suitable coefficients $\gamma_p$, $p = 0, ..., 2\ell - 1$ such that the functions $c_0, \dots, c_{2\ell-1}$ are polynomials in $E$ satisfying the relations

$$c_{2j-2}(\sigma) = \gamma_{2j-2}(\sigma) Q_2^{j-1} + O(E^{j-2}), \tag{2.45}$$

$$\gamma_{2j-2}(-s) = 0, \quad \gamma_{2j-2}'(-s) < 0, \tag{2.46}$$

$$c_{2j-1}(\sigma) = \gamma_{2j-1}(\sigma) Q_{2k+1} Q_2^{j-k-1} + O(E^{j-k-2}), \tag{2.47}$$

$$\gamma_{2j-1}(-s) = 0, \quad \gamma_{2j-1}'(-s) \leq 0 \tag{2.48}$$

for $1 \leq j \leq \ell$ as $E$ tends to infinity. Using the recursion relation (2.38) we then obtain that $c_{2\ell}(\sigma)$ and $c_{2\ell+1}(\sigma)$ are polynomials in $E$ and that

$$c_{2\ell}(\sigma) = \frac{\gamma_{2\ell-2}(\sigma)}{f_0(\sigma + 2\ell)} Q_2^{\ell} + O(E^{\ell-1}), \tag{2.49}$$

$$c_{2\ell+1}(\sigma) = \frac{\gamma_{2\ell-1}(\sigma) + \gamma_{2(\ell-k)}(\sigma)}{f_0(\sigma + 2\ell + 1)} Q_{2k+1} Q_2^{\ell-k} + O(E^{\ell-k-1}) \tag{2.50}$$

as $E$ tends to infinity. Letting $\gamma_{2\ell} = \gamma_{2\ell-2}/f_0(\cdot + 2\ell)$ and $\gamma_{2\ell+1} = (\gamma_{2\ell-1} + \gamma_{2(\ell-k)})/f_0(\cdot + 2\ell + 1)$ we find that relations (2.45), (2.46) and (2.47) are satisfied for $j = \ell + 1$. Moreover, relation (2.48) is also satisfied unless $\ell = s$. Hence we proved that $c_{2s+1}$ is a polynomial in $E$ and that

$$c_{2s+1}(\sigma) = \frac{\gamma_{2s-1}(\sigma) + \gamma_{2(s-k)}(\sigma)}{f_0(\sigma + 2s + 1)} Q_{2k+1} Q_2^{s-k} + O(E^{s-k-1}). \tag{2.51}$$

But both $\gamma_{2s-1}+\gamma_{2(s-k)}$ and $f_0(\cdot+2s+1)$ have simple zeros at $-s$. Therefore $\gamma_{2s+1}(-s)$ is different from zero. In fact

$$\gamma_{2s+1}(-s) = -\frac{\gamma'_{2s-1}(-s)+\gamma'_{2(s-k)}(-s)}{2s+1} > 0. \tag{2.52}$$

Lemma 2.10 then shows that $y''+qy=Ey$ has a solution which is not meromorphic whenever $E$ is not a root of the polynomial $c_{2s+1}(-s)$. This contradiction proves our assumption $Q_{2k+1}\neq 0$ wrong.

Since $Q_1=0$, we proved that if $q$ is a Picard potential with pairwise distinct poles $\zeta_1,\dots,\zeta_M$, then the principal part of $q$ about any pole $\zeta_\ell$ is of the form $-s_\ell(s_\ell+1)/(z-\zeta_\ell)^2$ for an appropriate positive integer $s_\ell$. Since $q$ is elliptic Theorem B.3 then proves (2.41). This immediately implies that for $z_0=\zeta_\ell$,

$$Q_{2k+1} = -\sum_{\substack{\ell'=1\\ \ell'\neq\ell}}^{M} s_{\ell'}(s_{\ell'}+1)\frac{1}{(2k-1)!}\mathcal{P}^{(2k-1)}(\zeta_\ell-\zeta_{\ell'}). \tag{2.53}$$

This proves necessity of the conditions (2.41) and (2.42) for $q$ to be a Picard potential. To prove their sufficiency we now assume that (2.41) and (2.42) hold. Then, if $z_0$ denotes any of the points $\zeta_\ell$, one infers that the corresponding $c_{2s_\ell+1}(-s_\ell)=0$. Lemma 2.10 then guarantees that all solutions of $y''+qy=Ey$ are meromorphic and hence that $q$ is a Picard potential.

The proof for simply periodic or rational potentials is virtually the same. Indeed, the proof presented in the elliptic case uses only the fact that $q$ is meromorphic and that elliptic functions allow a partial fractions expansion, which is true for simply periodic meromorphic functions, too. In particular, Lemma 2.10 does not rely on $q$ being elliptic. □

**Remark 2.12.** To the best of our knowledge, the explicit characterization (2.42) of the simply periodic and elliptic Calogero–Moser locus is new inspite of the considerable attention devoted to this circle of ideas. It solves a problem left open since the mid-1970s. The algebraic curves associated with various special cases of (2.41), (2.42) have been extensively studied and we refer, for instance, to [9, Sects. 7.7, 7.8], [10]–[13], [29]–[31], [35], [41]–[46], [56], [77], [78], [80], [81], [84]–[95].

**Remark 2.13.** *(i)* The necessary and sufficient conditions on $\zeta_\ell$ for $q$ in (2.41) to be a rational KdV potential were first obtained by Duistermaat and Grünbaum [24] in their analysis of bispectral pairs of differential operators. Our approach to proving the locus characterization (2.42) in [39] was based on Halphen's theorem and a direct Frobenius-type analysis exactly along the lines just presented in the elliptic case.
*(ii)* We note that the restrictions (2.42) simplify in the absence of collisions, where $s_\ell=1$, $1\le\ell\le N$. In this case (2.42) reduces to

$$\sum_{\substack{j'=1\\ j'\neq j}}^{N}\mathcal{P}'(z_j-z_{j'})=0, \quad 1\le j\le N, \tag{2.54}$$

which represents the well-known locus discussed by Airault, McKean, and Moser [8]. Equation (2.42) properly extends this locus to the case of collisions (i.e., to cases where some of the $s_\ell>1$). Historically, the locus defined by (2.54) is called the Calogero–Moser (CM) locus. Since the extended Calogero–Moser locus (2.42), that is,

$$\sum_{\substack{\ell'=1 \\ \ell'\neq\ell}}^{M} s_{\ell'}(s_{\ell'}+1)\mathcal{P}^{(2k-1)}(\zeta_\ell-\zeta_{\ell'})=0 \text{ for } 1\leq k\leq s_\ell \text{ and } 1\leq \ell\leq M, \tag{2.55}$$

was first derived by Duistermaat and Grünbaum in the rational case, from this point on we will call (2.55) the Duistermaat and Grünbaum (DG) locus. The CM and DG loci will be further explored in Sections 3 and 4 (cf. Theorems 3.7 and 4.3).
(*iii*) For $k=1$, conditions (2.42) coincide with the necessary conditions at collision points found by Airault, McKean, and Moser [8] in their Remark 1 on p. 113. However, since there are additional necessary conditions in (2.42) corresponding to $k\geq 2$, this disproves the conjecture made at the end of the proof of their Remark 1.
(*iv*) In the special elliptic case $N=3$, the DG locus (2.55) was explicitly determined by Airault, McKean, and Moser [8, p. 140] using a different method (in this case one simply joins the diagonal $z_1=z_2=z_3$ to the original CM locus, cf. (3.6)).
(*v*) In the rational case it is known that the CM locus is nonempty if and only if $N$ is of the type $N=g(g+1)/2$ for some $g\in\mathbb{N}$ (cf. [8], [75]). In the simply periodic case we will derive new results in Section 3. The analogous result in the elliptic case is more involved. Various examples in connection with Lamé and Treibich–Verdier potentials and their generalizations, in which the elliptic CM locus is nonempty, are discussed, for instance, in [8], [9, Sects. 7.7, 7.8], [10]–[13], [21], [23], [26]–[31], [35], [41]–[45], [47], [53], [56], [61], [77], [78], [80], [81], [84]–[95]. For a systematic treatment of the elliptic locus we refer, in particular, to [43], [84]–[87], and [89]–[94].

Next, we present a result on the KdV recursion coefficients $f_j$ (cf. Appendix A), extending Proposition 4 in [8].

**Theorem 2.14.** *Assume that $\{z_j\}_{1\leq j\leq N}\subset\mathbb{C}$ are pairwise distinct, $z_j\neq z_k$ for $j\neq k$, $1\leq j,k\leq N$ and suppose the CM locus conditions are satisfied, that is,*

$$\sum_{\substack{j=1 \\ j\neq k}}^{N} \mathcal{P}'(z_k-z_j)=0 \textit{ for } 1\leq k\leq N. \tag{2.56}$$

*In addition, let $q$ be a rational, simply periodic* (*bounded near the ends of the period strip*)*, or elliptic KdV potential of the form*

$$q(z)=q_0-2\sum_{j=1}^{N}\mathcal{P}(z-z_j). \tag{2.57}$$

*Then $q$ satisfies some of the equations of the stationary KdV hierarchy. Next, define the KdV recursion coefficients $f_j$ as in* (A.1)*. Then, $f_j$ are of the form*

$$f_0=1, \quad f_j(z)=d_j+\sum_{k=1}^{N} a_{j,k}\mathcal{P}(z-z_k), \quad j\in\mathbb{N} \tag{2.58}$$

*for some $\{a_{j,k}\}_{1\leq k\leq N}\subset\mathbb{C}$ and $d_j\in\mathbb{C}$, $j\in\mathbb{N}$. More precisely, $d_j$ is of the form*[2]

$$d_j=c_j(\underline{E})+\sum_{\ell=1}^{j} c_{j-\ell}(\underline{E})\frac{(2\ell-1)!!}{2^\ell \ell!}q_0^\ell, \quad j\in\mathbb{N} \tag{2.59}$$

[2]We use the standard abbreviations $(2q-1)!!=1\cdot 3\cdots(2q-1)$, $q\in\mathbb{N}$.

*with* $c_\ell(\underline{E})$, $\ell \in \mathbb{N}_0$ *given by* (A.26), *and* $a_{j,k}$ *satisfying the recursion relation*

$$a_{0,k} = 0,\ 1 \le k \le N, \quad d_0 = 1, \tag{2.60}$$

$$a_{j+1,k} = a_{j,k} q_0 - d_j - \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N} \big(a_{j,\ell} + 2a_{j,k}\big) \mathcal{P}(z_k - z_\ell), \quad j \in \mathbb{N}_0,\ 1 \le k \le N.$$

*Proof.* The choice

$$a_{0,k} = 0,\ 1 \le k \le N, \quad d_0 = 1 \tag{2.61}$$

proves (2.58) for $j = 0$. Next, assume that it is valid for some nonnegative integer $j$ and note that by (A.1)

$$f'_{j+1} = \frac{1}{4} f'''_j + q f'_j + \frac{1}{2} q' f_j, \quad j \in \mathbb{N}_0. \tag{2.62}$$

Next, we introduce the asymptotic expansion

$$\mathcal{P}(z) = z^{-2} + O(z^2) \text{ as } z \to 0, \tag{2.63}$$

and define the quantities

$$Q_{j,k,r} = a_{j,k} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N} \mathcal{P}^{(r)}(z_k - z_\ell), \quad R_{j,k,r} = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N} a_{j,\ell} \mathcal{P}^{(r)}(z_k - z_\ell), \tag{2.64}$$
$$j, r \in \mathbb{N}_0,\ 1 \le k \le N.$$

Then $Q_{j,k,1} = 0$, $j \in \mathbb{N}_0$, $1 \le k \le N$ by hypothesis (2.56) and one computes, as $z$ approaches $z_k$,

$$\frac{1}{4} f'''_j(z) = -6 a_{j,k} (z - z_k)^{-5} + O(1), \tag{2.65}$$

$$q(z) f'_j(z) = 4 a_{j,k} (z - z_k)^{-5} + (4Q_{j,k,0} - 2a_{j,k} q_0)(z - z_k)^{-3} - 2R_{j,k,1}(z - z_k)^{-2}$$
$$- 2(R_{j,k,2} - Q_{j,k,2})(z - z_k)^{-1} + O(1), \tag{2.66}$$

$$\frac{1}{2} q'(z) f_j(z) = 2a_{j,k}(z - z_k)^{-5} + 2(R_{j,k,0} + d_j)(z - z_k)^{-3} + 2R_{j,k,1}(z - z_k)^{-2}$$
$$+ (R_{j,k,2} - Q_{j,k,2})(z - z_k)^{-1} + O(1). \tag{2.67}$$

Since $f_{j+1}$ is a differential polynomial in $q$, it is a meromorphic function and hence the residues of its derivative are zero. This implies that

$$R_{j,k,2} - Q_{j,k,2} = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N} (a_{j,\ell} - a_{j,k}) \mathcal{P}''(z_k - z_\ell) = 0, \quad j \in \mathbb{N}_0,\ 1 \le k \le N. \tag{2.68}$$

Hence, as $z$ approaches $z_k$,

$$f'_{j+1} = (4Q_{j,k,0} + 2R_{j,k,0} + 2d_j - 2a_{j,k} q_0)(z - z_k)^{-3} + O(1). \tag{2.69}$$

Define

$$a_{j+1,k} = a_{j,k} q_0 - d_j - 2Q_{j,k,0} - R_{j,k,0} \tag{2.70}$$

and

$$p_{j+1}(z) = \sum_{k=1}^{N} a_{j+1,k} \mathcal{P}(z - z_k). \tag{2.71}$$

This implies that the function $f'_{j+1} - p'_{j+1}$, as well as its antiderivative $f_{j+1} - p_{j+1}$, are entire. Since $f_{j+1}$ is a differential polynomial in $q$, $f_{j+1} - p_{j+1}$ is equal to a constant, say $d_{j+1}$, in the elliptic case. In the simply periodic meromorphic, or rational case, $f_{j+1} - p_{j+1}$ is simply periodic meromorphic, or rational, and one arrives at the same conclusion by considering the behavior of $f_{j+1} - p_{j+1}$ at infinity. This proves

$$f_{j+1} = d_{j+1} + \sum_{k=1}^{N} a_{j+1,k}\mathcal{P}(z - z_k) \tag{2.72}$$

and hence (2.58). By induction on $j$ one verifies from (2.62) that $\hat{f}_j$, $j \in \mathbb{N}$, contains the term $\alpha_j q^j$, where

$$\alpha_{j+1} = \frac{2j+1}{2j+2}\alpha_j, \ j \in \mathbb{N}, \quad \alpha_1 = 1/2, \tag{2.73}$$

implying

$$\alpha_j = \frac{(2j-1)!!}{2^j j!}, \quad j \in \mathbb{N}. \tag{2.74}$$

Since according to (A.9),

$$f_j = \sum_{k=0}^{j} c_{j-k}\hat{f}_k \tag{2.75}$$

with $c_\ell = c_\ell(\underline{E})$ as defined in (A.28), one infers that the constant term in $f_j$ is of the form

$$c_j + \sum_{k=1}^{j} c_{j-k}\alpha_k q_0^k. \tag{2.76}$$

Together with (2.74) this proves (2.59), which in turn proves (2.60) because of (2.70). ☐

Finally, we derive the analog of (2.58) for $f_j$ in the presence of collisions. To the best of our knowledge, this is a new result. However, since the corresponding proof based on induction is a bit lengthy (even though the arguments involved are quite elementary), we defer its proof to Appendix D .

**Theorem 2.15.** *Assume $M \in \mathbb{N}$, $s_\ell \in \mathbb{N}$, $1 \le \ell \le M$, $q_0 \in \mathbb{C}$, and suppose $\zeta_\ell \in \mathbb{C}$, $\ell = 1, \dots, M$, are pairwise distinct. Consider*

$$q(z) = q_0 - \sum_{\ell=1}^{M} s_\ell(s_\ell + 1)\mathcal{P}(z - \zeta_\ell), \tag{2.77}$$

*and suppose the DG locus conditions*

$$\sum_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{M} s_{\ell'}(s_{\ell'} + 1)\mathcal{P}^{(2k-1)}(\zeta_\ell - \zeta_{\ell'}) = 0 \text{ for } 1 \le k \le s_\ell \text{ and } 1 \le \ell \le M \tag{2.78}$$

*are satisfied. Then*

$$f_0 = 1, \quad f_j(z) = d_j + \sum_{\ell=1}^{M} \sum_{k=1}^{\min(j,s_\ell)} a_{j,\ell,k}\mathcal{P}(z - \zeta_\ell)^k, \quad j \in \mathbb{N} \tag{2.79}$$

*for some $\{a_{j,\ell,k}\}_{1 \le k \le \min(j,s_\ell), 1 \le \ell \le M} \subset \mathbb{C}$ and $d_j \in \mathbb{C}$, $j \in \mathbb{N}$.*

## 3. Additional results on the CM and DG loci

The principal purpose of this section is a closer examination of the locus of poles with special emphasis on collisions. In particular, we will prove in the rational and simply periodic cases that the DG locus is the closure of the CM locus in an appropriate (in fact, canonical) topology.

Following our strategy of describing the rational, simply periodic, and elliptic cases simultaneously whenever possible, we first introduce

$$X = \begin{cases} \mathbb{C} & \text{in the rational case,} \\ \mathbb{C}/\Lambda_\omega & \text{in the simply periodic case,} \\ \mathbb{C}/\Lambda_{2\omega_1,2\omega_3} & \text{in the elliptic case,} \end{cases} \tag{3.1}$$

where $\Lambda_\omega$ denotes the period lattice

$$\Lambda_\omega = \{m\omega \in \mathbb{C} \,|\, m \in \mathbb{Z}\}, \quad \omega \in \mathbb{C}\backslash\{0\} \tag{3.2}$$

in the simply periodic case, and $\Lambda_{2\omega_1,2\omega_3}$ denotes the period lattice

$$\Lambda_{2\omega_1,2\omega_3} = \{2m\omega_1 + 2n\omega_3 \in \mathbb{C} \,|\, (m,n) \in \mathbb{Z}^2\}, \quad \omega_1, \omega_3 \in \mathbb{C}\backslash\{0\}, \ \mathrm{Im}(\omega_3/\omega_1) > 0 \tag{3.3}$$

in the elliptic case. In addition to the cartesian product $X^N = X \times \cdots \times X$ ($N$ factors), $N \in \mathbb{N}$, we also need to introduce the $N$th symmetric product $X^N/S_N$ of $X$ defined as in (C.1), with $S_N$ denoting the symmetric group on $N$ letters acting as the group of permutations of the factors in the cartesian product $X^N$. The elements of $X^N/S_N$ are denoted by $[z_1, \dots, z_N]$ and $X^N/S_N$ will be endowed with the quotient topology $\tau_{S_N}$ as discussed in Appendix C.

Next, we fix $N \in \mathbb{N}$ and define the corresponding Calogero–Moser (CM) locus of poles $\mathcal{L}_N \subset X^N/S_N$ by

$$\begin{aligned} \mathcal{L}_N = \Bigg\{ [z_1, \dots, z_N] \in X^N/S_N \,\Bigg|\, & \sum_{j'=1,\, j'\neq j}^{N} \mathcal{P}'(z_j - z_{j'}) = 0, \ 1 \leq j \leq N \\ & \text{and } z_j \neq z_{j'} \text{ for } j \neq j', \, 1 \leq j, j' \leq N \Bigg\} \end{aligned} \tag{3.4}$$

in the collisionless case.

In the presence of collisions, $\mathcal{L}_N$ needs to be extended to what we called the Duistermaat–Grünbaum (DG) locus in Remark 2.13 (ii), $\widehat{\mathcal{L}}_N$, defined by

$$\begin{aligned} &\widehat{\mathcal{L}}_1 = X = \mathcal{L}_1, \\ &\widehat{\mathcal{L}}_N = \mathcal{L}_N \cup \bigcup_{M=1}^{N-1} \bigcup_{\substack{s_1,\dots,s_M \in \mathbb{N} \\ \sum_{\ell=1}^{M} s_\ell(s_\ell+1) = 2N}} \mathcal{M}_{s_1,\dots,s_M}, \quad N \geq 2, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \mathcal{M}_{s_1} = \Big\{ [z_1, \dots, z_N] = [\underbrace{\zeta_1, \dots, \zeta_1}_{N}] \in X^N/S_N \Big\}, \\ s_1 \in \mathbb{N}, \ s_1(s_1+1) = 2N, \ M = 1, \end{aligned} \tag{3.6}$$

$$\mathcal{M}_{s_1,\dots,s_M} = \Bigg\{ [z_1,\dots,z_N] = [\underbrace{\zeta_1,\dots,\zeta_1}_{s_1(s_1+1)/2}, \underbrace{\zeta_2,\dots,\zeta_2}_{s_2(s_2+1)/2},\dots,\underbrace{\zeta_M,\dots,\zeta_M}_{s_M(s_M+1)/2}] \in X^N/S_N \,\Bigg|$$

$$\sum_{\ell'=1,\,\ell'\neq\ell}^{M} s_{\ell'}(s_{\ell'}+1)\mathcal{P}^{(2k-1)}(\zeta_\ell - \zeta_{\ell'}) = 0, \quad 1 \le k \le s_\ell,\ 1 \le \ell \le M$$

$$\text{and } \zeta_\ell \neq \zeta_{\ell'} \text{ for } \ell \neq \ell', 1 \le \ell, \ell' \le M \Bigg\}, \tag{3.7}$$

$$s_\ell \in \mathbb{N},\ 1 \le \ell \le M,\ \sum_{\ell=1}^{M} s_\ell(s_\ell+1) = 2N,\ M \ge 2.$$

In addition to the CM and DG loci we find it convenient to introduce the following additional locus

$$\mathcal{A}_N = \Bigg\{ [z_1,\dots,z_N] \in X^N/S_N \,\Bigg|\, q(z) = -2\sum_{j=1}^{N} \mathcal{P}(z-z_j) \text{ is an}$$

$$\text{algebro-geometric KdV potential} \Bigg\} \tag{3.8}$$

**Remark 3.1.** $(i)$ Some of the sets $\mathcal{M}_{s_1,\dots,s_M}$ in the decomposition (3.5) of the DG locus (3.5) may of course be empty. To illustrate this fact it suffices to consider the simple $g = 2$ ($N = 4$) elliptic example

$$q_2(z) = -2\sum_{j=1}^{4} \wp(z-\omega_j) \tag{3.9}$$

(here $\omega_4 = 0$, cf. Appendix B for the notation employed in connection with elliptic functions), a special case of the family of Treibich–Verdier examples analyzed in detail in [42]. In this case it is clear that the isospectral manifold of KdV potentials of $q_2$ contains no element of the form $\tilde q_2(z) = -8\wp(z-\zeta_1)$ for some $\zeta_1 \in \mathbb{C}$ since $8 \neq s_1(s_1+1)$ for any $s_1 \in \mathbb{N}$. In particular, there exists no possibility in the corresponding DG locus associated with the isospectral class of KdV potentials of the form $\hat q(z) = -2\sum_{j=1}^{4} \wp(z-z_j)$ for all $z_1,\dots,z_4$ to collide at a point $\zeta_1 \in \mathbb{C}$ and hence $\mathcal{M}_{s_1} = \emptyset$ in (3.5), (3.6) in connection with example (3.9). This simple example also shows that for fixed genus $g$, the corresponding set of elliptic KdV potentials correspond to several DG loci $\widehat{\mathcal{L}}_N$ for different values of $N$, in stark contrast to the rational case.
$(ii)$ Actually, it is easily seen that the situation is even more complicated in the elliptic case. An analysis of the following KdV potentials (cf. [42])

$$q_4(z) = -20\wp(z-\omega_j) - 12\wp(z-\omega_k), \tag{3.10}$$

$$\hat q_4(z) = -20\wp(z-\omega_j) - 6\wp(z-\omega_k) - 6\wp(z-\omega_\ell), \tag{3.11}$$

$$q_5(z) = -30\wp(z-\omega_j) - 2\wp(z-\omega_k), \tag{3.12}$$

$$\hat q_5(z) = -12\wp(z-\omega_j) - 12\wp(z-\omega_k) - 6\wp(z-\omega_\ell) - 2\wp(z-\omega_m), \tag{3.13}$$

where $j,\ k,\ \ell,\ m \in \{1,2,3,4\}$ are mutually distinct, then shows the following: The potentials $q_4$ and $\hat q_4$ correspond to genus $g = 4$ while $q_5$ and $\hat q_5$ correspond to $g = 5$. However, we note that all four potentials correspond to $N = 16$ in (2.57).

In addition, it can be shown that $q_5$ and $\hat{q}_5$ are isospectral while $q_4$ and $\hat{q}_4$ are not. In particular, since $q_4$ and $\hat{q}_4$ are not isospectral, there is no KdV flow that deforms $q_4$ into $\hat{q}_4$, and one infers that in the elliptic case the DG locus for fixed $N$ in general consists of several disconnected components. The latter fact is again in sharp contrast to the rational case where for fixed $N$ all potentials (with asymptotic value $q_0$ as $|z| \to \infty$) flow out of $q_g(z) = q_0 - g(g+1)z^{-2}$, $N = g(g+1)/2$.
(*iii*) The simply periodic case is somewhat intermediate between the rational and elliptic cases. While it is clearly more complex than the rational case (e.g., not all simply periodic KdV potentials flow out of a single potential such as $q_0 - g(g+1)z^{-2}$ in the simpler rational case), it is still possible to describe explicitly the connected components of the DG locus for fixed genus $g$ (cf. Theorem 3.16). This is related to the facts described in Remark 2.7.
(*iv*) We note that by Theorem 2.11,

$$\mathcal{L}_N \subset \widehat{\mathcal{L}}_N = \mathcal{A}_N. \tag{3.14}$$

Next we closely investigate the case of rational KdV potentials. In this case $X = \mathbb{C}$ and $\mathcal{P}(z) = z^{-2}$. We start with the following known result relating the coefficients and the roots of a polynomial.

**Lemma 3.2.** *Fix $N \in \mathbb{N}$, assume $r_0 = 1$, $(r_1, \dots, r_N) \in \mathbb{C}^N$ and let*

$$\tau_N(z) = \sum_{k=0}^{N} r_{N-k} z^k = \prod_{j=1}^{N} (z - z_j), \quad z \in \mathbb{C} \tag{3.15}$$

*be a monic polynomial of degree $N$ with divisor of zeros $[z_1, \dots, z_N] \in \mathbb{C}^N / S_N$. Introduce the map*

$$\Phi_{\tau_N} : \begin{cases} \mathbb{C}^N \to \mathbb{C}^N / S_N \\ (r_1, \dots, r_N) \mapsto [z_1(r_1, \dots, r_N), \dots, z_N(r_1, \dots, r_N)]. \end{cases} \tag{3.16}$$

*Then $\Phi_{\tau_N}$ is a homeomorphism.*

*Let $\Omega \subseteq \mathbb{C}^N$ and $\Phi_{\tau_N, \Omega} \colon \Omega \to \Phi_{\tau_N}(\Omega)$ be the restriction of $\Phi_{\tau_N}$ to $\Omega$. If $\Omega$ and $\Phi_{\tau_N}(\Omega)$ are both equipped with their relative topologies, then $\Phi_{\tau_N, \Omega}$ is a homeomorphism.*

*Proof.* Although the first part of this lemma is well-known, we briefly sketch a proof for completeness: As zeros of a polynomial vary continuously with the coefficients of a polynomial, $\Phi$ is continuous. The map $\Phi$ is clearly a bijection. The continuity of the inverse of $\Phi$ is obvious since the coefficients $r_\ell$ are polynomials (in fact, elementary symmetric functions) of the roots $[z_1, \dots, z_N]$.

It is clear that $\Phi_{\tau_N, \Omega}$ is a bijection. Let $V$ be an open set in $\Phi_{\tau_N}(\Omega)$. Then there is an open set $U \subset \mathbb{C}^N / S_N$ such that $V = U \cap \Phi_{\tau_N}(\Omega)$. The preimage of $V$ under $\Phi_{\tau_N, \Omega}$ equals $\Phi_{\tau_N}^{-1}(U) \cap \Omega$. Since $\Phi_{\tau_N}^{-1}(U)$ is open in $\mathbb{C}^N$, the preimage $\Phi_{\tau_N, \Omega}^{-1}(V)$ is open in $\Omega$. Thus $\Phi_{\tau_N, \Omega}$ is continuous. The continuity of $\Phi_{\tau_N, \Omega}^{-1}$ is shown analogously. $\square$

Next, let $R = \mathbb{C}[t_0, \dots, t_{g-1}]$ denote the ring of polynomials in $t_0, \dots, t_{g-1}$ with coefficients in $\mathbb{C}$ and $\tau_N$ a monic polynomial in $R[z]$ (the ring of polynomials in $z$ with coefficients in $R$) of degree $N = g(g+1)/2$ for some $g \in \mathbb{N}$. The polynomial

$\tau_N$ induces a map

$$\Psi_{\tau_N}\colon \begin{cases} \mathbb{C}^g \to X^N/S_N \\ (t_0,\dots,t_{g-1}) \mapsto [z_1,\dots,z_N], \end{cases} \tag{3.17}$$
$$z_j = z_j(r_1(t_0,\dots,t_{g-1}),\dots,r_N(t_0,\dots,t_{g-1})),\ 1\le j\le N,$$

where

$$\begin{aligned}\tau_N(t_0,\dots,t_{g-1},z) &= \sum_{k=0}^{N} r_{N-k}(t_0,\dots,t_{g-1})z^k \\ &= \prod_{j=1}^{N}(z - z_j(r_1(t_0,\dots,t_{g-1}),\dots,r_N(t_0,\dots,t_{g-1}))), \\ &\qquad r_0 = 1,\ z\in\mathbb{C}.\end{aligned} \tag{3.18}$$

We note that

$$\Psi_{\tau_N} = \Phi_{\tau_N,\Theta_{\tau_N}(\mathbb{C}^g)} \circ \Theta_{\tau_N}, \tag{3.19}$$

where

$$\Theta_{\tau_N}\colon \begin{cases} \mathbb{C}^g \to \mathbb{C}^N \\ (t_0,\dots,t_{g-1}) \mapsto (r_1(t_0,\dots,t_{g-1}),\dots,r_N(t_0,\dots,t_{g-1})) \end{cases} \tag{3.20}$$

and

$$\Phi_{\tau_N,\Theta_{\tau_N}(\mathbb{C}^g)} = \Phi_{\tau_N}|_{\Theta_{\tau_N}(\mathbb{C}^g)}. \tag{3.21}$$

Next, we list the following known results (we recall that $N\in\mathbb{N}$ is called triangular if there is a $g\in\mathbb{N}$ such that $N = g(g+1)/2$).

**Theorem 3.3.** (*Airault, McKean, and Moser* [8, Prop. 2.2, Cor. 3.2]) *If $N$ is triangular then $\mathcal{L}_N$ (and hence $\mathcal{A}_N$) is not empty. If $N$ is not triangular then $\mathcal{A}_N$ (and hence $\mathcal{L}_N$) is empty.*

**Theorem 3.4.** (*Airault, McKean, and Moser* [8, Thms. 3.2], *Adler and Moser* [3, Sect. 4]) *Suppose $g\in\mathbb{N}$ and $N = g(g+1)/2$. Then there exists a unique monic polynomial $\tau_N\in\mathbb{C}[t_0,t_1,\dots,t_{g-1}][z]$ of degree $N$ such that the map $\Psi_{\tau_N}\colon\mathbb{C}^g\to\mathcal{A}_N$, defined in* (3.17), (3.18)*, is a surjection. The algebro-geometric KdV potential $q_{\tau_N}$ associated with the divisor of zeros $[z_1,\dots,z_N]\in\mathcal{A}_N$ of $\tau_N$ is of the type*

$$q_{\tau_N}(t_0,\dots,t_{g-1},z) = q_0 + 2[\ln(\tau_N(t_0,\dots,t_{g-1},z))]'' \tag{3.22}$$

*with $q_0 = \lim_{|z|\to\infty} q_{\tau_N}(t_0,\dots,t_{g-1},z)$.*

**Theorem 3.5.** (*Adler and Moser* [3, Lemmas 2.2 and 2.3]) *The unique monic polynomial $\tau_N$ in* (3.22),

$$\tau_N(t_0,\dots,t_{g-1},z) = \sum_{k=0}^{N} r_{N-k}(t_0,\dots,t_{g-1})z^k, \quad r_0 = 1,\ z\in\mathbb{C}, \tag{3.23}$$

*has the following properties:*
*(i) Giving $t_m$ weight $2m+1$, $0\le m\le g-1$, then $r_j$ is isobaric of weight $j$, $1\le j\le N$.*
*(ii) The coefficient of $t_m$ in $r_{2m+1}$ is not equal to zero.*

The first part of Theorem 3.4 can be strengthened as follows.

**Theorem 3.6.** (*Airault, McKean, and Moser* [8, Thm. 3.2], *Adler and Moser* [3, Sect. 4]) *Let $g$ be a positive integer and $N = g(g+1)/2$. Then $\Psi_{\tau_N} : \mathbb{C}^g \to \mathcal{A}_N$, defined in* (3.17), (3.18), *is a homeomorphism.*

*Proof.* For completeness we sketch a proof. It was proven in Lemma 3.2 that $\Phi_{\tau_N,\Theta(\mathbb{C}^g)}$ is a homeomorphism from $\Theta_{\tau_N}(\mathbb{C}^g)$ to $\Phi_{\tau_N}(\Theta_{\tau_N}(\mathbb{C}^g))$. By Theorem 3.4, $\Phi_{\tau_N}(\Theta_{\tau_N}(\mathbb{C}^g)) = \Psi_{\tau_N}(\mathbb{C}^g) = \mathcal{A}_N$. Thus, we only have to show that $\Theta_{\tau_N}$ is a homeomorphism from $\mathbb{C}^g$ to $\Theta_{\tau_N}(\mathbb{C}^g)$.

Since the $r_j$ are polynomials in $t_0, \dots, t_{g-1}$, continuity of $\Theta_{\tau_N}$ is obvious. Next we prove by induction that $t_p \in \mathbb{C}[r_1, \dots, r_{2p+1}]$. By Theorem 3.5 one infers that $r_1 = \alpha_0 t_0$ with $\alpha_0 \neq 0$. Hence $t_0 = r_1/\alpha_0$. So the claim holds for $p = 0$. Next we assume it holds for $p = 0, \dots, m-1$. Again by Theorem 3.5 one infers that $t_m = (r_{2m+1} - \tilde{r}_{2m+1})/\alpha_m$, where $\tilde{r}_{2m+1}$ is a suitable polynomial in $\mathbb{C}[t_0, \dots, t_{m-1}]$. By the induction hypothesis $t_0, \dots, t_{m-1}$ are in turn polynomials in $r_1, \dots, r_{2m-1}$. This completes the induction step. Thus $\Theta_{\tau_N}$ is injective and $\Theta_{\tau_N}^{-1}$ is continuous. □

The next theorem contains our principal result in the case of rational KdV potentials; it details discussions in the literature concerning the closure of the Calogero–Moser locus (cf., e,g., [8], [65]). To the best of our knowledge, this is the first explicit characterization of the closure of the rational CM locus.

**Theorem 3.7.** *The DG locus $\widehat{\mathcal{L}}_N$ is the closure of the CM locus $\mathcal{L}_N$ in the quotient topology $\tau_{S_N}$ of $\mathbb{C}^N/S_N$,*

$$\mathcal{A}_N = \widehat{\mathcal{L}}_N = \overline{\mathcal{L}_N}. \tag{3.24}$$

*Proof.* The statement is trivial if $N$ is not triangular (since all sets are empty in this case). Hence we suppose for the rest of this proof that $N = g(g+1)/2$ for some $g \in \mathbb{N}$. The first equality in (3.24) is then the content of Theorem 2.11.

Let $\tau_N$ be the polynomial whose unique existence is guaranteed by Theorem 3.4. First we will prove that $\overline{\mathcal{L}_N} \subseteq \widehat{\mathcal{L}}_N$. Since, obviously, $\mathcal{L}_N \subseteq \widehat{\mathcal{L}}_N$, this follows provided that $\widehat{\mathcal{L}}_N$ is closed. But by Theorem 3.6 $\widehat{\mathcal{L}}_N = \Psi_{\tau_N}(\mathbb{C}^g)$ is the preimage of the closed set $\mathbb{C}^g$ under the continuous map $\Psi_{\tau_N}^{-1}$ and hence closed.

Next we prove that $\widehat{\mathcal{L}}_N \subseteq \overline{\mathcal{L}_N}$. Let

$$\Xi = \Big[\underbrace{\zeta_1, \dots, \zeta_1}_{s_1(s_1+1)/2}, \dots, \underbrace{\zeta_M, \dots, \zeta_M}_{s_M(s_M+1)/2}\Big], \qquad \sum_{\ell=1}^{M} s_\ell(s_\ell+1) = 2N \tag{3.25}$$

be an arbitrary point in $\widehat{\mathcal{L}}_N$. By Theorem 3.4 there is a point $\widetilde{T} = (\tilde{t}_0, \dots, \tilde{t}_{g-1}) \in \mathbb{C}^g$ such that $\Xi$ represents the roots of $\tau_N(\widetilde{T}, \cdot)$. The discriminant $\Delta_{\tau_N}$ of the polynomial $\tau_N(t_0, \dots, t_{g-1}, \cdot)$ is in turn a polynomial in $\mathbb{C}[t_0, \dots, t_{g-1}]$. By Theorem 3.3, $\Delta_{\tau_N}$ is not identically equal to zero, because otherwise $\mathcal{L}_N$ would be empty. Let $m$ denote an index for which $\Delta_{\tau_N}$ actually depends on $t_m$ and define $\delta \in \mathbb{C}[s]$ by

$$\delta(s) = \Delta_{\tau_N}(\tilde{t}_0, \dots, \tilde{t}_{m-1}, s, \tilde{t}_{m+1}, \dots, \tilde{t}_{g-1}). \tag{3.26}$$

Then there is a neighborhood of $\tilde{t}_m$ which contains only one zero of $\delta$ (namely, $\tilde{t}_m$). Let $t_{n,m} \in \mathbb{C}\backslash\{\tilde{t}_m\}$, $n \in \mathbb{N}$, be a sequence of points in this neighborhood which converges to $\tilde{t}_m$ as $n \to \infty$. Then

$$\Xi_n = \Psi_{\tau_N}(\tilde{t}_0, \dots, \tilde{t}_{m-1}, t_{n,m}, \tilde{t}_{m+1}, \dots, \tilde{t}_{g-1}) \tag{3.27}$$

is in $\mathcal{L}_N$ and converges to $\Xi$ as $n \to \infty$ by the continuity of $\Psi_{\tau_N}$. This proves the second equality in (3.24). $\square$

In the rational case the issue of the closure of the CM locus $\mathcal{L}_N$ can also be approached in an alternative manner. Since the actual details are rather involved, we describe the special case $N = 3$ which reveals some of the underlying mechanism. For this purpose we briefly recall some facts on elementary symmetric functions. Given $x_j \in \mathbb{C}$, $1 \le j \le N$, the elementary symmetric functions $\sigma_j = \sigma_j(x_1, \dots, x_N)$ of $x_1, \dots, x_N$ are defined by

$$\sigma_0(x_1, \dots, x_N) = 1, \quad \sigma_j(x_1, \dots, x_N) = \begin{cases} \sum_{\substack{\ell_1=1,\dots,\ell_j=1 \\ \ell_1 < \dots < \ell_j}}^N \prod_{k=1}^j x_{\ell_k}, & 1 \le j \le N, \\ 0, & j \ge N+1. \end{cases} \tag{3.28}$$

Alternatively, one can consider $s_j = s_j(x_1, \dots, x_N)$ defined by

$$s_0(x_1, \dots, x_N) = N, \quad s_j(x_1, \dots, x_N) = \sum_{k=1}^N x_k^j, \quad j \in \mathbb{N}. \tag{3.29}$$

We note the following well-known result.

**Lemma 3.8.** *Let $x_j \in \mathbb{C}$, $j = 1, \dots, N$, and define the elementary symmetric functions $\sigma_j = \sigma_j(x_1, \dots, x_N)$ and $s_j = s_j(x_1, \dots, x_N)$, $j \in \mathbb{N}_0$ as in* (3.28) *and* (3.29)*. Then*

$$\sum_{k=0}^{j-1} (-1)^k \sigma_k s_{j-k} + (-1)^j \sigma_j j = 0, \quad j \in \mathbb{N}. \tag{3.30}$$

*In particular, for $j \in \{1, \dots, N\}$, $s_j$ are polynomials in $\sigma_1, \dots, \sigma_j$, and for $j \ge N+1$, $s_j$ are polynomials in $\sigma_1, \dots, \sigma_N$. Conversely, for $j \in \{1, \dots, N\}$, $\sigma_j$ are polynomials in $s_1, \dots, s_j$.* (*All these polynomials are without constant term.*)

Given these preliminaries we return to the CM locus conditions in (2.54): They explicitly read for $N = 3$,

$$\sum_{\substack{j'=1 \\ j' \ne j}}^3 (z_j - z_{j'})^{-3} = 0, \quad 1 \le j \le 3, \text{ assuming } z_j \ne z_{j'} \text{ for } j \ne j',\ 1 \le j, j' \le 3. \tag{3.31}$$

Rewriting them in the form

$$[(z_3 - z_2)(z_3 - z_1)(z_2 - z_1)]^{-3} \gamma_j(z_1, z_2, z_3) = 0, \quad 1 \le j \le 3, \tag{3.32}$$

where the numerators $\gamma_j$ are certain polynomials in $z_1, z_2, z_3$. Using the fact that

$$\gamma_j(z_1, z_2, z_3) = 0,\ 1 \le j \le 3 \text{ is equivalent to } s_k(\gamma_1, \gamma_2, \gamma_3) = 0,\ 1 \le k \le 3, \tag{3.33}$$

one infers that

$$s_1(\gamma_1, \gamma_2, \gamma_3) = 0 \text{ is automatically satisfied by symmetry}, \tag{3.34}$$

$$s_2(\gamma_1, \gamma_2, \gamma_3) = 0 \text{ is equivalent to } s_1^2(z_1, z_2, z_3) - 3 s_2(z_1, z_2, z_3) = 0, \tag{3.35}$$

$$\begin{split} s_3(\gamma_1, \gamma_2, \gamma_3) = 0 \text{ is equivalent to } [s_1^2(z_1, z_2, z_3) - 3 s_2(z_1, z_2, z_3)]^3 \\ \times [2 s_1^2(z_1, z_2, z_3)^3 - 9 s_1^2(z_1, z_2, z_3) s_2^2(z_1, z_2, z_3) + 9 s_3^2(z_1, z_2, z_3)] = 0. \end{split} \tag{3.36}$$

Thus, the CM conditions (3.31) reduce to

$$s_1^2(z_1,z_2,z_3) - 3s_2(z_1,z_2,z_3) = 0 \text{ assuming } z_j \neq z_{j'} \text{ for } j \neq j',\ 1 \le j, j' \le 3. \tag{3.37}$$

One readily verifies that $s_1^2 - 3s_2 = 0$ is satisfied in particular on the diagonal, where all $z_j$ confluent to some point $z_0 \in \mathbb{C}$, that is, $z_1 = z_2 = z_3 = z_0$. Since confluence of only two points $z_j = z_k = z_0$, $z_\ell \neq z_0$ with $j,k,\ell$ pairwise distinct, is clearly impossible, this readily leads to the fact that the closure of the CM condition (3.37) in $\mathbb{C}^3/S_3$ is simply given by

$$s_1^2(z_1,z_2,z_3) - 3s_2(z_1,z_2,z_3) = 0. \tag{3.38}$$

In other words, the closure of the CM locus $\mathcal{L}_3$ is obtained by joining the diagonal $z_1 = z_2 = z_3$ to $\mathcal{L}_3$, in agreement with (3.24) and the description of $\widehat{\mathcal{L}}_3 = \mathcal{L}_3 \cup \mathcal{M}_2$ in (3.5), (3.7) (cf. also Remark 2.11 (*iv*)).

Next, we turn to the case of simply periodic meromorphic KdV potentials of period $\omega \in \mathbb{C}\backslash\{0\}$ bounded near the ends of the period strip $\mathcal{S}_\omega$. In this case

$$X = \mathbb{C}/\Lambda_\omega \text{ and } \mathcal{P}(z) = \frac{\pi^2}{\omega^2}\Big([\sin(\pi z/\omega)]^{-2} - \frac{1}{3}\Big). \tag{3.39}$$

We denote by $\mathbb{C}^*$ the set of nonzero complex numbers $\mathbb{C}\backslash\{0\}$ equipped with the relative topology in $\mathbb{C}$. We start with the following result.

**Lemma 3.9.** *Fix $N \in \mathbb{N}$, assume $r_0 = 1$, $(r_1,\dots,r_{N-1}) \in \mathbb{C}^{N-1}$, $r_N \in \mathbb{C}\backslash\{0\}$ and let*

$$\tau_N(u) = \sum_{k=0}^N r_k u^k = r_N \prod_{j=1}^N \big(u - e^{2\pi i z_j/\omega}\big), \quad u \in \mathbb{C} \tag{3.40}$$

*be a polynomial of degree $N$ with divisor of zeros $\big[e^{2\pi i z_1/\omega},\dots,e^{2\pi i z_N/\omega}\big] \in X^N/S_N$. Then all zeros of $\tau$ are nonzero and each has a logarithm*[3]*. In particular, $z_j \in X$, $1 \le j \le N$. Introduce the map*

$$\Phi_{\tau_N}\colon \begin{cases} \mathbb{C}^{N-1} \times \mathbb{C}^* \to X^N/S_N \\ (r_1,\dots,r_N) \mapsto [z_1(r_1,\dots,r_N),\dots,z_N(r_1,\dots,r_N)]. \end{cases} \tag{3.41}$$

*Then $\Phi_{\tau_N}$ is a homeomorphism.*

*Let $\Omega \subseteq \mathbb{C}^{N-1} \times \mathbb{C}^*$ and $\Phi_{\tau_N,\Omega}\colon \Omega \to \Phi_{\tau_N}(\Omega)$ be the restriction of $\Phi_{\tau_N}$ to $\Omega$. If $\Omega$ and $\Phi_{\tau_N}(\Omega)$ are both equipped with their relative topologies, then $\Phi_{\tau_N,\Omega}$ is a homeomorphism.*

*Proof.* The proof is similar to that of Lemma 3.2. □

In the following let $g \in \mathbb{N}$. Introducing the $g \times g$ Vandermonde matrix

$$\mathcal{V}(a_1,\dots,a_g) = \Big(a_p^{p'-1}\Big)_{1 \le p,p' \le g}, \quad a_p \in \mathbb{C},\ 1 \le p \le g, \tag{3.42}$$

and denoting its determinant by $\vartheta(a_1,\dots,a_g)$, that is,

$$\vartheta(a_1,\dots,a_g) = \det(\mathcal{V}(a_1,\dots,a_g)) = \prod_{\substack{p,p'=1 \\ p<p'}}^g (a_{p'} - a_p), \tag{3.43}$$

it is clear that $\vartheta(a_1,\dots,a_g) \neq 0$ if and only if the $a_p$ are pairwise distinct.

[3] We note that the logarithm of $e^{2\pi i z/\omega}$ is well-defined for $z \in X$.

Next, define the sets[4]

$$\mathcal{N}_g = \{(n_1,\dots,n_g)\in\mathbb{N}^g \,|\, n_1<n_2<\dots<n_g,\ \gcd(n_1,\dots,n_g)=1\} \tag{3.44}$$

and

$$\mathcal{N} = \bigcup_{g=1}^{\infty} \mathcal{N}_g. \tag{3.45}$$

For $\underline{n}=(n_1,\dots,n_g)\in\mathcal{N}$ we denote the number of its components by

$$\#(\underline{n}) = g. \tag{3.46}$$

For $\underline{n}=(n_1,\dots,n_g)\in\mathcal{N}_g$ and $\underline{v}=(v_1,\dots,v_g)\in\mathbb{C}^{*g}$ we also introduce the $g\times g$ matrix $T(\underline{n},\underline{v},u)$ by

$$T(\underline{n},\underline{v},u) = \Big(n_{p'}^{p-1}\Big[v_{p'}u^{n_{p'}} - (-1)^p\Big]\Big)_{1\le p,p'\le g}. \tag{3.47}$$

Moreover, we define[5]

$$\tau_N(\underline{n},\underline{v},u) = (-1)^{\lfloor g/2\rfloor}\frac{\det(T(\underline{n},\underline{v},u))}{\vartheta(n_1,\dots,n_g)}. \tag{3.48}$$

**Lemma 3.10.** *Suppose $\underline{n}\in\mathcal{N}_g$ and $\underline{v}\in\mathbb{C}^{*g}$. Then*

$$\tau_N(\underline{n},\underline{v},u) = (-1)^{\lfloor g/2\rfloor}\frac{\det(T(\underline{n},\underline{v},u))}{\vartheta(n_1,\dots,n_g)} = \sum_{k=0}^{N} r_k(\underline{v})u^k, \quad N=\sum_{p=1}^{g} n_p, \tag{3.49}$$

*where*

$$r_k(\underline{v}) = \sum_{\substack{\sigma_1=0,\dots,\sigma_g=0\\ \underline{n}\cdot\underline{\sigma}=k}}^{1} \frac{\vartheta\big((-1)^{\sigma_1}n_1,\dots,(-1)^{\sigma_g}n_g\big)}{\vartheta(n_1,\dots,n_g)} v_1^{\sigma_1}\cdots v_g^{\sigma_g}, \quad 0\le k\le N. \tag{3.50}$$

*In particular, assigning the weight $n_p$ to $v_p$, one infers the following properties of the coefficients $r_k(\underline{v})$, $0\le k\le N$:*
*(i) $r_k(\underline{v})$ is a polynomial of the variables $v_1,\dots,v_g$ isobaric of weight $k$.*
*(ii) $r_k(\underline{v})$ has degree at most one if it is considered as a polynomial of $v_p$ only.*
*(iii) $r_0(\underline{v})=1$ and $r_N(\underline{v})=(-1)^{\lfloor g/2\rfloor}v_1\cdots v_g$.*
*(iv) The coefficient of $v_1^{\sigma_1}\cdots v_g^{\sigma_g}$ in $r_k$ is different from zero for any $\underline{\sigma}\in\{0,1\}^g$ such that $\underline{n}\cdot\underline{\sigma}=k$.*

*Proof.* The $p$th column of $T(\underline{n},\underline{v},u))$ can be written as

$$v_p u^{n_p}\underline{\alpha}_p + \underline{\beta}_p, \tag{3.51}$$

where

$$\underline{\alpha}_p = \big(1,n_p,n_p^2,\dots,n_p^{g-1}\big)^\top, \tag{3.52}$$

$$\underline{\beta}_p = \big(1,-n_p,(-n_p)^2,\dots,(-n_p)^{g-1}\big)^\top, \quad 1\le p\le g. \tag{3.53}$$

Hence, for $\sigma_p\in\{0,1\}$, one infers that

$$\underline{\gamma}_p(\sigma_p) = \sigma_p\underline{\alpha}_p + (1-\sigma_p)\underline{\beta}_p \tag{3.54}$$

[4]Here $\gcd(n_1,\dots,n_g)$ abbreviates the greatest common divisor of $(n_1,\dots,n_g)\in\mathbb{N}^g$.
[5]Here $\lfloor x\rfloor$ denotes the greatest integer less or equal to $x\in\mathbb{R}$.

either equals $\underline{\alpha}_p$ or $\underline{\beta}_p$, $1 \leq p \leq g$. The multilinearity of the determinant then implies

$$\begin{aligned} \det(T(\underline{n}, \underline{v}, u))) &= \sum_{\sigma_1=0}^{1} \cdots \sum_{\sigma_g=0}^{1} (v_1 u^{n_1})^{\sigma_1} \cdots (v_g u^{n_g})^{\sigma_g} \det\big(\underline{\gamma}_1(\sigma_1), \dots, \underline{\gamma}_g(\sigma_g)\big) \\ &= \sum_{\sigma_1=0}^{1} \cdots \sum_{\sigma_g=0}^{1} v_1^{\sigma_1} \cdots v_g^{\sigma_g} u^{n_1\sigma_1 + \cdots + n_g \sigma_g} \det\big(\underline{\gamma}_1(\sigma_1), \dots, \underline{\gamma}_g(\sigma_g)\big). \end{aligned} \tag{3.55}$$

Next, noticing

$$\begin{aligned} \det\big(\underline{\gamma}_1(\sigma_1), \dots, \underline{\gamma}_g(\sigma_g)\big) &= \vartheta\big((-1)^{1+\sigma_1} n_1, \dots, (-1)^{1+\sigma_g} n_g\big) \\ &= (-1)^{\lfloor g/2 \rfloor} \vartheta\big((-1)^{\sigma_1} n_1, \dots, (-1)^{\sigma_g} n_g\big), \end{aligned} \tag{3.56}$$

one computes

$$\begin{aligned} \tau_N(\underline{n}, \underline{v}, u) &= (-1)^{\lfloor g/2 \rfloor} \frac{\det(T(\underline{n}, \underline{v}, u))}{\vartheta(n_1, \dots, n_g)} \\ &= \Big[(-1)^{\lfloor g/2 \rfloor} \vartheta\big(n_1, \dots, n_g\big)\Big]^{-1} \\ &\quad \times \sum_{\sigma_1=0}^{1} \cdots \sum_{\sigma_g=0}^{1} v_1^{\sigma_1} \cdots v_g^{\sigma_g} u^{\underline{n}\cdot\underline{\sigma}} \det\big(\underline{\gamma}_1(\sigma_1), \dots, \underline{\gamma}_g(\sigma_g)\big) \\ &= \sum_{k=0}^{N} \Bigg( \sum_{\substack{\sigma_1=0,\dots,\sigma_g=0 \\ \underline{n}\cdot\underline{\sigma}=k}}^{1} \frac{\vartheta\big((-1)^{\sigma_1} n_1, \dots, (-1)^{\sigma_g} n_g\big)}{\vartheta(n_1, \dots, n_g)} v_1^{\sigma_1} \cdots v_g^{\sigma_g} \Bigg) u^k. \end{aligned} \tag{3.57}$$

This implies all statements made in the lemma. ☐

Next we introduce the map

$$\Theta_{\underline{n}} \colon \begin{cases} \mathbb{C}^{*g} \to \mathbb{C}^{N-1} \times \mathbb{C}^* \\ (v_1, \dots, v_g) \mapsto (r_1, \dots, r_N). \end{cases} \tag{3.58}$$

**Lemma 3.11.** *$\Theta_{\underline{n}}$ is a homeomorphism from $\mathbb{C}^{*g}$ to $\Theta_{\underline{n}}(\mathbb{C}^{*g})$.*

*Proof.* Since the $r_k$ are polynomials in terms of the $v_p$, continuity of $\Theta_{\underline{n}}$ is obvious. Next we prove by induction that $v_p \in \mathbb{C}[r_1, \dots, r_{n_p}]$. By Lemma 3.10 one infers $r_{n_1} = \alpha_1 v_1$ with $\alpha_1 \neq 0$. Hence $v_1 = r_{n_1}/\alpha_1$ and the claim holds for $p = 1$. Assume it holds for $p = 1, \dots, m-1$. Again by Lemma 3.10 one infers that $v_m = (r_{n_m} - \tilde{r}_{n_m})/\alpha_m$, where $\tilde{r}_{n_m}$ is a suitable polynomial in $\mathbb{C}[v_1, \dots, v_{m-1}]$. By the induction hypothesis $v_1,\dots, v_{m-1}$ are, in turn, polynomials in terms of $r_1,\dots,r_{n_{m-1}}$. This completes the induction proof. This proves both that $\Theta_{\underline{n}}$ is injective and that $\Theta_{\underline{n}}^{-1}$ is continuous. ☐

**Lemma 3.12.** *Fix $\underline{n} \in \mathcal{N}$ with $\#(\underline{n}) = g$. Then the discriminant of $\tau_N(\underline{n}, \underline{v}, \cdot)$ is a nonzero polynomial in $v_1,\dots,v_g$.*

*Proof.* It is clear that the discriminant of $\tau_N(\underline{n}, \underline{v}, \cdot)$ is a polynomial in $v_1,\dots, v_g$. Next we will prove that it is not identically zero.

Let $\underline{v}_k = (v_1, \dots, v_k, 0, \dots, 0)$ and $f_k(u) = \tau_N(\underline{n}, \underline{v}_k, u)$. We will prove by induction that there is a choice of $v_1,\dots, v_k$ such that $f_k$ has $n_1 + \cdots + n_k$ simple zeros.

In particular, there is a choice of $\underline{v}$ such that $\tau_N(\underline{n}, \underline{v}, \cdot) = f_g$ has $N$ simple zeros and hence its discriminant is not identically equal to zero.

First one notes that $f_1(u) = 1 + c_1 v_1 u^{n_1}$ for some nonzero constant $c_1$. Thus, $f_1$ has $n_1$ simple zeros for any $v_1 \in \mathbb{C}^*$. Next, assume that $f_k$ has $n_1 + \cdots + n_k$ simple zeros. Choose $v_{k+1} = \varepsilon$ and define

$$\tilde{f}_{k+1}(\varepsilon, t) = t^{n_1+\cdots+n_{k+1}} f_{k+1}(1/t). \tag{3.59}$$

Then there are polynomials $g_{k+1}$ and $h_{k+1}$ of degree $n_1 + \cdots + n_k$ such that

$$\tilde{f}_{k+1}(\varepsilon, t) = t^{n_{k+1}} g_{k+1}(t) + \varepsilon h_{k+1}(t). \tag{3.60}$$

One notes that $g_{k+1}(0)$ is the coefficient of $u^{n_1+\cdots+n_k}$ in $f_k(u)$ and that $h_{k+1}(0)$ is the coefficient of $u^{n_1+\cdots+n_{k+1}}$ in $f_{k+1}(u)$. By statement (iv) of Lemma 3.10 both $g_{k+1}(0)$ and $h_{k+1}(0)$ are different from zero.

$\tilde{f}_{k+1}(0, \cdot)$ has $n_1 + \cdots + n_k$ simple zeros away from zero and a zero of multiplicity $n_{k+1}$ at zero. As the zeros of a polynomial are continuous functions of the coefficients, one infers for $\varepsilon$ sufficiently small, that $\tilde{f}_{k+1}(\varepsilon, \cdot)$ has $n_{k+1}$ zeros in some small disk $D_0$ centered at zero and $n_1 + \cdots + n_k$ simple zeros outside $D_0$. The zeros in $D_0$ have Puiseux expansions whose leading term is given by $\gamma \varepsilon^{1/n_{k+1}}$, where $\gamma$ is any of the $n_{k+1}$st roots of $-h_{k+1}(0)/g_{k+1}(0)$. This implies that there are $n_{k+1}$ simple roots in $D_0$. Thus all roots of $\tilde{f}_{k+1}$ and hence all roots of $f_{k+1}$ are simple. □

We briefly illustrate $\tau_N(\underline{n}, \underline{v}, u)$ with a few explicit examples.

**Example 3.13.**
$g = 1$: Then necessarily $n_1 = N = 1$ and

$$\tau_1(1, v_1, u) = 1 + v_1 u. \tag{3.61}$$

$g = 2$, $N = n_1 + n_2$:

$$\tau_{n_1+n_2}(\underline{n}, \underline{v}, u) = 1 - \frac{n_1 + n_2}{n_2 - n_1} v_1 u^{n_1} + \frac{n_1 + n_2}{n_2 - n_1} v_2 u^{n_2} - v_1 v_2 u^{n_1+n_2}. \tag{3.62}$$

$g = 3$, $n_1 = 1$, $n_2 = 2$, and $n_3 = 3$, $N = 6$:

$$\begin{aligned}\tau_6(\underline{n}, \underline{v}, u) = {} & 1 + 6 v_1 u - 15 v_2 u^2 + (10 v_3 - 10 v_1 v_2) u^3 + 15 v_1 v_3 u^4 \\ & - 6 v_2 v_3 u^5 - v_1 v_2 v_3 u^6.\end{aligned} \tag{3.63}$$

$g = 3$, $n_1 = 1$, $n_2 = 3$, and $n_3 = 4$, $N = 8$:

$$\begin{aligned}\tau_8(\underline{n}, \underline{v}, u) = {} & 1 + 10/3 v_1 u - 14 v_2 u^3 + 35/3 (v_3 - v_1 v_2) u^4 + 14 v_1 v_3 u^5 \\ & - 10/3 v_2 v_3 u^7 - v_1 v_2 v_3 u^8.\end{aligned} \tag{3.64}$$

$g = 4$, $n_1 = 1$, $n_2 = 3$, $n_3 = 4$, and $n_4 = 6$, $N = 14$:

$$\begin{aligned}\tau_{14}(\underline{n}, \underline{v}, u) = {} & 1 - 14/3 v_1 u + 42 v_2 u^3 - (175/3 v_3 + 49 v_1 v_2) u^4 + 98 v_1 v_3 u^5 \\ & + 21 v_4 u^6 - 50 (v_2 v_3 + v_1 v_4) u^7 + 21 v_1 v_2 v_3 u^8 + 98 v_2 v_4 u^9 \\ & - (175/3 v_1 v_2 v_4 + 49 v_3 v_4) u^{10} + 42 v_1 v_3 v_4 u^{11} \\ & - 14/3 v_2 v_3 v_4 u^{13} + v_1 v_2 v_3 v_4 u^{14}.\end{aligned} \tag{3.65}$$

Given these preparations we can now characterize the class of simply periodic meromorphic KdV potentials of period $\omega \in \mathbb{C}^*$, bounded near the ends of the period strip $\mathcal{S}_\omega$, as follows.

**Theorem 3.14.** *Let $g \in \mathbb{N}$ and assume $q$ is a simply periodic, meromorphic KdV potential of period $\omega \in \mathbb{C}^*$, bounded near the ends of the period strip $\mathcal{S}_\omega$, corresponding to the singular hyperelliptic curve $\mathcal{K}_g$ in* (2.26)*. Then $q$ is of the form*

$$q(z) = e_0 + 2\big[\ln\big(\tau_N\big(\underline{n}, \underline{v}, e^{2\pi i z/\omega}\big)\big)\big]'' \\ \text{for some } \underline{n} = (n_1, \dots, n_g) \in \mathcal{N}_g,\ \underline{v} = (v_1, \dots, v_g) \in \mathbb{C}^{*g},\ N = \sum_{p=1}^{g} n_p. \tag{3.66}$$

*Conversely, every $q$ of the form* (3.66) *is a simply periodic meromorphic KdV potential of period $\omega$, bounded near the ends of the period strip $\mathcal{S}_\omega$, corresponding to a singular hyperelliptic curve $\mathcal{K}_g$ of the form* (2.26)*.*

*Proof.* Suppose $q$ satisfies the hypotheses of the theorem. By [36, Theorem 2.3] (see also [37, App. G]), $g$ Darboux transformations at the mutually distinct energy parameters $e_p \in \mathbb{C}$, $1 \le p \le g$ (all different from $e_0 \in \mathbb{C}$), reduce $q$ to the constant potential $q_0 = e_0$. Reversing the $g$ Darboux transformations, using the Crum–Darboux approach discussed, for instance, in [38, Appendix A], then shows that $q$ is of the form

$$q(z) = e_0 + 2[\ln(W(\psi_1(e_1, z), \dots, \psi_g(e_g, z)))]'', \tag{3.67}$$

where

$$\psi_p(e_p, z) = A_p e^{(e_p - e_0)^{1/2} z} + B_p e^{-(e_p - e_0)^{1/2} z}, \quad e_p \ne e_{p'} \text{ for } p \ne p',\ 0 \le p, p' \le g \tag{3.68}$$

for some choice of $A_p, B_p \in \mathbb{C}^*$, $1 \le p \le g$ and $W(\psi_1, \dots, \psi_g)$ denotes the Wronskian of $\psi_1, \dots, \psi_g$. Introducing

$$v_p = A_p/B_p, \quad 1 \le p \le g, \tag{3.69}$$

$$\widetilde{T}(\underline{v}, z) = \Big(\big[(e_{p'} - e_0)^{1/2}\big]^{(p-1)}\Big[v_{p'} e^{2(e_{p'} - e_0)^{1/2} z} - (-1)^p\Big]\Big)_{1 \le p, p' \le g}, \tag{3.70}$$

a direct computation confirms that (3.67) can be rewritten as

$$q(z) = e_0 + 2\big[\ln\big(\det\big(\widetilde{T}(\underline{v}, z)\big)\big)\big]''. \tag{3.71}$$

In general, expressions such as (3.71) exhibit no periodicity properties with respect to $z$. Periodicity of $q$ is obtained as follows. By (3.70), $\det(\widetilde{T}(\underline{v}, z))$ is of the form

$$F\big(e^{2\pi i z_1/\omega_1}, \dots, e^{2\pi i z_g/\omega_g}\big)\big|_{z_1 = \dots = z_g = z}, \quad \omega_p = \pi i/(e_p - e_0)^{1/2},\ 1 \le p \le g \tag{3.72}$$

for some continuous function $F\colon \mathbb{C}^g \to \mathbb{C}$. Thus, $\det(\widetilde{T}(\underline{v}, \cdot))$ (and hence $q$) becomes periodic with respect to $z$ of period $\omega \in \mathbb{C}^*$ if and only if

$$\omega_p = \omega/n_p, \text{ that is, } (e_p - e_0)^{1/2} = \pi i n_p/\omega, \quad 1 \le p \le g \tag{3.73}$$

for some integers $n_p \in \mathbb{N}$, $1 \le p \le g$. By (3.68), the integers $n_p$ are necessarily mutually distinct. In addition, $\omega$ is a fundamental period[6] of $q$ if and only if $\gcd(n_1, \dots, n_g) = 1$. Thus, observing

$$\big[\ln\big(\det\big(\widetilde{T}(\underline{v}, z)\big)\big)\big]'' = \big[\ln\big(\tau_N\big(\underline{n}, \underline{v}, e^{2\pi i z/\omega}\big)\big)\big]'' \tag{3.74}$$

yields

$$q(z) = e_0 + 2\big[\ln\big(\tau_N\big(\underline{n}, \underline{v}, e^{2\pi i z/\omega}\big)\big)\big]'' \tag{3.75}$$

[6] $\omega$ is a fundamental period of $q$ if and only if every period $\Omega$ of $q$ is of the form $\Omega = m\omega$ for some $m \in \mathbb{Z}\backslash\{0\}$

and hence (3.66).

Conversely, suppose $q$ is of the form (3.66). Then,

$$q(z) = q^*\big(e^{2\pi iz/\omega}\big), \tag{3.76}$$

where

$$q^*(u) = e_0 - 8\pi^2\omega^{-2}\,\frac{u^2\tilde\tau_N''(u)\tilde\tau_N(u) + u\tilde\tau_N'(u)\tilde\tau_N(u) - u^2\tilde\tau_N'(u)^2}{\tilde\tau_N(u)^2}, \quad u = e^{2\pi iz/\omega}, \tag{3.77}$$

and we denoted $\tilde\tau_N(u) = \tau_N(\underline{n}, \underline{v}, e^{2\pi iz/\omega})$. We recall that $\tilde\tau_N(\cdot)$ is a polynomial of degree $N$ and hence it has precisely $N$ zeros counting multiplicities. As discussed above these correspond to precisely $N$ poles of the meromorphic function $q$ in each period strip. By inspection of (3.77) one confirms that the degree of the numerator of $q^*$ is less than $2N$ and that $q^*(0) = e_0$. Thus, $q^* - e_0$ tends to zero at the ends of the period strip $\mathcal{S}_\omega$. Reversing the argument leading from (3.67) to (3.75) then yields

$$q^*(u) = q(z) = e_0 + 2[\ln(W(\psi_1(e_1, z), \dots, \psi_g(e_g, z)))]'', \quad u = e^{2\pi iz/\omega}, \tag{3.78}$$

where

$$\psi_p(e_p, z) = \widetilde{A}_p e^{(e_p-e_0)^{1/2}z} + \widetilde{B}_p e^{-(e_p-e_0)^{1/2}z} \tag{3.79}$$

for some choice of $\widetilde{A}_p, \widetilde{B}_p \in \mathbb{C}^*$ satisfying

$$v_p = \widetilde{A}_p/\widetilde{B}_p, \quad 1 \le p \le g. \tag{3.80}$$

This proves that $q$ is obtained from the constant potential $q_0 = e_0$ by precisely $g$ Darboux transformations. Again applying [36, Theorem 2.3] then shows that $q$ is a simply periodic, meromorphic KdV potential of period $\omega$, bounded near the ends of the period strip, and associated with the singular hyperelliptic curve (2.26). $\square$

Summarizing, one obtains the following result.

**Corollary 3.15.** *The set*

$$\mathcal{S} = \{q(z) = C + 2[\ln(\tau_N(\underline{n}, \underline{v}, \exp(2\pi iz/\omega)))]'' \,|\, C \in \mathbb{C},\ \underline{n} \in \mathcal{N}_g,\ \underline{v} \in \mathbb{C}^{*g},\ g \in \mathbb{N}_0\} \tag{3.81}$$

*is precisely the set of simply periodic meromorphic KdV potentials of period $\omega$, bounded near the ends of the period strip $\mathcal{S}_\omega$.*

**Theorem 3.16.** *For $N \in \mathbb{N} - \{2\}$ there are finitely many $\underline{n} \in \mathcal{N}$ such that $\sum_{p=1}^{\#(\underline{n})} n_p = N$. For each of these $\underline{n}$, the map $\Phi \circ \Theta_{\underline{n}}$ is a homeomorphism from $\mathbb{C}^{*g}$ to its image, a closed subset of $\mathcal{A}_N$. Furthermore $\mathcal{A}_N$ is the union of these images over all possible choices of $\underline{n}$. In particular, $\mathcal{A}_N$ is a finite union of closed connected sets.*

*Proof.* The first statement is obvious. Next, we denote by $g = \#(\underline{n})$ the number of components in $\underline{n}$. By Lemma 3.11, $\Theta_{\underline{n}}$ is a homeomorphism from $\mathbb{C}^{*g}$ to $\Theta_{\underline{n}}(\mathbb{C}^{*g})$ and by Lemma 3.9, $\Phi$ restricted to this set is also a homeomorphism. Clearly, the image of $\Phi \circ \Theta_{\underline{n}}$ is a closed set in $\mathcal{A}_N$. This proves the second statement. Finally, pick any element $\Xi$ in $\mathcal{A}_N$ and let $q$ be the associated potential. By Corollary 3.15, $q \in \mathcal{S}$, that is, there exist $C \in \mathbb{C}$, $g \in \mathbb{N}$, $\underline{n} \in \mathcal{N}_g$ and $\underline{v} \in \mathbb{C}^{*g}$ such that $q(z) = C + 2[\ln(\tau_N(\underline{n}, \underline{v}, \exp(2\pi iz/\omega)))]''$. Since the number of poles of $q$ per period strip is $N$, we must have $n_1 + \cdots + n_g = N$. Thus, $\Xi = \Phi(\Theta_{\underline{n}}(\underline{v}))$. $\square$

**Theorem 3.17.** *The DG locus $\widehat{\mathcal{L}}_N$ is the closure of the CM locus $\mathcal{L}_N$ in the quotient topology $\tau_{S_N}$ of $X^N/S_N$,*

$$\mathcal{A}_N = \widehat{\mathcal{L}}_N = \overline{\mathcal{L}_N}. \tag{3.82}$$

*Proof.* The statement is trivial if $N = 2$ (all sets are empty). Hence we suppose for the rest of this proof that $N \neq 2$. The first equality in (3.68) is then the content of Theorem 2.11.

Since by Theorem 3.16, $\hat{\mathcal{L}}_N$ is closed, and since $\mathcal{L}_N \subseteq \hat{\mathcal{L}}_N$ it follows that $\overline{\mathcal{L}_N} \subseteq \hat{\mathcal{L}}_N$.

Next we prove that $\hat{\mathcal{L}}_N \subseteq \overline{\mathcal{L}_N}$. Let

$$\Xi = \Bigg[\underbrace{\zeta_1, \dots, \zeta_1}_{s_1(s_1+1)/2}, \dots, \underbrace{\zeta_M, \dots, \zeta_M}_{s_M(s_M+1)/2}\Bigg], \qquad \sum_{\ell=1}^{M} s_\ell(s_\ell + 1) = 2N \tag{3.83}$$

be an arbitrary point in $\hat{\mathcal{L}}_N$. By Theorem 3.16, there is an $\underline{n} \in \mathcal{N}$ and a $\underline{\tilde{v}} = (\tilde{v}_1, \dots, \tilde{v}_g) \in \mathbb{C}^{*g}$ such that $\Xi$ represents $\omega/(2\pi i)$ times the logarithm of the roots of $\tau_N(\underline{n}, \underline{\tilde{v}}, \cdot)$. The discriminant $\Delta_{\tau_N}$ of $\tau_N(\underline{n}, \underline{v}, \cdot)$ is a polynomial in $\mathbb{C}[v_1, \dots, v_g]$ which is not identically equal to zero according to Lemma 3.12. Let $m$ denote an index for which $\Delta_{\tau_N}$ actually depends on $v_m$ and define $\delta \in \mathbb{C}[w]$ by

$$\delta(w) = \Delta_{\tau_N}(\tilde{v}_1, \dots, \tilde{v}_{m-1}, w, \tilde{v}_{m+1}, \dots, \tilde{v}_g). \tag{3.84}$$

Then there is a neighborhood of $\tilde{v}_m$ which contains only one zero of $\delta$ (namely, $\tilde{v}_m$). Let $v_{n,m} \in \mathbb{C}^* \backslash \{\tilde{v}_m\}$, $n \in \mathbb{N}$, be a sequence of points in this neighborhood which converges to $\tilde{v}_m$ as $n \to \infty$. Then

$$\Xi_n = (\Phi \circ \Theta_{\underline{n}})(\tilde{v}_1, \dots, \tilde{v}_{m-1}, v_{n,m}, \tilde{v}_{m+1}, \dots, \tilde{v}_g) \tag{3.85}$$

is in $\mathcal{L}_N$ and converges to $\Xi$ as $n \to \infty$ by the continuity of $\Phi \circ \Theta_{\underline{n}}$. This proves the second equality in (3.82). $\square$

To the best of our knowledge, the precise structure of the isospectral set of simply periodic meromorphic KdV potentials bounded near the ends of the period strip as described in Theorem 3.16 and the explicit characterization of the closure of the simply periodic CM locus are new.

**Remark 3.18.** The corresponding results in the elliptic case require different techniques since elliptic KdV potentials cannot be constructed by finitely many Darboux transformations starting from constant potentials.

## 4. Some applications to the time-dependent KdV hierarchy

Rational, simply periodic, and elliptic KdV solutions are frequently discussed in a time-dependent setting and the dynamics of their poles is well-known to be in an intimate relationship with completely integrable systems of the Calogero–Moser-type. In our discussion below, the time-dependence (and the ensuing isospectral deformations of the KdV hierarchy) will be approached from the point of view of tracing trajectories in the DG locus (3.5) (the appropriate extension of the CM locus (3.4)), which permits an efficient description of the behavior of solutions in a neighborhood of collisions of their poles.

We start with a brief summary of the time-dependent setup and freely employ the notation used in Appendix A. Fix $r \in \mathbb{N}_0$ and suppose $q = q(x, t_r)$ satisfies the $r$th time-dependent KdV equation with initial condition $q^{(0)} = q^{(0)}\big(x, t_r^{(0)}\big)$ a solution of the $n$th stationary KdV equation for some $n \in \mathbb{N}$,

$$\widetilde{\mathrm{KdV}}_r(q) = q_{t_r} - 2\tilde{f}_{r+1,z} = 0, \tag{4.1}$$

$$q\big|_{t_r = t_r^{(0)}} = q^{(0)},$$

$$\text{s-KdV}_n(q^{(0)}) = -2f_{n+1,z} = 0, \tag{4.2}$$

where

$$q\big(z, t_r^{(0)}\big) = q^{(0)}(z) = q_0 - \sum_{\ell=1}^{M^{(0)}} s_\ell^{(0)}\big(s_\ell^{(0)} + 1\big)\mathcal{P}\big(z - \zeta_\ell^{(0)}\big) \tag{4.3a}$$

$$= q_0 - 2\sum_{j=1}^{N} \mathcal{P}\big(z - z_j^{(0)}\big) \tag{4.3b}$$

and

$$s_\ell^{(0)} \in \mathbb{N},\ 1 \leq s_\ell^{(0)} \leq M^{(0)}, \quad \sum_{\ell=1}^{M^{(0)}} s_\ell^{(0)}\big(s_\ell^{(0)} + 1\big) = 2N. \tag{4.4}$$

Here we assume in accordance with the paragraph following (A.30) that the set of integration constants $\tilde{c}_s$, $1 \leq s = 1 \leq r$ in $\tilde{f}_{r+1,z}$ and $c_j = c_j(\underline{E})$, $1 \leq j \leq n$ (cf. (A.28)) in $f_{n+1,z}$ are independent of each other as discussed in the paragraph following (A.30).

Next, taking advantage of the isospectral property of KdV flows, one can replace (4.1)–(4.4) by

$$\widetilde{\mathrm{KdV}}_r(q) = q_{t_r} - 2\tilde{f}_{r+1,z} = 0, \tag{4.5}$$

$$\text{s-KdV}_n(q) = -2f_{n+1,z} = 0, \tag{4.6}$$

or equivalently, by the pair of equations

$$q_{t_r} = \frac{1}{2}\widetilde{F}_{r,zzz} + 2(q - E)\widetilde{F}_{r,z} + q_z\widetilde{F}_r, \tag{4.7}$$

$$-\frac{1}{2}F_{n,zz}F_n + \frac{1}{4}F_{n,z}^2 + (E - q)F_n^2 = \prod_{m=0}^{2n}(E - E_m) \text{ for some } \{E_m\}_{m=0}^{2n} \subset \mathbb{C}, \tag{4.8}$$

where

$$q(z, t_r) = q_0 - 2\sum_{j=1}^{N} \mathcal{P}(z - z_j(t_r)) \tag{4.9a}$$

$$= q_0 - \sum_{\ell=1}^{M(t_r)} s_\ell(t_r)(s_\ell(t_r) + 1)\mathcal{P}(z - \zeta_\ell(t_r)) \tag{4.9b}$$

and for each $t_r \in \mathbb{C}$,

$$s_\ell(t_r) \in \mathbb{N},\ 1 \leq \ell \leq M(t_r), \quad \sum_{\ell=1}^{M(t_r)} s_\ell(t_r)(s_\ell(t_r) + 1) = 2N. \tag{4.10}$$

Below we will show that $z_j(t_r)$ locally have an algebraic behavior so that they can be labelled in such a manner that they remain continuous functions of $t_r$ even through the process of collisions. On the other hand, $s_\ell(t_r) \in \mathbb{N}$ are integer-valued and discontinuous with respect to $t_r$ at instances of collisions.

First we turn to a determination of the time-dependence of $z_j(t_r)$ in the absence of collisions.

**Lemma 4.1.** *Let $\Omega \subset \mathbb{C}^2$ be open and assume $q$ in (4.9b) satisfies (4.7), (4.8) on $\Omega$ for some set of constants $\tilde c_s$, $1 \le s \le r$. In addition suppose that $z_j(t_r)$ are pairwise disjoint for $q|_\Omega$. Then $z_j$ is analytic with respect to $t_r$ for $(z_j(t_r), t_r)$ in a sufficiently small neighborhood of any point $\big(z_0, t_r^{(0)}\big) \in \Omega$. Moreover, introducing the recursion relation*

$$\begin{aligned}
a_{0,j}(t_r) &= 0,\ 1 \le j \le N, \quad \tilde c_0 = 1, \\
a_{s+1,j}(t_r) &= a_{s,j}(t_r) q_0 - \tilde c_s - \sum_{p=1}^{s} \tilde c_{s-p} \alpha_p q_0^p \\
&\quad - \sum_{\substack{k=1 \\ k \ne j}}^{N} \big(a_{s,k}(t_r) + 2a_{s,j}(t_r)\big) \mathcal{P}(z_j(t_r) - z_k(t_r)), \qquad (4.11) \\
&\qquad 0 \le s \le r,\ 1 \le j \le N
\end{aligned}$$

*with $\alpha_p = 2^{-p}((2p-1)!!)/p!$, one obtains,*

$$\frac{dz_j}{dt_r}(t_r) = a_{r+1,j}(t_r), \quad 1 \le j \le N. \qquad (4.12)$$

*Proof.* By results of [75], the $\tau$-function for algebro-geometric (and hence for rational, simply periodic, and elliptic) solutions of the KdV hierarchy is entire with respect to $(z, \underline{t})$, where $\underline{t} = (t_0, t_1, t_2, \dots)$ comprises all time variables in Hirota's notation. Thus, choosing $t_s = \tilde c_{r-s} t_r$, $0 \le s \le r$ with $\tilde c_0 = 1$, and $t_s = 0$, $s \ge r+1$, the resulting $\tau$-function is entire in $(z, t_r)$ and hence $q$ is analytic in $(z, t_r)$ as long as $(z, t_r) \in \Omega$, that is, as long as collisions of the $z_j$ are avoided. Hence the implicit function theorem applied to the $\tau$-function (2.32), (2.33) yields analyticity of $z_j$ with respect to $t_r$ for $(z_j(t_r), t_r)$ in a sufficiently small neighborhood of any point $\big(z_0, t_r^{(0)}\big) \in \Omega$. Using (2.58) and (4.9a) one then computes

$$\begin{aligned}
q_{t_r}(z, t_r) &= 2 \sum_{j=1}^{N} \frac{dz_j}{dt_r}(t_r) \mathcal{P}'(z - z_j(t_r)) = 2\tilde f_{r+1,z}(z, t_r) \\
&= 2 \sum_{j=1}^{N} a_{r+1,j}(t_r) \mathcal{P}'(z - z_j(t_r)), \quad 1 \le j \le N, \qquad (4.13)
\end{aligned}$$

implying (4.12). □

Next we illustrate Theorem 2.14 and Lemma 4.1 with the simplest nontrivial rational example.

**Example 4.2.** The genus $g = 2$ ($N = 3$) example with $r = 1$ (see, e.g., [7], [24]). In this case one verifies

$$\begin{aligned}
q(z, t_1) &= q_0 + 2\partial_z^2[\ln(z^3 - 3t_1)] \qquad (4.14a) \\
&= q_0 - \frac{6z(z^3 + 6t_1)}{(z^3 - 3t_1)^2} \qquad (4.14b)
\end{aligned}$$

$$= q_0 - 2\sum_{j=1}^{3} \frac{1}{[z-(3t_1)^{1/3}\varepsilon_j]^2}, \quad t_1 \in \mathbb{C}, \tag{4.14c}$$

and hence

$$z_j(t_1) = (3t_1)^{1/3}\varepsilon_j, \quad \varepsilon_j = \exp(2\pi i j/3), \ \ j = 1,2,3, \tag{4.15}$$

and

$$\tau(z; z_1(t_1), z_2(t_1), z_3(t_1)) = \prod_{j=1}^{3}[z - z_j(t_1)] = z^3 - 3t_1, \tag{4.16}$$

explicitly illustrates the CM (respectively, DG) locus of poles in (2.54) (respectively, (2.55)). Moreover, one computes for the symmetric functions

$$\sigma_k = \sigma_k(z_1(t_1), z_2(t_1), z_3(t_1)), \quad 1 \le k \le 3 \tag{4.17}$$

and

$$s_\ell = s_\ell(z_1(t_1), z_2(t_1), z_3(t_1)), \quad \ell \in \mathbb{N}, \tag{4.18}$$

of $(z_1(t_1), z_2(t_1), z_3(t_1)$ that

$$\sigma_1 = \sigma_2 = 0, \ \sigma_3 = 3t_1, \tag{4.19}$$

$$s_{3k+1} = s_{3k+2} = 0, \ s_{3k} = 3(3t_1)^k, \ k \in \mathbb{N}_0. \tag{4.20}$$

In addition, one verifies the following facts,

$$c_0 = 1, \quad c_1 = -\frac{5}{2}q_0, \quad c_2 = \frac{15}{8}q_0^2, \tag{4.21}$$

$$\begin{aligned}
f_0(z,t_1) &= 1,\\
f_1(z,t_1) &= \frac{1}{2}q(z,t_1) + c_1 = -2q_0 - \frac{3z\,(z^3+6t_1)}{(z^3-3t_1)^2},\\
&= -2q_0 - \sum_{j=1}^{3}\frac{1}{[z-(3t_1)^{1/3}\varepsilon_j]^2},
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
f_2(z,t_1) &= \frac{1}{8}q_{zz}(z,t_1) + \frac{3}{8}q^2(z,t_1) - \frac{5}{4}q(z,t_1)q_0 + \frac{15}{8}q_0^2\\
&= q_0^2 + \frac{3\,q_0 z\,\left(z^3+6\,t_1\right)}{\left(z^3-3\,t_1\right)^2} + \frac{9\,z^2}{\left(z^3-3\,t_1\right)^2},\\
&= q_0^2 + \sum_{j=1}^{3}\frac{q_0 + (3t_1)^{-2/3}\varepsilon_j}{[z-(3t_1)^{1/3}\varepsilon_j]^2},
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
F_2(E,z,t_1) &= E^2 + f_1(z,t_1)E + f_2(z,t_1)\\
&= E^2 - \left(2q_0 + \frac{3z\,(z^3+6t_1)}{(z^3-3t_1)^2}\right)E\\
&\quad + q_0^2 + \frac{3\,q_0\,\left(6\,t_1\,z + z^4\right)}{\left(z^3-3\,t_1\right)^2} + \frac{9\,z^2}{\left(z^3-3\,t_1\right)^2},\\
&= (E-\mu_1(z,t_1))(E-\mu_2(z,t_1)),
\end{aligned} \tag{4.24}$$

$$\mu_{1,2}(z,t_1) = q_0 + \frac{3z\,(6\,t_1+z^3) \pm 3\,\sqrt{3z^5(12\,t_1 - z^3)}}{2\,\left(z^3-3\,t_1\right)^2}. \tag{4.25}$$

Finally, $q$ satisfies the following 2nd stationary nonhomogeneous KdV equation,

$$\text{s-KdV}_2(q) = \text{s-}\widehat{\text{KdV}}_2(q) - \frac{5}{2}q_0\,\text{s-}\widehat{\text{KdV}}_1(q) + \frac{15}{8}q_0^2\,\text{s-}\widehat{\text{KdV}}_0(q) = 0, \tag{4.26}$$

as well as the 1st nonhomogeneous time-dependent KdV equation

$$\text{KdV}_1(q) = q_{t_1} - \frac{1}{4}q_{zzz} - \frac{3}{2}qq_z + \frac{3}{2}q_0q_z = 0. \tag{4.27}$$

The following result explicitly connects the DG locus with rational, simply periodic, and elliptic solutions of the KdV hierarchy and also describes the local behavior of $z_j(t_r)$ as a function of $t_r$, including the case of collisions.

**Theorem 4.3.** *Fix $N \in \mathbb{N}$ and suppose $\widehat{\mathcal{L}}_N$ to be nonempty.*
*(i) Consider KdV solutions $q = q(z,t_r)$ of* (4.5), (4.6) *(for some set of constants $\tilde{c}_s$, $1 \leq s \leq r$) of the type* (4.9a)–(4.10)*. Then*

$$[z_1(t_r),\dots,z_N(t_r)] \subset \widehat{\mathcal{L}}_N, \quad t_r \in \mathbb{C}. \tag{4.28}$$

*(ii) Fix $N \in \mathbb{N}$ and $t_r^{(0)} \in \mathbb{C}$ and consider KdV solutions $q = q(z,t_r)$ of* (4.5), (4.6) *(for some set of constants $\tilde{c}_s$, $1 \leq s \leq r$) of the type* (4.9a)–(4.10)*, such that for $t_r = t_r^{(0)}$,*

$$q(z,t_r^{(0)}) = q_0 - \sum_{\ell=1}^{M^{(0)}} s_\ell^{(0)}\big(s_\ell^{(0)}+1\big)\mathcal{P}(z-\zeta_\ell^{(0)}). \tag{4.29a}$$

$$s_\ell^{(0)} \in \mathbb{N},\ 1 \leq \ell \leq M^{(0)}, \quad \sum_{\ell=1}^{M^{(0)}} s_\ell^{(0)}\big(s_\ell^{(0)}+1\big) = 2N. \tag{4.29b}$$

*Then, in a sufficiently small neighborhood of $\big(\zeta_\ell^{(0)}, t_r^{(0)}\big) \in \mathbb{C}^2$, $1 \leq \ell \leq M^{(0)}$, there exist precisely $s_\ell^{(0)}\big(s_\ell^{(0)}+1\big)/2$ points $z_{j_k}(t_r)$ (not necessarily distinct) such that*

$$q(z,t_r) \underset{\substack{z\to\zeta_\ell^{(0)}\\ t_r\to t_r^{(0)}}}{=} -2 \sum_{k=1}^{s_\ell^{(0)}(s_\ell^{(0)}+1)/2} \mathcal{P}(z-z_{j_k}(t_r)) + O(1) \tag{4.30}$$

*and each $z_{j_k}(t_r)$ has a Puiseux expansion of the type*

$$z_{j_k}(t_r) \underset{t_r\to t_r^{(0)}}{=} \zeta_\ell^{(0)} + \sum_{p=1}^{\infty} C_{j_k,\ell,p}\big(t_r - t_r^{(0)}\big)^{p/q_k} \tag{4.31}$$

*for some constants $C_{j_k,\ell,p} \in \mathbb{C}$, $p \in \mathbb{N}$, and appropriate $q_k \in \big\{1,\dots,s_\ell^{(0)}\big(s_\ell^{(0)}+1\big)/2\big\}$, $1 \leq k \leq s_\ell^{(0)}\big(s_\ell^{(0)}+1\big)/2$, $1 \leq \ell \leq M^{(0)}$. In particular, all elementary symmetric functions of the $z_{j_k}$, $1 \leq k \leq s_\ell^{(0)}\big(s_\ell^{(0)}+1\big)/2$ (cf.* (3.28), (3.29)*) are analytic in a neighborhood of $t_r^{(0)}$. Finally, in the special rational case, the $z_{j_k}$, $1 \leq k \leq s_\ell^{(0)}\big(s_\ell^{(0)}+1\big)/2$, are algebraic functions (on an appropriate compact Riemann surface).*
*(iii) $q$ defined by*

$$q(z) = q_0 - 2\sum_{j=1}^{N} \mathcal{P}(z-z_j) \tag{4.32}$$

*satisfies a particular stationary KdV equation* (*and hence is an algebro-geometric KdV potential*) *if and only if*

$$[z_1, \dots, z_N] \subset \widehat{\mathcal{L}}_N. \tag{4.33}$$

*Proof.* (*i*) The DG locus $\widehat{\mathcal{L}}_N$ as defined in (3.5) is a consequence of Theorem 2.11, the isospectral property of KdV flows (cf. (4.6), (4.8)), and the fact that any potential $q = q(z)$ in (2.41), (2.42) can be chosen as the initial value $q^{(0)}$ in (4.1), (4.2). Put differently, the poles $\{z_j(t_r)\}_{1\le j\le N}$ of any rational, simply periodic, and elliptic solution of (4.5), (4.6), of the form (4.9b) for fixed $N \in \mathbb{N}$, trace out curves on the DG locus (3.5) as $t_r$ varies in $\mathbb{C}$ as described in (4.28).
(*ii*) As mentioned in the proof of Lemma 4.1, the $\tau$-function associated with $q(z, t_r)$,

$$\tilde\tau(z, t_r) = \tau(z; z_1(t_r), \dots, z_N(t_r)) = \prod_{j=1}^{N} \nu(z - z_j(t_r)), \tag{4.34}$$

where

$$\nu(z) = \begin{cases} z & \text{in the rational case,} \\ (\omega/\pi)\sin(\pi z/\omega)\exp[\pi^2 z^2/(6\omega)^2] & \text{in the simply periodic case,} \\ \sigma(z) & \text{in the elliptic case,} \end{cases} \tag{4.35}$$

is entire in $(z, t_r)$. By (4.29),

$$\tilde\tau\big(z, t_r^{(0)}\big) = \big(z - \zeta_\ell^{(0)}\big)^{s_\ell^{(0)}(s_\ell^{(0)}+1)/2}\,\tilde{\tilde\tau}\big(z, t_r^{(0)}\big) \tag{4.36}$$

and

$$\tilde{\tilde\tau}\big(\cdot, t_r^{(0)}\big) \text{ is analytic and nonvanishing in a neighborhood of } \zeta_\ell^{(0)}. \tag{4.37}$$

Applying the Weierstrass preparation theorem (cf., e.g., [74, Sect. III.14]), one obtains

$$\begin{aligned} \tilde\tau(z, t_r) &= \sum_{k=0}^{s_\ell^{(0)}(s_\ell^{(0)}+1)/2} A_k(t_r)\big(z - \zeta_\ell^{(0)}\big)^{s_\ell^{(0)}(s_\ell^{(0)}+1)/2-k}\,\tilde{\tilde\tau}(z, t_r) && (4.38\text{a}) \\ &= \Bigg(\prod_{k=1}^{s_\ell^{(0)}(s_\ell^{(0)}+1)/2} [z - z_{j_k}(t_r)]\Bigg)\tilde{\tilde\tau}(z, t_r), && (4.38\text{b}) \end{aligned}$$

where

$$A_0(t_r) = 1, \quad A_k\big(t_r^{(0)}\big) = 0, \;\; 1 \le k \le s_\ell^{(0)}\big(s_\ell^{(0)} + 1\big)/2 \tag{4.39}$$

and

$$\tilde{\tilde\tau} \text{ is analytic and nonvanishing in a neighborhood of } \big(\zeta_\ell^{(0)}, t_r^{(0)}\big). \tag{4.40}$$

In particular, the elementary symmetric functions

$$A_k(t_r) = (-1)^k \sigma_k\Big(z_{j_1}(t_r), \dots, z_{j_{s_\ell^{(0)}(s_\ell^{(0)}+1)/2}}(t_r)\Big) \tag{4.41}$$

of the roots $z_{j_k}$ are all analytic in a neighborhood of $t_r^{(0)}$. By Lemma 3.8, also the corresponding symmetric functions $s_j$, $j \in \mathbb{N}$, of the $z_{j_k}$ are analytic at $t_r^{(0)}$. The roots $z_{j_k}$ of $\prod_{k=1}^{s_\ell^{(0)}(s_\ell^{(0)}+1)/2}(z - z_{j_k}(t_r))$ then permit a Puiseux expansion of the type

(4.31) (see, e.g., [64, p. 303–304]). In the special rational case the corresponding $\tau$-function is of the type (cf. [3])

$$\tilde{\tau}(z,t_r) = \sum_{k=0}^{s_\ell^{(0)}(s_\ell^{(0)}+1)/2} A_k(t_r)\big(z-\zeta_\ell^{(0)}\big)^{s_\ell^{(0)}(s_\ell^{(0)}+1)/2-k} \tag{4.42a}$$

$$= \prod_{k=1}^{s_\ell^{(0)}(s_\ell^{(0)}+1)/2} (z - z_{j_k}(t_r)) \tag{4.42b}$$

with $A_k(t_r)$ polynomials in $t_r$. Hence $z_{j_k}$ are (globally) algebraic functions of $t_r$. Part (iii) is just a reformulation of (parts of) Theorem 2.11. $\square$

**Remark 4.4.** $(i)$ In the elementary case of Example 4.2, where $g = 2$ ($N = 3$) and $r = 1$, Theorem 4.3 is explicitly illustrated by the results (4.19) and (4.20).
$(ii)$ In the case of the classical elliptic $N$-particle Calogero–Moser system on the circle (cf. [17, Ch. 2], [82]), a model that differs from the one describing the motion of poles of KdV solutions considered in this paper, it was shown in [35] that every symmetric elliptic function of the $N$ coordinates is meromorphic with respect to time.

**ACKNOWLEDGEMENTS.** We are indebted to Gilbert Weinstein and Finn Faye Knudsen for helpful discussions.

## Appendix A. The KdV hierarchy

In this section we review basic facts on the KdV hierarchy. Since this material is well-known, we confine ourselves to a brief account (for a detailed treatment, see, e.g., [37, Ch. 1]). Assuming for simplicity $q(\cdot,t)$ to be meromorphic in $\mathbb{C}$ for all $t \in \mathbb{C}$ and $q(x,\cdot)$ to be $C^1$ with respect to $t \in \mathbb{C}$ (except possibly for a discrete set) for all $x \in \mathbb{C}$, consider the recursion relation

$$f_0(z,t) = 1,$$
$$f_{j+1,z}(z,t) = \tfrac{1}{4} f_{j,zzz}(z,t) + q(z,t) f_{j,z}(z,t) + \tfrac{1}{2} q_z(z,t) f_j(z,t), \quad j \in \mathbb{N}_0 \tag{A.1}$$

(with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) and the associated differential expressions (Lax pair)

$$L_2(t) = \frac{d^2}{dz^2} + q(z,t), \tag{A.2}$$

$$P_{2n+1}(t) = \sum_{j=0}^{n} \left[ -\frac{1}{2} f_{j,z}(z,t) + f_j(z,t) \frac{d}{dz} \right] L_2^{n-j}(t), \quad g \in \mathbb{N}_0. \tag{A.3}$$

One can show that

$$[P_{2n+1}(t), L_2(t)] = 2 f_{n+1,z}(\cdot,t) \tag{A.4}$$

($[\cdot,\cdot]$ the commutator symbol) and explicitly computes from (A.1),

$$f_0 = 1,\ f_1 = \tfrac{1}{2} q + c_1,\ f_2 = \tfrac{1}{8} q_{zz} + \tfrac{3}{8} q^2 + c_1 \tfrac{1}{2} q + c_2, \text{ etc.}, \tag{A.5}$$

where $c_j \in \mathbb{C}$, $j \in \mathbb{N}$, are integration constants. For subsequent purposes we also introduce the corresponding homogeneous coefficients $\hat{f}_j$ defined by the vanishing of all integration constants $c_\ell = 0$, $1 \le \ell \le j$,

$$\hat{f}_0 = f_0 = 1, \quad \hat{f}_j = f_j\big|_{c_\ell = 0,\, \ell = 1,\dots,j}, \quad j \in \mathbb{N}. \tag{A.6}$$

If one assigns to $q^{(\ell)} = d^\ell q/dz^\ell$ the degree $\deg(q^{(\ell)}) = \ell + 2$, $\ell \in \mathbb{N}_0$, then the homogeneous differential polynomial $\hat{f}_j$ with respect to $q$ turns out to have degree $2j$, that is,

$$\deg(\hat{f}_j) = 2j, \quad j \in \mathbb{N}_0. \tag{A.7}$$

Introducing

$$c_0 = 1, \tag{A.8}$$

one verifies,

$$f_0 = \hat{f}_0 = 1, \quad f_j = \sum_{\ell=0}^{j} c_{j-\ell} \hat{f}_\ell, \quad j \in \mathbb{N}. \tag{A.9}$$

The KdV hierarchy is then defined as the sequence of evolution equations

$$\mathrm{KdV}_n(q) = L_{2,t} - [P_{2n+1}, L_2] = q_t - 2f_{n+1,z} = 0, \quad n \in \mathbb{N}_0. \tag{A.10}$$

Explicitly one obtains,

$$\begin{aligned} &\mathrm{KdV}_0(q) = q_t - q_z = 0, \\ &\mathrm{KdV}_1(q) = q_t - \tfrac{1}{4} q_{zzz} - \tfrac{3}{2} q q_z + c_1(-q_z) = 0, \text{ etc.} \end{aligned} \tag{A.11}$$

with $\mathrm{KdV}_1(.)|_{c_1=0}$ the usual KdV functional. Moreover, one verifies,

$$\mathrm{KdV}_n(q) = q_t - 2\sum_{\ell=0}^{n+1} c_{n-\ell} \hat{f}_{\ell,z} = 0, \quad n \in \mathbb{N}. \tag{A.12}$$

Next, introducing the polynomial $F_n(\cdot, z, t)$ of degree $n$,

$$F_n(E, z, t) = \sum_{\ell=0}^{n} f_{n-\ell}(z,t) E^\ell = \prod_{p=1}^{n} [E - \mu_p(z,t)], \tag{A.13}$$

(A.10) becomes

$$q_t = \frac{1}{2} F_{n,zzz} + 2(q - E) F_{n,z} + q_z F_n. \tag{A.14}$$

In the following we turn to the stationary case characterized by $q_t = 0$, or equivalently, by

$$[P_{2n+1}, L_2] = 0. \tag{A.15}$$

The corresponding stationary KdV hierarchy is then defined as the sequence of equations

$$\text{s-KdV}_n(q) = -[P_{2n+1}, L_2] = -2f_{n+1,z} = 0, \quad n \in \mathbb{N}_0. \tag{A.16}$$

Explicitly, this yields

$$\begin{aligned} &\text{s-KdV}_0(q) = -q_z = 0, \\ &\text{s-KdV}_1(q) = -\tfrac{1}{4} q_{zzz} - \tfrac{3}{2} q q_z + c_1(-q_z) = 0, \text{ etc.} \end{aligned} \tag{A.17}$$

Similarly, the corresponding homogeneous stationary KdV equations are then defined by

$$\text{s-}\widehat{\mathrm{KdV}}_n(q) = -2\hat{f}_{n+1,z} = 0, \quad n \in \mathbb{N}_0 \tag{A.18}$$

and one thus obtains from (A.12),

$$\text{s-KdV}_n(q) = \sum_{\ell=0}^{n} c_{n-\ell}\, \text{s-}\widehat{\mathrm{KdV}}_\ell(q). \tag{A.19}$$

Since $f_0(z) = 1$,

$$-\frac{1}{2}F_{n,zz}(E,z)F_n(E,z)+\frac{1}{4}F_{n,z}(E,z)^2+(E-q(z))F_n(E,z)^2 = R_{2n+1}(E,z) \quad \text{(A.20)}$$

is a monic polynomial in $E$ of degree $2n+1$. However, equations (A.1) and (A.16) imply that

$$\frac{1}{2}F_{n,zzz} - 2(E-q)F_{n,z} + q_z F_n = 0 \quad \text{(A.21)}$$

and this shows that $R_{2n+1}(E,z)$ is in fact independent of $z$. Hence it can be written as

$$R_{2n+1}(E) = \prod_{m=0}^{2n}(E-E_m) \text{ for some } \{E_m\}_{0\leq m\leq 2n} \subset \mathbb{C} \quad \text{(A.22)}$$

and (A.20) becomes

$$\begin{aligned} &-\frac{1}{2}F_{n,zz}(E,z)F_n(E,z)+\frac{1}{4}F_{n,z}(E,z)^2+(E-q(z))F_n(E,z)^2 \\ &= R_{2n+1}(E) = \prod_{m=0}^{2n}(E-E_m). \end{aligned} \quad \text{(A.23)}$$

By (A.15) the stationary KdV equation (A.16) is equivalent to the commutativity of $L_2$ and $P_{2n+1}$ and therefore, if $L_2\psi = E\psi$ one infers $P_{2n+1}^2\psi = R_{2n+1}(E)\psi$. Thus $[P_{2n+1}, L_2] = 0$ implies $P_{2n+1}^2 = R_{2n+1}(L_2)$ by the Burchnall–Chaundy theorem. This illustrates the intimate connection between the stationary KdV equation $f_{n+1,z} = 0$ in (A.16) and the compact (possibly singular) hyperelliptic curve $\overline{\mathcal{K}_n}$ of (arithmetic) genus $n$ obtained upon one-point compactification of the curve

$$\mathcal{K}_n\colon y^2 = R_{2n+1}(E) = \prod_{m=0}^{2n}(E-E_m) \quad \text{(A.24)}$$

by joining the point at infinity, denoted by $P_\infty$. Points $P \in \mathcal{K}_n$ are denoted by $P = (E,y)$.

The above formalism leads to the following standard definition.

**Definition A.1.** Any solution $q$ of one of the stationary KdV equations (A.16) is called an *algebro-geometric KdV potential*.

For brevity of notation we will occasionally call such $q$ simply *KdV potentials*.

Next, denoting $\underline{E} = (E_0,\dots,E_{2n})$, we consider

$$\left(\prod_{m=0}^{2n}\left(1-\frac{E_m}{z}\right)\right)^{1/2} = \sum_{k=0}^{\infty} c_k(\underline{E})z^{-k}, \quad \text{(A.25)}$$

where

$$\begin{aligned} &c_0(\underline{E}) = 1, \\ &c_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{2n}=0 \\ j_0+\cdots+j_{2n}=k}}^{k} \frac{(2j_0-3)!!\cdots(2j_{2n}-3)!!}{2^k j_0!\cdots j_{2n}!} E_0^{j_0}\cdots E_{2n}^{j_{2n}}, \ k\in\mathbb{N}, \end{aligned} \quad \text{(A.26)}$$

and hence the first few coefficients explicitly read

$$c_0(\underline{E}) = 1, \quad c_1(\underline{E}) = -\frac{1}{2}\sum_{m=0}^{2n} E_m,$$

$$c_2(\underline{E}) = \frac{1}{4} \sum_{\substack{m_1,m_2=0 \\ m_1<m_2}}^{2n} E_{m_1} E_{m_2} - \frac{1}{8} \sum_{m=0}^{2n} E_m^2, \quad \text{etc.} \tag{A.27}$$

Assuming that $q$ satisfies the $g$th stationary (nonhomogeneous) KdV equation (A.10), the corresponding integration constants $c_\ell$ in (A.5) become symmetric functions of the branch points $E_0, \dots, E_{2n}$ of the underlying curve (A.24) and one verifies (cf., e.g., [37, Sect. 1.2])

$$c_\ell = c_\ell(\underline{E}), \quad 1 \le \ell \le n. \tag{A.28}$$

Finally, we return to the general time-dependent setup and briefly recall the algebro-geometric KdV initial value problem, where by definition $q$ satisfies the $r$th time-dependent KdV equation

$$\widetilde{\mathrm{KdV}}_r(q) = q_{t_r} - 2\tilde{f}_{r+1,z} = 0, \quad (z, t_r) \in \mathbb{C}^2, \tag{A.29a}$$

$$q\big|_{t_r = t_r^{(0)}} = q^{(0)} \tag{A.29b}$$

with initial value $q^{(0)}$ satisfying the $n$th stationary KdV equation

$$\text{s-KdV}_n(q^{(0)}) = -2f_{n+1,z} = 0 \tag{A.30}$$

for fixed $n, r \in \mathbb{N}_0$ and some $t_r^{(0)} \in \mathbb{C}$. Here we replaced $t$ by $t_r$ to emphasize the $r$th KdV flow. Moreover, since the integration constants in (A.29a) and (A.30) are independent of each other, we denote the ones in $f_k$ by $c_\ell$, $1 \le \ell \le k$ as before and the ones in the right-hand side of (A.29a) by $\tilde{c}_s$, $1 \le s \le r$. Similarly, $\tilde{f}_j$, $\widetilde{F}_r$, $\widetilde{P}_{2r+1}$, $\widetilde{\mathrm{KdV}}_r$ are constructed as $f_j$, $F_r$, $P_{2r+1}$, $\mathrm{KdV}_r$ in (A.1), (A.3), (A.10), (A.13), replacing $c_\ell$ by $\tilde{c}_s$, etc. The isospectral property of KdV flows then permits one to replace (A.29) and (A.30) by the following pair of equations

$$q_{t_r} = \frac{1}{2}\widetilde{F}_{r,zzz} + 2(q-E)\widetilde{F}_{r,z} + q_z \widetilde{F}_r, \tag{A.31}$$

$$-\frac{1}{2}F_{n,zz}F_n + \frac{1}{4}F_{n,z}^2 + (E-q)F_n^2 = R_{2n+1}, \tag{A.32}$$

or in terms of Lax differential expressions, by

$$L_{2,t_r}(t_r) - [\widetilde{P}_{2r+1}(t_r), L_2(t_r)] = 0, \tag{A.33a}$$

$$[P_{2n+1}(t_r), L_2(t_r)] = 0. \tag{A.33b}$$

Because of (A.33), the common eigenfunction $\psi(P)$ of $L_2$ and $P_{2n+1}$, the Baker–Akhiezer function, will satisfy

$$L_2(t_r)\psi(P,z,t_r) = E\psi(P,z,t_r), \tag{A.34}$$

$$P_{2n+1}(t_r)\psi(P,z,zt_r) = y\psi(P,z,t_r), \tag{A.35}$$

$$\psi_{t_r}(P,z,t_r) = \widetilde{P}_{2r+1}(t_r)\psi(P,z,t_r) \tag{A.36}$$

$$= \widetilde{F}_r(E,z,t_r)\psi_z(P,z,t_r) - \frac{1}{2}\widetilde{F}_{r,z}(E,z,t_r)\psi(P,z,t_r), \tag{A.37}$$

$$P = (E,y).$$

## Appendix B. A Few Basic Results on Elliptic Functions

For convenience of the reader we recall some theorems representing an arbitrary elliptic function in terms of $\sigma$- and $\zeta$-functions which are used in this text. For general references see, for instance, Akhiezer [4], Chandrasekharan [18], Markushevich [64], and Whittaker and Watson [103] (for connoisseurs we recommend, in particular, Krause's two volume treatise [57], [58]).

A function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ with two periods $a, b \in \mathbb{C}\backslash\{0\}$, the ratio of which is not real, is called doubly periodic. If all its periods are of the form $m_1 a + m_2 b$, where $m_1, m_2 \in \mathbb{Z}$, then $a$ and $b$ are called fundamental periods of $f$. A doubly periodic meromorphic function is called *elliptic*. It is customary to denote the fundamental periods of an elliptic function by $2\omega_1$ and $2\omega_3$ with $\mathrm{Im}(\omega_3/\omega_1) > 0$. We also introduce $\omega_2 = \omega_1 + \omega_3$ and $\omega_4 = 0$. The numbers $\omega_1, \ldots, \omega_4$ are called half-periods. The fundamental period parallelogram $\Delta$ is the half-open region consisting of the line segments $[0, 2\omega_1)$, $[0, 2\omega_3)$ and the interior of the parallelogram with vertices 0, $2\omega_1$, $2\omega_2$, and $2\omega_3$.

The function

$$\wp(z; \omega_1, \omega_3) = \frac{1}{z^2} + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \left( \frac{1}{(z - 2m\omega_1 - 2n\omega_3)^2} - \frac{1}{(2m\omega_1 + 2n\omega_3)^2} \right), \tag{B.1}$$

or $\wp(z)$ for short, was introduced by Weierstrass. It is an even elliptic function of order 2 with fundamental periods $2\omega_1$ and $2\omega_3$. Its derivative $\wp'$ is an odd elliptic function of order 3 with fundamental periods $2\omega_1$ and $2\omega_3$. Every elliptic function may be written as $R_1(\wp(z)) + R_2(\wp(z))\wp'(z)$ where $R_1$ and $R_2$ are rational functions of $\wp$.

The numbers

$$g_2 = 60 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(2m\omega_1 + 2n\omega_3)^4}, \quad g_3 = 140 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(2m\omega_1 + 2n\omega_3)^6} \tag{B.2}$$

are called the invariants of $\wp$. Since the coefficients of the Laurent expansions of $\wp(z)$ and $\wp'(z)$ at $z = 0$ are polynomials of $g_2$ and $g_3$ with rational coefficients, the function $\wp(z; \omega_1, \omega_3)$ is also uniquely characterized by its invariants $g_2$ and $g_3$. One also frequently uses the notation $\wp(z|g_2, g_3)$.

The function $\wp(z)$ satisfies the first-order differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{B.3}$$

and hence the equations

$$\wp''(z) = 6\wp(z)^2 - g_2/2 \text{ and } \wp'''(z) = 12\wp'(z)\wp(z). \tag{B.4}$$

Thus, $-2\wp$ is a stationary solution of the first KdV equation, s-KdV$_1(q) = 0$ in (A.11) with $c_1 = 0$.

The function $\wp'$, being of order 3, has three zeros in $\Delta$. Since $\wp'$ is odd and elliptic it is obvious that these zeros are the half-periods $\omega_1, \omega_2 = \omega_1 + \omega_3$ and $\omega_3$. Let $e_j = \wp(\omega_j)$, $j = 1, 2, 3$. Then (B.3) implies that $4e_j^3 - g_2 e_j - g_3 = 0$ for $j = 1, 2, 3$. Therefore

$$0 = e_1 + e_2 + e_3, \tag{B.5}$$

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) = 2(e_1^2 + e_2^2 + e_3^2), \tag{B.6}$$

$$g_3 = 4e_1e_2e_3 = \frac{4}{3}(e_1^3 + e_2^3 + e_3^3). \tag{B.7}$$

Weierstrass also introduced two other functions denoted by $\zeta$ and $\sigma$. The Weierstrass $\zeta$-function is defined by

$$\frac{d}{dz}\zeta(z) = -\wp(z), \quad \lim_{z\to 0}\left(\zeta(z) - \frac{1}{z}\right) = 0. \tag{B.8}$$

It is a meromorphic function with simple poles at $2m\omega_1 + 2n\omega_3, m, n \in \mathbb{Z}$ having residues 1. It is not periodic but quasi-periodic in the sense that

$$\zeta(z + 2\omega_j) = \zeta(z) + 2\eta_j, \quad 1 \le j \le 4, \tag{B.9}$$

where $\eta_j = \zeta(\omega_j)$ for $j = 1, 2, 3$ and $\eta_4 = 0$.

The Weierstrass $\sigma$-function is defined by

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z), \quad \lim_{z\to 0}\frac{\sigma(z)}{z} = 1. \tag{B.10}$$

$\sigma$ is an entire function with simple zeros at the points $2m\omega_1 + 2n\omega_3, m, n \in \mathbb{Z}$. Under translation by a period $\sigma$ behaves according to

$$\sigma(z + 2\omega_j) = -\sigma(z)e^{2\eta_j(z+\omega_j)}, \quad j = 1, 2, 3. \tag{B.11}$$

**Theorem B.1.** ([47]) *Given numbers $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_m$ such that $\beta_k \ne \beta_\ell \pmod{\Delta}$ for $k \ne \ell$, the following identity holds*

$$\prod_{j=1}^m \frac{\sigma(z-\alpha_j)}{\sigma(z-\beta_j)} = \sum_{j=1}^m \frac{\prod_{k=1}^m \sigma(\beta_j - \alpha_k)}{\prod_{\ell=1,\ell\ne j}^m \sigma(\beta_j - \beta_\ell)} \frac{\sigma(z - \beta_j + \beta - \alpha)}{\sigma(z-\beta_j)\sigma(\beta-\alpha)}, \tag{B.12}$$

*where*

$$\alpha = \sum_{j=1}^m \alpha_j \text{ and } \beta = \sum_{j=1}^m \beta_j \tag{B.13}$$

*and $\sigma$ is constructed from the fundamental periods $2\omega_1$ and $2\omega_3$.*

**Theorem B.2.** ([64, p. 182, Theorem 5.12]) *Given an elliptic function $f$ of order $n$ with fundamental periods $2\,\omega_1$ and $2\,\omega_3$, let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be the zeros and poles of $f$ in the fundamental period parallelogram $\Delta$ repeated according to their multiplicities. Then*

$$f(z) = C\frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_{n-1})\sigma(z-b_n')}, \tag{B.14}$$

*where $C \in \mathbb{C}$ is a suitable constant, $\sigma$ is constructed from the fundamental periods $2\,\omega_1$ and $2\,\omega_3$, and where*

$$b_n' - b_n = (a_1 + \cdots + a_n) - (b_1 + \cdots + b_n) \tag{B.15}$$

*is a period of $f$. Conversely, every such function is an elliptic function.*

**Theorem B.3.** ( [64, p. 182, Theorem 5.13]) *Given an elliptic function $f$ with fundamental periods $2\,\omega_1$ and $2\,\omega_3$, let $b_1, \dots, b_r$ be the distinct poles of $f$ in $\Delta$. Suppose the principal part of the Laurent expansion near $b_k$ is given by*

$$\sum_{j=1}^{\beta_k} \frac{A_{j,k}}{(z-b_k)^j}, \quad 1 \le k \le r. \tag{B.16}$$

*Then*

$$f(z) = A_0 + \sum_{k=1}^{r}\sum_{j=1}^{\beta_k}(-1)^{j-1}\frac{A_{j,k}}{(j-1)!}\zeta^{(j-1)}(z-b_k), \tag{B.17}$$

*where* $A_0 \in \mathbb{C}$ *is a suitable constant and* $\zeta$ *is constructed from the fundamental periods* $2\,\omega_1$ *and* $2\,\omega_3$. *Conversely, every such function is an elliptic function if* $\sum_{k=1}^{r} A_{1,k} = 0$.

One notes that this theorem resembles the partial fraction expansions for rational functions.

Finally, we turn to elliptic functions of the second kind, the central object in our analysis. A meromorphic function $\psi : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ for which there exist two complex constants $\omega_1$ and $\omega_3$ with non-real ratio and two complex constants $\rho_1$ and $\rho_3$ such that

$$\psi(z + 2\,\omega_j) = \rho_j \psi(z), \quad j = 1, 3, \tag{B.18}$$

is called elliptic of the second kind. We call $2\,\omega_1$ and $2\,\omega_3$ the quasi-periods of $\psi$. Together with $2\,\omega_1$ and $2\,\omega_3$, $2\,m_1\omega_1 + 2\,m_3\omega_3$ are also quasi-periods of $\psi$ if $m_1, m_3 \in \mathbb{Z}$. If every quasi-period of $\psi$ can be written as an integer linear combination of $2\,\omega_1$ and $2\,\omega_3$, then these are called fundamental quasi-periods.

**Theorem B.4.** *A function* $\psi$ *which is elliptic of the second kind and has fundamental quasi-periods* $2\,\omega_1$ *and* $2\,\omega_3$ *can always be put into the form*

$$\psi(z) = C\exp(\lambda z)\frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)} \tag{B.19}$$

*for suitable constants* $C$, $\lambda$, $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$. *Here* $\sigma$ *is constructed from the fundamental periods* $2\,\omega_1$ *and* $2\,\omega_3$. *Conversely, every such function is elliptic of the second kind.*

## Appendix C. Symmetric products

Let $X$ be a Riemann surface. In addition to the cartesian product $X^N = X \times \cdots \times X$ ($N$ factors), $N \in \mathbb{N}$, we also introduce the $N$th symmetric product of $X$ defined as the quotient space

$$X^N/S_N. \tag{C.1}$$

Here $S_N$ denotes the symmetric group on $N$ letters acting as the group of permutations of the factors in the cartesian product $X^N$, that is,

$$\pi(x_1, \ldots, x_n) = (x_{\pi(1)}, \ldots, x_{\pi(N)}), \quad \pi \in S_N. \tag{C.2}$$

Thus, the points in $X^N/S_N$ can be considered as $N$-tuples of points of $X$ without regard to their order. $X^N/S_N$ inherits the topology from $X^N$ (the quotient topology) and the canonical projection (quotient map)

$$\nu\colon \begin{cases} X^N \to X^N/S_N \\ (x_1, \ldots, x_N) \mapsto [x_1, \ldots, x_N] = \{\pi(x_1, \ldots, x_N) \in X^N \mid \pi \in S_N\} \end{cases} \tag{C.3}$$

defines a complex structure on $X^N/S_N$ as follows. Consider a point $[p_1, \ldots, p_N] \in X^N/S_N$, let $x_j$ be a local coordinate in an open neighborhood $U_j$ of $p_j \in X$, assuming $U_j \cap U_k = \emptyset$ if $p_j \neq p_k$ and $x_j = x_k$ in $U_j = U_k$ for $p_j = p_k$. Denote by $\sigma_1, \dots, \sigma_N$ the elementary symmetric functions of $x_1, \dots, x_N$, then the map

$$\begin{aligned} &\nu(U_1 \times \cdots \times U_N) \to \mathbb{C}^N \\ &[q_1, \dots, q_N] \mapsto (\sigma_1(x_1(q_1), \dots, x_N(q_N)), \dots, \sigma_N(x_1(q_1), \dots, x_N(q_N))) \end{aligned} \tag{C.4}$$

provides a coordinate chart on $\nu(U_1 \times \cdots \times U_N)$. In this manner, $X^N/S_N$ (like $X^N$) becomes an $N$-dimensional complex manifold with $X^N$ an $N!$-sheeted branched analytic covering of $X^N/S_N$.

Away from the branch locus the map $\nu$ is a covering map and one can take

$$(x_1(q_1), \dots, x_N(q_N)) \tag{C.5}$$

as coordinates on $X^N/S_N$ (here the points $p_j$, corresponding to the charts $(U_j, x_j)$, are mutually distinct). At the other extreme, where $p_1 = p_2 = \cdots = p_N$, local coordinates are given by $(\sigma_1(x_1(q_1), \dots, x_N(q_N)), \dots, \sigma_N(x_1(q_1), \dots, x_N(q_N)))$, that is, by

$$\left( \sum_{j=1}^N x_j(q_j), \dots, \prod_{j=1}^N x_j(q_j) \right). \tag{C.6}$$

Next, assume the topological space $(X^N, \tau)$ is generated by the metric $d$ on $X^N$. We then write $\tau = (d)$ and hence $(X^N, \tau) = (X^N, (d))$. In addition, let $(X^N/S_N, \tau_{S_N})$ denote the topological space equipped with the quotient topology of $X^N/S_N$ relative to $(X, \tau)$,

$$\tau_{S_N} = \{U \subseteq X^N/S_N \,|\, \nu^{-1}(U) \in \tau\}, \tag{C.7}$$

We now investigate a case in which $(X^N/S_N, \tau_{S_N})$ is also generated by a metric $D$ on $X^N/S_N$. For this purpose we now assume that the metric $d$ is such that each permutation in $S_N$ is an isometry[7], that is,

$$\text{for all } \pi \in S_N\colon\ d(\pi(x), \pi(y)) = d(x, y), \quad x, y \in X^N \tag{C.8}$$

(here $x = (x_1, \dots, x_N) \in X^N$, etc.). A standard situation in which (C.8) can be verified is as follows: Suppose $\delta$ is a metric on $X$. Then for any fixed $r \in [1, \infty)$, $d_r\colon X^N \times X^N \to [0, \infty)$, defined as

$$d_r(x, y) = \left( \sum_{j=1}^N \delta(x_j, y_j)^r \right)^{1/r}, \quad x = (x_1, \dots, x_N),\, y = (y_1, \dots, y_N) \in X^N, \tag{C.9}$$

defines a metric on $X^N$ satisfying (C.8) (and similarly in the case $r = \infty$ using the supremum over $j \in \{1, \dots, N\}$).

Since $S_N$ is transitive, the expression $\min_{\sigma, \rho \in S_N}\{d(\sigma(x), \rho(y))\}$ does not change when $x$ and $y$ are replaced by other representatives in their respective equivalence classes, that is, it depends only on $[x]$ and $[y]$. Hence, we may define

$$D([x], [y]) = \min_{\sigma, \rho \in S_N}\{d(\sigma(x), \rho(y))\}, \quad [x], [y] \in X^N/S_N. \tag{C.10}$$

The assumption that the permutations are isometries then yields

$$D([x], [y]) = \min_{\rho \in S_N}\{d(x, \rho(y))\}, \quad [x], [y] \in X^N/S_N. \tag{C.11}$$

---

[7] This holds for $X = \mathbb{C}$, $X = \mathbb{C}/\Lambda_\omega$, and $X = \mathbb{C}/\Lambda_{2\omega_1, 2\omega_3}$ and the usual metrics on them (cf. Remark C.2).

**Theorem C.1.** *Let* $(X^N, d)$ *be a metric space and suppose that every permutation in* $S_N$ *is an isometry on* $X^N$*. Define* $D$ *as in* (C.11)*. Then* $(X^N/S_N, D)$ *is a metric space and the topology* $(D)$ *induced by the metric* $D$ *on* $X^N/S_N$ *is the quotient topology* $\tau_{S_N}$*,* $(X^N/S_N, (D)) = (X^N/S_N, \tau_{S_N})$*.*

*Proof.* Clearly $D$ assumes non-negative real values only and symmetry of $D$ follows immediately from (C.10). If $[x] = [y]$ then there is a $\rho \in S_N$ such that $x = \rho(y)$. Hence $d(x, \rho(y)) = 0$ and thus $D([x], [y]) = 0$. Next, suppose that $D([x], [y]) = 0$. Then there exists a $\rho \in S_N$ such that $x = \rho(y)$, that is, $y$ is equivalent to $x$ and hence $[x] = [y]$. For the triangle inequality one notes that, given $z \in X^N$,

$$\begin{aligned} D([x], [y]) &= \min_{\rho \in S_N} \{d(x, \rho(y))\} \\ &\leq \min_{\rho \in S_N} \{d(x, \sigma(z)) + d(\sigma(z), \rho(y))\} \\ &= d(x, \sigma(z)) + \min_{\rho \in S_N} \{d(\sigma(z), \rho(y))\} \\ &= d(x, \sigma(z)) + D([z], [y]), \quad \sigma \in S_N. \end{aligned} \tag{C.12}$$

In particular (C.12) holds for that $\sigma$ which yields the minimum of the right-hand side of (C.12) and hence $D$ is a metric on $X^N/S_N$.

The metric $D$ induces a topology $\tilde{\tau}$ on $X^N/S_N$ and we denote the resulting topological space by $(X^N/S_N, \tilde{\tau})$. Let $\nu : X^N \to X^N/S_N$, $x \mapsto [x]$ denote the canonical projection. We will next show that the map $\nu : (X^N, d) \to (X^N/S_N, \tilde{\tau})$ is open and continuous. It is obviously surjective. By [101, Theorem 6.5.1] we then conclude that $\tau_{S_N} = \tilde{\tau}$.

To prove that $\nu$ is continuous, let $U$ be an open set in $(X^N/S_N, \tilde{\tau})$. We want to show that $\nu^{-1}(U)$ is open. Let $x$ be a point in $\nu^{-1}(U)$. Then $[x]$ is in $U$ and there is an $\varepsilon > 0$ such that $B([x], \varepsilon)$, the ball of radius $\varepsilon$ centered at $[x]$, is a subset of $U$. Pick $y \in B(x, \varepsilon) \subset X^N$. We note that

$$D([x], [y]) \leq d(x, y) < \varepsilon, \tag{C.13}$$

that is, $[y] \in B([x], \varepsilon) \subset U$ and thus $y \in \nu^{-1}(U)$. Since $y$ is arbitrary, one infers $B(x, \varepsilon) \subset \nu^{-1}(U)$.

To prove that $\nu$ is open, let $V$ be an open set in $X^N$. We want to show that $\nu(V)$ is open. Let $[x]$ be a point in $\nu(V)$. Then there is a point in the equivalence class of $x$ which is in $V$. Without loss of generality we may assume that $x$ is that point. In addition, there is an $\varepsilon > 0$ such that $B(x, \varepsilon)$ is a subset of $V$. Pick $[y] \in B([x], \varepsilon) \subset (X^N/S_N, \tilde{\tau})$. Note that this is equivalent to $D([x], [y]) < \varepsilon$, which in turn means that there is a $\rho$ in $S_N$ such that $d(x, \rho(y)) < \varepsilon$. Hence $\rho(y) \in B(x, \varepsilon) \subset V$ and thus $[y] = \nu(y) = \nu(\rho(y)) \in \nu(V)$. Since $[y]$ is arbitrary, one concludes $B([x], \varepsilon) \subset \nu(V)$. ☐

**Remark C.2.** The results of this appendix apply in the three cases $X = \mathbb{C}$, $X = \mathbb{C}/\Lambda_\omega$, $X = \mathbb{C}/\Lambda_{2\omega_1, 2\omega_3}$ considered in Section 3. For brevity we just take a quick look at the simply periodic case $X = \mathbb{C}/\Lambda_\omega$: Consider the equivalence classes $[x] = \{x + m\omega \,|\, x \in \mathbb{C}, \, m \in \mathbb{Z}\} \in \mathbb{C}/\Lambda_\omega$, then the quotient topology on $\mathbb{C}/\Lambda_\omega$ is seen to be generated by the following metric $\delta \colon \mathbb{C}/\Lambda_\omega \times \mathbb{C}/\Lambda_\omega \to [0, \infty)$ on $\mathbb{C}/\Lambda_\omega$,

$$\delta([x], [y]) = \inf_{m,n \in \mathbb{Z}} |x + m\omega - (y + n\omega)|, \quad [x], [y] \in \mathbb{C}/\Lambda_\omega. \tag{C.14}$$

Analogous considerations apply to the elliptic case $X = \mathbb{C}/\Lambda_{2\omega_1, 2\omega_3}$.

## Appendix D. The proof of Theorem 2.15

In this appendix we provide the proof of Theorem 2.15.

**Theorem D.1.** *Assume $M \in \mathbb{N}$, $s_\ell \in \mathbb{N}$, $1 \le \ell \le M$, $q_0 \in \mathbb{C}$, and suppose $\zeta_\ell \in \mathbb{C}$, $\ell = 1, \dots, M$, are pairwise distinct. Consider*

$$q(z) = q_0 - \sum_{\ell=1}^{M} s_\ell(s_\ell+1)\mathcal{P}(z-\zeta_\ell), \tag{D.1}$$

*and suppose the DG locus conditions*

$$\sum_{\substack{\ell'=1\\ \ell'\neq\ell}}^{M} s_{\ell'}(s_{\ell'}+1)\mathcal{P}^{(2k-1)}(\zeta_\ell-\zeta_{\ell'}) = 0 \text{ for } 1 \le k \le s_\ell \text{ and } 1 \le \ell \le M \tag{D.2}$$

*are satisfied. Then*

$$f_0 = 1, \quad f_j(z) = d_j + \sum_{\ell=1}^{M} \sum_{k=1}^{\min(j,s_\ell)} a_{j,\ell,k}\mathcal{P}(z-\zeta_\ell)^k, \quad j \in \mathbb{N} \tag{D.3}$$

*for some $\{a_{j,\ell,k}\}_{1\le k\le \min(j,s_\ell), 1\le\ell\le M} \subset \mathbb{C}$ and $d_j \in \mathbb{C}$, $j \in \mathbb{N}$.*

*Proof.* By equation (2.17) we can treat the rational, simply periodic, and elliptic cases simultaneously.
(1) $j = 1$: Then

$$f_1(z) = c_1 + \frac{1}{2}\,q(z) = c_1 + \frac{1}{2}\,q_0 - \sum_{\ell=1}^{M}\frac{1}{2}s_\ell(s_\ell+1)\,\mathcal{P}(z-\zeta_\ell) \tag{D.4}$$

is of the form (D.3) with $d_1 = c_1 + \frac{1}{2}q_0$ and $a_{1,\ell,1} = -\frac{1}{2}s_\ell(s_\ell+1)$.
(2) We assume (D.3) holds for some $j \in \mathbb{N}$, that is,

$$f_j(z) = d_j + \sum_{\ell=1}^{M}\sum_{k=1}^{\min(j,s_\ell)} a_{j,\ell,k}\mathcal{P}(z-\zeta_\ell)^k, \tag{D.5}$$

or equivalently,

$$f_j'(z) = \sum_{\ell=1}^{M}\sum_{k=1}^{\min(j,s_\ell)} a_{j,\ell,k}\,k\,\mathcal{P}(z-\zeta_\ell)^{k-1}\,\mathcal{P}'(z-\zeta_\ell). \tag{D.6}$$

We now start the proof of (D.3) for $j+1$: First, we recall the recurrence relation (A.1),

$$f_{j+1}'(z) = \frac{1}{4}f_j(z)''' + q(z)f_j'(z) + \frac{1}{2}q'(z)f_j(z) \tag{D.7}$$

$$= \frac{1}{4}f_j(z)''' + (q(z)f_j(z))' - \frac{1}{2}q'(z)f_j(z) \tag{D.8}$$

$$= \frac{1}{4}f_j(z)''' + \frac{1}{2}q(z)f_j'(z) + \frac{1}{2}(q(z)f_j(z))'. \tag{D.9}$$

Since $q$ is elliptic, so are $f_k$ for all $k \in \mathbb{N}$ by the recursion relation (A.1) as the latter implies that each $f_k$ is a differential polynomial in $q$. Equations (D.8) and (D.9) then imply that as $z \to \zeta_\ell$, none of the terms in (D.7) can have a constant term or a term of the form $(z-\zeta_\ell)^{-1}$ in the Laurent expansion around $\zeta_\ell$. This fact will be used repeatedly in the remainder of this proof.

Next we separately investigate each of the three terms on the right-hand side of (D.7). For brevity we denote $\min(j, s_\ell)$ by $m$ in the following.
$(i)$ Considering $f_j'''$ one computes

$$f_j'''(z) = \sum_{\ell=1}^{M}\sum_{k=1}^{m} a_{j,\ell,k}\frac{d^3}{dz^3}\mathcal{P}(z-\zeta_\ell)^k \tag{D.10}$$

and

$$\begin{aligned}\frac{d^3}{dz^3}\mathcal{P}(z-\zeta_\ell)^k =& \big[k(2k+1)(2k+2)\mathcal{P}(z-\zeta_\ell)^k\mathcal{P}'(z-\zeta_\ell)\\ &- g_2k(k-\frac{1}{2})(k-1)\mathcal{P}(z-\zeta_\ell)^{k-2}\mathcal{P}'(z-\zeta_\ell)\\ &- g_3k(k-1)(k-2)\mathcal{P}(z-\zeta_\ell)^{k-3}\mathcal{P}'(z-\zeta_\ell)\big],\end{aligned} \tag{D.11}$$

using (B.3) and (B.4). (For $k=2$ the term $\mathcal{P}(z-\zeta_\ell)^{k-3}$ does not occur in (D.11), for $k=1$ the terms $\mathcal{P}(z-\zeta_\ell)^{k-3}$ and $\mathcal{P}(z-\zeta_\ell)^{k-2}$ do not occur in (D.11).) Thus, $\frac{1}{4}f_j'''$ is of the expected form (D.3),

$$\frac{1}{4}f_j'''(z) = \frac{1}{4}\sum_{\ell=1}^{M}\sum_{k=1}^{m+1}\tilde{a}_{j+1,\ell,k}\mathcal{P}(z-\zeta_\ell)^{k-1}\mathcal{P}'(z-\zeta_\ell). \tag{D.12}$$

Moreover, the highest-order pole of $\frac{1}{4}f_j'''$ at $\zeta_\ell$ reads

$$\frac{1}{4}\,m(4m+2)(m+1)\,\frac{(-2)a_{j,\ell,m}}{(z-\zeta_\ell)^{2m+3}}. \tag{D.13}$$

$(ii)$ Considering $qf_j'$ one obtains

$$\begin{aligned}&q(z)f_j'(z)\\ &= \Big(q_0 - \sum_{\ell=1}^{M}s_\ell(s_\ell+1)\mathcal{P}(z-\zeta_\ell)\Big)\Big(\sum_{\ell=1}^{M}\sum_{k=1}^{m}a_{j,\ell,k}k\mathcal{P}(z-\zeta_\ell)^{k-1}\mathcal{P}'(z-\zeta_\ell)\Big)\\ &= q_0\sum_{\ell=1}^{M}\sum_{k=1}^{m}a_{j,\ell,k}k\mathcal{P}(z-\zeta_\ell)^{k-1}\mathcal{P}'(z-\zeta_\ell)\\ &\quad - \sum_{\ell=1}^{M}\sum_{k=1}^{m}s_\ell(s_\ell+1)a_{j,\ell,k}k\mathcal{P}(z-\zeta_\ell)^{k}\mathcal{P}'(z-\zeta_\ell)\\ &\quad - \sum_{\ell=1}^{M}\Big[\Big(\sum_{k=1}^{m}a_{j,\ell,k}k\mathcal{P}(z-\zeta_\ell)^{k-1}\mathcal{P}'(z-\zeta_\ell)\Big)\Big(\sum_{\substack{\ell'=1\\ \ell'\neq\ell}}^{M}s_{\ell'}(s_{\ell'}+1)\mathcal{P}(z-\zeta_{\ell'})\Big)\Big].\end{aligned} \tag{D.14}$$

The first two terms on the right-hand side of (D.14) are already of the expected form (D.3). Next, we investigate the third term in (D.14). Let

$$g_{1,\ell}(z) = \sum_{k=1}^{m}a_{j,\ell,k}\,k\,\mathcal{P}(z-\zeta_\ell)^{k-1}\,\mathcal{P}'(z-\zeta_\ell), \quad h_{1,\ell}(z) = \sum_{\substack{\ell'=1\\ \ell'\neq\ell}}^{M}s_{\ell'}(s_{\ell'}+1)\mathcal{P}(z-\zeta_{\ell'}). \tag{D.15}$$

Then the third term in (D.14) equals $-\sum_{\ell=1}^{M} g_{1,\ell}h_{1,\ell}$. Next we recall (cf. (B.17)) that any elliptic function $f$ can be written in the form

$$f(z) = A_0 + \sum_{\ell=1}^{M}\sum_{k=1}^{s}(-1)^{k-1}\frac{A_{\ell,k}}{(k-1)!}\zeta^{(k-1)}(z-\zeta_\ell), \quad s \in \mathbb{N} \tag{D.16}$$

for appropriate $M, s \in \mathbb{N}$, $A_0, A_{\ell,k} \in \mathbb{C}$, $1 \le \ell \le M$, $1 \le k \le s$. Here $\zeta(\cdot) = \zeta(\cdot\,|g_2, g_3)$ abbreviates the Weierstrass $\zeta$-function in the elliptic case associated with the invariants $g_2$ and $g_3$ (see [2, Sect. 18.1]) and

$$\zeta(z) = \begin{cases} \zeta(z|0,0) = 1/z & \text{in the rational case,} \\ \zeta\big(z|[2\pi^2/\omega^2]^2/3, [2\pi^2/\omega^2]^3/27\big) & \\ = [\pi^2 z/(3\omega^2)] + (\pi/\omega)\cot(\pi z/\omega) & \text{in the simply periodic case} \end{cases} \tag{D.17}$$

(cf. [2, p. 652]). Since $g_{1,\ell}$ and $h_{1,\ell}$ are elliptic, we thus have

$$\sum_{\ell=1}^{M} g_{1,\ell}(z) = G_{1,0} + \sum_{\ell=1}^{M}\sum_{k=1}^{2m+1}(-1)^{k-1}\frac{G_{1,\ell,k}}{(k-1)!}\zeta^{(k-1)}(z-\zeta_\ell), \tag{D.18}$$

$$\sum_{\ell=1}^{M} g_{1,\ell}(z)h_{1,\ell}(z) = B_0 + \sum_{\ell=1}^{M}\sum_{k=1}^{2m+1}(-1)^{k-1}\frac{B_{\ell,k}}{(k-1)!}\zeta^{(k-1)}(z-\zeta_\ell). \tag{D.19}$$

To calculate $B_{\ell,k}$ we expand $g_{1,\ell}$ and $h_{1,\ell}$ at $z = \zeta_\ell$ using (D.2). First we recall (cf. (2.16) and [2, Sect. 18.5])

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{r=2}^{\infty} c_r z^{2r-2}. \tag{D.20}$$

Thus, $\mathcal{P}^k$ admits the Laurent expansion

$$(\mathcal{P}(z))^k = \frac{1}{z^{2k}} + \frac{1}{z^{2k-4}}\sum_{s=0}^{\infty} d_s z^{2s} \tag{D.21}$$

with only even orders of $z$ occurring in the expansion of $\mathcal{P}^k$ since $\mathcal{P}$ is an even function. For the derivative of $\mathcal{P}^k$ one computes

$$\frac{d}{dz}(\mathcal{P}(z))^k = (-2k)\frac{1}{z^{2k+1}} + (-2k+4)\frac{1}{z^{2k-3}}\sum_{s=0}^{\infty} d_s z^{2s} + \frac{1}{z^{2k-4}}\sum_{s=1}^{\infty} d_s 2s z^{2s-1}$$

and hence only odd orders of $z$ occur in the expansion of $\frac{d}{dz}(\mathcal{P}(z))^k$. Thus, one concludes that only odd orders of $z$ occur in the expansion of $g_{1,\ell}$ at $z = \zeta_\ell$.

On the other hand any elliptic function $f$, whose residue at $\zeta_\ell$ vanishes and whose principal part of its Laurent expansion at $z = \zeta_\ell$ contains only odd terms, can be written in the form

$$f(z) = \sum_{k=1}^{n_\ell} \tilde{d}_k \frac{d}{dz}(\mathcal{P}(z-\zeta_\ell))^k + O(1) \tag{D.22}$$

for $z$ in a neighborhood of $\zeta_\ell$. Here $n_\ell \in \mathbb{N}$ depends on the order of the pole of $f$ at $\zeta_\ell$.

By (D.2) the odd powers of $(z-\zeta_\ell)^j$ in the expansion of $h_{1,\ell}(z)$ at $z=\zeta_\ell$ up to order $(2s_\ell-1)$ are zero and hence

$$\begin{aligned}
h_{1,\ell}(z) &= h_{1,\ell,0} + \sum_{k=1}^{\infty} \frac{h_{1,\ell}^{(k)}(\zeta_\ell)}{k!}(z-\zeta_\ell)^k = \sum_{\substack{\ell'=1\\ \ell'\neq\ell}}^{M} s_{\ell'}(s_{\ell'}+1)\mathcal{P}(\zeta_\ell-\zeta_{\ell'}) \\
&\quad + \sum_{\substack{\ell'=1\\ \ell'\neq\ell}}^{M} s_{\ell'}(s_{\ell'}+1)\mathcal{P}'(\zeta_\ell-\zeta_{\ell'})(z-\zeta_\ell) + \frac{1}{2}\sum_{\substack{\ell'=1\\ \ell'\neq\ell}}^{M} s_{\ell'}(s_{\ell'}+1)\mathcal{P}''(\zeta_\ell-\zeta_{\ell'})(z-\zeta_\ell)^2 \\
&\quad + \frac{1}{6}\sum_{\substack{\ell'=1\\ \ell'\neq\ell}}^{M} s_{\ell'}(s_{\ell'}+1)\mathcal{P}^{(3)}(\zeta_\ell-\zeta_{\ell'})(z-\zeta_\ell)^3 + \dots \\
&= h_{1,\ell,0} + h_{1,\ell,2}(z-\zeta_\ell)^2 + h_{1,\ell,4}(z-\zeta_\ell)^4 + \dots + h_{1,\ell,2s_\ell}(z-\zeta_\ell)^{2s_\ell} \\
&\quad + O\big((z-\zeta_\ell)^{2s_\ell+1}\big).
\end{aligned} \tag{D.23}$$

Expanding $g_{1,\ell}h_{1,\ell}$ at $z=\zeta_\ell$ then yields

$$\begin{aligned}
g_{1,\ell}(z)h_{1,\ell}(z) &= b_{-2m-1}\frac{1}{(z-\zeta_\ell)^{2m+1}} + b_{-2m+1}\frac{1}{(z-\zeta_\ell)^{2m-1}} + \dots \\
&\quad + b_{2s_\ell-2m-1}(z-\zeta_\ell)^{2s_\ell-2m-1} + O\big((z-\zeta_\ell)^{2s_\ell-2m}\big).
\end{aligned} \tag{D.24}$$

By (D.22) we can write (D.24) as

$$\begin{aligned}
g_{1,\ell}(z)h_{1,\ell}(z) &= \sum_{k=1}^{m} e_{j,\ell,k} k\mathcal{P}(z-\zeta_\ell)^{k-1}\,\mathcal{P}'(z-\zeta_\ell) + \frac{c_{1,\ell}}{z-\zeta_\ell} + c_{0,\ell} \\
&\quad + O\big((z-\zeta_\ell)^1\big).
\end{aligned} \tag{D.25}$$

Since no terms of the form $(z-\zeta_\ell)^{-1}$ and no constant term can occur in $\sum_{\ell=1}^{M} g_{1,\ell}h_{1,\ell}$ by the comment following (D.9), the coefficients $c_{1,\ell}$ of $(z-\zeta_\ell)^{-1}$, $\ell=1,\dots,M$, in (D.25), as well as the constant term $\sum_{\ell=1}^{M} c_{0,\ell}$, must be zero and we arrive at the expected form (D.3),

$$\sum_{\ell=1}^{M} g_{1,\ell}(z)h_{1,\ell}(z) = \sum_{\ell=1}^{M}\sum_{k=1}^{m} e_{j,\ell,k}\, k\,\mathcal{P}(z-\zeta_\ell)^{k-1}\,\mathcal{P}'(z-\zeta_\ell) \tag{D.26}$$

of the third term in (D.14). The highest-order pole of $qf_j'$ at $\zeta_\ell$ reads

$$-s_\ell(s_\ell+1)m\frac{(-2)\,a_{j,\ell,m}}{(z-\zeta_\ell)^{2m+3}}. \tag{D.27}$$

$(iii)$ Considering $\frac{1}{2}q'f_j$ one obtains

$$\begin{aligned}
\frac{1}{2}q'(z)f_j(z) &= -\frac{1}{2}\left(\sum_{\ell=1}^{M} s_\ell(s_\ell+1)\mathcal{P}'(z-\zeta_\ell)\right)\left(d_j + \sum_{\ell=1}^{M}\sum_{k=1}^{m} a_{j,\ell,k}\,\mathcal{P}(z-\zeta_\ell)^k\right) \\
&= -\frac{1}{2}\,d_j\sum_{\ell=1}^{M} s_\ell(s_\ell+1)\mathcal{P}'(z-\zeta_\ell) \\
&\quad - \frac{1}{2}\sum_{\ell=1}^{M}\sum_{k=1}^{m} s_\ell(s_\ell+1)a_{j,\ell,k}\,\mathcal{P}(z-\zeta_\ell)^k\,\mathcal{P}'(z-\zeta_\ell)
\end{aligned} \tag{D.28}$$

$$- \frac{1}{2} \sum_{\ell=1}^{M} \bigg[ \bigg( \sum_{k=1}^{m} a_{j,\ell,k}\, \mathcal{P}(z-\zeta_\ell)^k \bigg) \bigg( \sum_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{M} s_{\ell'}(s_{\ell'}+1)\mathcal{P}'(z-\zeta_{\ell'}) \bigg) \bigg].$$

The first two terms in (D.28) are already of the expected form (D.3). Next we investigate the third term in (D.28). Let

$$g_{2,\ell}(z) = \sum_{k=1}^{m} a_{j,\ell,k}\, \mathcal{P}(z-\zeta_\ell)^k, \quad h_{2,\ell}(z) = \bigg( \sum_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{M} s_{\ell'}(s_{\ell'}+1)\mathcal{P}'(z-\zeta_{\ell'}) \bigg). \tag{D.29}$$

Then the third term in (D.28) equals $-\frac{1}{2}\sum_{\ell=1}^{M} g_{2,\ell}h_{2,\ell}$. Since $g_{2,\ell}$ and $h_{2,\ell}$ are elliptic, one has

$$\sum_{\ell=1}^{M} g_{2,\ell}(z) = G_{2,0} + \sum_{\ell=1}^{M}\sum_{k=1}^{2m} (-1)^{k-1} \frac{G_{2,\ell,k}}{(k-1)!}\zeta^{(k-1)}(z-\zeta_\ell), \tag{D.30}$$

$$\sum_{\ell=1}^{M} g_{2,\ell}(z)h_{2,\ell}(z) = D_0 + \sum_{\ell=1}^{M}\sum_{k=1}^{2m-1} (-1)^{k-1} \frac{D_{\ell,k}}{(k-1)!}\zeta^{(k-1)}(z-\zeta_\ell). \tag{D.31}$$

From (D.21) one concludes that only even orders in $z$ can occur in the expansion of $g_{2,\ell}$ at $z=\zeta_\ell$. Next we expand $h_{2,\ell}$ at $z=\zeta_\ell$. By (D.2), the even powers of $(z-\zeta_\ell)^k$ in the expansion of $h_{2,\ell}$ at $z=\zeta_\ell$ up to order $(2s_\ell-2)$ are zero and hence,

$$\begin{aligned} h_{2,\ell}(z) &= \sum_{k=0}^{\infty} \frac{h_{2,\ell}^{(k)}(\zeta_\ell)}{k!}(z-\zeta_\ell)^k \\ &= h_{1,\ell,1}(z-\zeta_\ell) + h_{1,\ell,3}(z-\zeta_\ell)^3 + \ldots + h_{1,\ell,2s_\ell-1}(z-\zeta_\ell)^{2s_\ell-1} \\ &\quad + O\big((z-\zeta_\ell)^{2s_\ell}\big). \end{aligned} \tag{D.32}$$

Expanding $g_{2,\ell}\, h_{2,\ell}$ at $z=\zeta_\ell$ then yields

$$\begin{aligned} g_{2,\ell}(z)h_{2,\ell}(z) &= \tilde b_{-2m+1}\frac{1}{(z-\zeta_\ell)^{2m-1}} + \tilde b_{-2m+3}\frac{1}{(z-\zeta_\ell)^{2m-3}} + \ldots \\ &\quad + \tilde b_{2s_\ell-2m-1}(z-\zeta_\ell)^{2s_\ell-2m-1} + O\big((z-\zeta_\ell)^{2s_\ell-2m}\big). \end{aligned} \tag{D.33}$$

By (D.22) we can write (D.33) as

$$g_{2,\ell}(z)h_{2,\ell}(z) = \sum_{k=1}^{m-1} \tilde e_{j,\ell,k}\, k\, \mathcal{P}(z-\zeta_\ell)^{k-1}\, \mathcal{P}'(z-\zeta_\ell) + \frac{\tilde c_{1,\ell}}{z-\zeta_\ell} + \tilde c_{0,\ell} + O\big((z-\zeta_\ell)^1\big). \tag{D.34}$$

Since no terms of the form $(z-\zeta_\ell)^{-1}$ and no constant term can occur in $\sum_{\ell=1}^{M} g_{2,\ell}h_{2,\ell}$ by the comment following (D.9), the coefficients $\tilde c_{1,\ell}$ of $(z-\zeta_\ell)^{-1}$, $1 \leq \ell \leq M$, in (D.34), as well as the constant term $\sum_{\ell=1}^{M}\tilde c_{0,\ell}$, must vanish and we arrive at the expected form (D.3),

$$\sum_{\ell=1}^{M} g_{2,\ell}(z)h_{2,\ell}(z) = \sum_{\ell=1}^{M}\sum_{k=1}^{m-1} \tilde e_{j,\ell,k} k\mathcal{P}(z-\zeta_\ell)^{k-1}\mathcal{P}'(z-\zeta_\ell).$$

The highest-order pole of $\frac{1}{2}q'f_n$ at $\zeta_\ell$ reads

$$-\frac{1}{2}\, s_\ell(s_\ell+1)\frac{(-2)\, a_{j,\ell,m}}{(z-\zeta_\ell)^{2m+3}}. \tag{D.35}$$

Summing up (D.13), (D.27), and (D.35) yields

$$\left[\frac{1}{4}\, m(4m+2)(m+1) - s_\ell(s_\ell+1)m - \frac{1}{2}s_\ell(s_\ell+1)\right]\frac{-2a_{j,\ell,m}}{(z-\zeta_\ell)^{2m+3}}. \tag{D.36}$$

This term becomes zero as soon as $m = s_\ell$. Summing up our analysis of the three terms in (D.7), each term has the form (D.3) and the index $k$ does not exceed $\min(j+1, s_\ell)$, because of (D.36). □

## References

[1] M. J. Ablowitz and J. Satsuma, *Solitons and rational solutions of nonlinear evolution equations*, J. Math. Phys. **19**, 2180–2186 (1978).

[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.

[3] M. Adler and J. Moser, *On a class of polynomials connected with the Korteweg-de Vries equation*, Commun. Math. Phys. **61**, 1–30 (1978).

[4] N. I. Akhiezer, *Elements of the Theory of Elliptic Functions*, Amer. Math. Soc., Providence, RI, 1990.

[5] A. A. Akhmetshin and Yu. S. Volvovsky, *The dynamics of zeros of finite-gap solutions of the Schrödinger equation*, Funct. Anal. Appl. **35**, 247–256 (2001).

[6] A. A. Akhmetshin, I. Krichever, and Yu. S. Volvovsky, *Elliptic families of solutions of the Kadomtsev–Petviashvili equation and the field elliptic Calogero–Moser system*, preprint, arXiv:hep-th/0203192.

[7] H. Airault, *Poles of nonlinear evolution equations and integrable dynamical systems*, in *Nonlinear Evolution Equations Solvable by the Spectral Transform*, F. Calogero (ed.), Pitman, London, 1978, p. 244–251.

[8] H. Airault, H. P. McKean, and J. Moser, *Rational and elliptic solutions of the Korteweg-deVries equation and a related many-body problem*, Commun. Pure Appl. Math. **30**, 95–148 (1977).

[9] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.

[10] E. D. Belokolos, A. I. Bobenko, V. B. Matveev, and V. Z. Enol'skii, *Algebraic-geometric principles of superposition of finite-zone solutions of integrable non-linear equations*, Russian Math. Surv. **41:2**, 1–49 (1986).

[11] E. D. Belokolos and V. Z. Enol'skii, *Verdier elliptic solitons and the Weierstrass theory of reduction*, Funct. Anal. Appl. **23**, 46–47 (1989).

[12] ———, *Isospectral deformations of elliptic potentials*, Russ. Math. Surv. **44:5**, 191–193 (1989).

[13] ———, *Reduction of theta functions and elliptic finite-gap potentials*, Acta Appl. Math. **36**, 87–117 (1994).

[14] B. Birnir, *An example of blow-up, for the complex KdV equation and existence beyond blow-up,* SIAM J. Appl. Math. **47**, 710–725 (1987).

[15] H. Burkhardt, *"Elliptische Funktionen"*, Verlag von Veit, Leipzig, 2nd ed., 1906.

[16] F. Calogero, *Motion of poles and zeros of special solutions of nonlinear and linear partial differential equations and related "solvable" many-body problems*, Nuovo Cim. **43B**, 177–241 (1978).

[17] F. Calogero, *Classical Many-Body Problems Amenable to Exact Treatments*, Springer, Berlin, 2001.

[18] K. Chandrasekharan, *Elliptic Functions*, Springer, Berlin, 1985.

[19] D. V. Choodnovsky and G. V. Choodnovsky, *Pole expansions of nonlinear partial differential equations*, Nuovo Cim. **40B**, 339–353 (1977).

[20] D. V. Chudnovsky, *Meromorphic solutions of nonlinear partial differential equations and many-particle completely integrable systems*, J. Math. Phys. **20**, 2416–2422 (1979).

[21] B. Deconinck and H. Segur, *Pole dynamics for elliptic solutions of the Korteweg-de Vries equation*, Math. Phys. Anal. Geom. **3**, 49–74 (2000).

[22] R. Y. Donagi and E. Previato, *Abelian solitons*, Math. Comp. Simulation **55**, 407–418 (2001).

[23] B. A. Dubrovin and S. P. Novikov, *Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg-de Vries equation*, Sov. Phys.-JETP **40**, 1058–1063 (1975).

[24] J. J. Duistermaat and F. A. Grünbaum, *Differential equations in the spectral parameter*, Commun. Math. Phys. **103**, 177–240 (1986).

[25] F. Ehlers and H. Knörrer, *An algebro-geometric interpretation of the Bäcklund transformation for the Korteweg-de Vries equation*, Comment. Math. Helv. **57**, 1–10 (1982).

[26] V. Z. Enol'skii, *On the solutions in elliptic functions of integrable nonlinear equations*, Phys. Lett. **96A**, 327–330 (1983).

[27] ———, *On the two-gap Lamé potentials and elliptic solutions of the Kovalevskaja problem connected with them*, Phys. Lett. **100A**, 463–466 (1984).

[28] ———, *On solutions in elliptic functions of integrable nonlinear equations associated with two-zone Lamé potentials*, Soc. Math. Dokl. **30**, 394–397 (1984).

[29] V. Z. Enol'skii and J. C. Eilbeck, *On the two-gap locus for the elliptic Calogero–Moser model*, J. Phys. **A 28**, 1069–1088 (1995).

[30] V. Z. Enol'skii, J. C. Eilbeck, and E. Previato, *Varieties of elliptic solitons*, J. Phys. **A 34**, 2215–2227 (2001).

[31] V. Z. Enol'skii and N. A. Kostov, *On the geometry of elliptic solitons*, Acta Appl. Math. **36**, 57–86 (1994).

[32] G. Floquet, *"Sur les équations différentielles linéaires à coefficients doublement périodiques"*, C. R. Acad. Sci. Paris **98**, 38–39, 82–85 (1884).

[33] ———, *"Sur les équations différentielles linéaires à coefficients doublement périodiques"*, Ann. l'École Normale Sup. **1**, 181–238 (1884).

[34] ———, *"Addition a un mémorie sur les équations différentielles linéaires"*, Ann. l'École Normale Sup., **1**, 405–408 (1884).

[35] L. Gavrilov and A. M. Perelomov, *On the explicit solutions of the elliptic Calogero system*, J. Math. Phys. **40**, 6339–6352 (1999).

[36] F. Gesztesy and H. Holden, *Darboux-type transformations and hyperelliptic curves*, J. Reine Angew. Math. **527**, 151–183 (2000).

[37] F. Gesztesy and H. Holden, *Soliton Equations and Their Algebro-Geometric Solutions. Vol. I:* $(1+1)$*-Dimensional Continuous Models,* Cambridge Studies in Advanced Mathematics, Vol. 79, Cambridge Univ. Press, 2003.

[38] F. Gesztesy and R. Svirsky, *(m)KdV solitons on the background of quasi-periodic finite-gap solutions*, Memoirs Amer. Math. Soc. **118**, No. 563, 1–88 (1995).

[39] F. Gesztesy, K. Unterkofler, and R. Weikard, *On a theorem of Halphen and its application to integrable systems*, J. Math. Anal. Appl. **251**, 504–526 (2000).

[40] F. Gesztesy and W. Sticka, *On a theorem of Picard*, Proc. Am. Math. Soc. **126**, 1089–1099 (1998).

[41] F. Gesztesy and R. Weikard, *Lamé potentials and the stationary (m)KdV hierarchy*, Math. Nachr. **176**, 73–91 (1995).

[42] ———, *Treibich-Verdier potentials and the stationary (m)KdV hierarchy*, Math. Z. **219**, 451–476 (1995).

[43] ———, *On Picard potentials*, Diff. Int. Eqs. **8**, 1453–1476 (1995).

[44] ———, *A characterization of elliptic finite-gap potentials*, C. R. Acad. Sci. Paris **321**, 837–841 (1995).

[45] ———, *Picard potentials and Hill's equation on a torus*, Acta Math. **176**, 73–107 (1996).

[46] ———, *A characterization of all elliptic algebro-geometric solutions of the AKNS hierarchy*, Acta Math. **181**, 63-108 (1998).

[47] ———, *Elliptic algebro-geometric solutions of the KdV and AKNS hierarchies – an analytic approach*, Bull. Amer. Math. Soc. **35**, 271–317 (1998).

[48] P. G. Grinevich, *Rational solutions for the equation of commutation of differential operators*, Funct. Anal. Appl. **16**, 15–19 (1982).

[49] G.-H. Halphen, *Sur une nouvelle classe d'équations différentielles linéaires intégrables*, C. R. Acad. Sc. Paris **101**, 1238–1240 (1885).

[50] ———, *"Traité des Fonctions Elliptiques"*, tome 2, Gauthier-Villars, Paris, 1888.

[51] C. Hermite, *"Oeuvres"*, tome 3, Gauthier-Villars, Paris, 1912.

[52] E. L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.

[53] A. R. Its and V. Z. Enol'skii, *Dynamics of the Calogero–Moser system and the reduction of hyperelliptic integrals to elliptic integrals* , Funct. Anal. Appl. **20**, 62–64 (1986).

[54] A. R. Its and V. B. Matveev, *Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg-de Vries equation*, Theoret. Math. Phys. **23**, 343–355 (1975).

[55] A. Kasman, *Bispectral KP solutions and linearization of Calogero–Moser particle systems*, Commun. Math. Phys. **172**, 427–448 (1995).

[56] N. A. Kostov and V. Z. Enloskii, *Spectral characteristics of elliptic solitons*, Math. Notes **53**, 287–2293 (1993).

[57] M. Krause, *Theorie der doppeltperiodischen Funktionen einer veränderlichen Grösse*, Vol. 1, Teubner, Leipzig, 1895.

[58] ———, *Theorie der doppeltperiodischen Funktionen einer veränderlichen Grösse*, Vol. 2, Teubner, Leipzig, 1897.

[59] I. M. Krichever, *Rational solutions of the Kadomtsev-Petviashvili equation and integrable systems of N particles on a line*, Funct. Anal. Appl. **12**, 59–61 (1978).

[60] ———, *Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles*, Funct. Anal. Appl. **14**, 282–290 (1980).

[61] ———, *Elliptic solutions of nonlinear integrable equations and related topics*, Acta Appl. Math. **36**, 7–25 (1994).

[62] I. Krichever, O. Babelon, E. Billey, and M. Talon, *Spin generalization of the Calogero–Moser system and the matrix KP equation,* Amer. Math. Soc. Transl. (2) **170**, 83–119 (1995).

[63] M. D. Kruskal, *The Korteweg-de Vries equation and related evolution equations*, in *Nonlinear Wave Motion*, A. C. Newell (ed.), Amer. Math. Soc., Providence, RI, Lectures in Appl. Math. **15**, 61–83 (1974).

[64] A. I. Markushevich, *Theory of Functions of a Complex Variable*, 2nd. ed., Chelsea, New York, 1985.

[65] H. P. McKean, *Integrable Systems and Algebraic Curves*, in *Global Analysis*, M. Grmela and J. E. Marsden (eds.), Lecture Notes in Mathematics **755**, Springer, Berlin, 1979, p. 83–200.

[66] G. Mittag-Leffler, *"Sur les équations différentielles linéaires à coefficients doublement périodiques"*, C.R. Acad. Sci. Paris **90**, 299-300 (1880).

[67] J. Moser, *On a class of polynomials connected with the Korteweg-de Vries equation*, in *Differential Equations*, G. Berg, M. Essén, and Å. Pleijel (eds.), Almquist & Wiksell Int., Stockholm, 1977, p. 144–154.

[68] M. Ohmiya, *On the Darboux transformation of the second order differential operator of Fuchsian type on the Riemann sphere*, Osaka J. Math. **25**, 607–632 (1988).

[69] A. Yu. Orlov, *Remarks about the Calogero–Moser system and the KP equation*, in *The Bispecral Problem*, J. Harnad and A. Kasman (eds.), CRM Proceedings and Lecture Notes, Vol. 14, Amer. Math. Soc., Providence, RI, 1998, p. 227–234.

[70] D. Pelinovsky, *Rational solutions of the Kadomtsev-Petviashvili hierarchy and the dynamics of their poles.I. New form of a general rational solution*, J. Math. Phys. **35**, 5820–5830 (1994).

[71] E. Picard, *"Sur une généralisation des fonctions périodiques et sur certaines équations différentielles linéaires"*, C. R. Acad. Sci. Paris **89**, 140–144 (1879).

[72] ———, *"Sur une classe d'équations différentielles linéaires"*, C. R. Acad. Sci. Paris **90**, 128–131 (1880).

[73] ———, *"Sur les équations différentielles linéaires à coefficients doublement périodiques"*, J. reine angew. Math. **90**, 281–302 (1881).

[74] S. Saks and A. Zygmund, *Analytic Functions*, Elsevier Publ., Amsterdam, 1971.

[75] G. Segal and G. Wilson, *Loop groups and equations of KdV type*, Publ. Math. IHES **61**, 5–65 (1985).

[76] T. Shiota, *Calogero–Moser hierarchy and KP hierarchy,* J. Math. Phys. **35**, 5844–5849 (1994).

[77] A. O. Smirnov, *Elliptic solutions of the Korteweg-de Vries equation*, Math. Notes **45**, 476–481 (1989).

[78] ———, *Finite-gap elliptic solutions of the KdV equation*, Acta Appl. Math. **36**, 125–166 (1994).

[79] V. V. Sokolov, *Examples of commutative rings of differential operators,* Funct. Anal. Appl. **12**, 65–66 (1978).

[80] I. A. Taimanov, *Elliptic solutions of nonlinear equations*, Theoret. Math. Phys. **84**, 700–706 (1990).

[81] ———, *On the two-gap elliptic potentials*, Acta Appl. Math. **36**, 119–124 (1994).

[82] K. Takasaki, *Elliptic Calogero–Moser systems and isomonodromic deformations*, J. Math. Phys. **40**, 5787–5821 (1999).

[83] W. R. Thickstun, *A system of particles equivalent to solitons*, J. Math. Anal. Appl. **55**, 335–346 (1976).

[84] A. Treibich, *Tangential polynomials and elliptic solitons,* Duke Math. J. **59**, 611–627 (1989).

[85] ———, *Compactified Jacobians of Tangential Covers*, in *Integrable Systems: The Verdier Memorial Conference*, O. Babelon, P. Cartier, and Y. Kosmann-Schwarzbach (eds.), Birkhäuser, Boston, 1993, p. 39–60.

[86] ———, *Rêvetements tangentiels et condition de Brill-Noether*, C. R. Acad. Sci. Paris **316**, 815–817 (1993).

[87] ———, *New elliptic potentials*, Acta Appl. Math. **36**, 27–48 (1994).

[88] ———, *Matrix elliptic solitons,* Duke Math. J. **90**, 523–547 (1997).

[89] ———, *Hyperelliptic tangential covers and finite-gap potentials*, Russ. Math. Surv. **56**, 1107–1151 (2001).

[90] A. Treibich and J.-L. Verdier, *Solitons elliptiques*, in *The Grothendieck Festschrift, Volume III*, P. Cartier, L. Illusie, N. M. Katz, G. Laumon, Y. Manin, and K. A. Ribet (eds.), Birkhäuser, Basel, 1990, p. 437–480.

[91] ———, *Revêtements tangentiels et sommes de 4 nombres triangulaires*, C. R. Acad. Sci. Paris **311**, 51–54 (1990).

[92] ———, *Revêtements exceptionnels et sommes de 4 nombres triangulaires*, Duke Math. J. **68**, 217–236 (1992).

[93] ———, *Variétés de Kritchever des solitons elliptiques de KP,* in *Proceedings of the Indo-French Conference on Geometry (Bombay, 1989)*, Hindustan Book Agency, Delhi, 1993, p. 187–232.

[94] ———, *Au-delà des potentiels et rêvetements tangentiels hyperelliptiques exceptionnels,* C. R. Acad. Sci. Paris **325**, 1101–1106 (1997).

[95] J. F. van Diejen and H. Puschmann, *Reflectionless Schrödinger operators, the dynamics of zeros, and the solitonic Sato formula*, Duke Math. J. **104**, 269–318 (2000).

[96] K. L. Vaninsky, *Trace formula for a system of particles with elliptic potential,* Pac. J. Math. **189**, 159–178 (1999).

[97] ———, *On explicit parametrisation of spectral curves for Moser–Calogero particles and its applications*, Internat. Math. Res. Notices 1999, no. 10, 509–529.

[98] R. Weikard, *On rational and periodic solutions of stationary KdV equations,* Doc. Math. **4**, 109–126 (1999).

[99] ———, *On commuting differential operators*, Electron. J. Diff. Eq. 2000, No. 19, 1–11.

[100] ———, *Floquet theory for linear differential equations with meromorphic solutions*, E. J. Qualitative Theory of Diff. Equ., No. 8., 1–6 (2000).

[101] A. Wilansky, *Topology for Analysis*, Krieger, Malabar, Fl, 1983.

[102] G. Wilson, *Collisions of Calogero–Moser particles and an adelic Grassmannian*, Invent. Math. **133**, 1–41 (1998).

[103] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, Cambridge, 1986.

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
*E-mail address*: `fritz@math.missouri.edu`
*URL*: `http://www.math.missouri.edu/people/fgesztesy.html`

Department of Computer Science, Applied Mathematics Group, FH-Vorarlberg, A–6850 Dornbirn, Austria
*E-mail address*: `karl.unterkofler@fh-vorarlberg.ac.at`

Department of Mathematics, University of Alabama at Birmingham,, Birmingham, AL 35294–1170, USA
*E-mail address*: `rudi@math.uab.edu`
*URL*: `http://www.math.uab.edu/rudi`