A New Approach to the Boussinesq Hierarchy and its Algebro-Geometric Solutions

Habilitationsschrift

Karl Unterkofler

Karl Unterkofler
Institut für Theoretische Physik
Petersgasse 16
Technische Universität Graz
A–8010 Graz, Austria, Europe

E-mail address: `karl@itp.tu-graz.ac.at`
URL: `http://fampm201.tu-graz.ac.at`

1991 Mathematics subject classification. 35Q58, 35Q53

Abstract. We develop a new systematic approach to the Boussinesq (Bsq) hierarchy based on elementary algebraic methods. In particular, we recursively construct Lax pairs for the Bsq hierarchy by introducing a fundamental polynomial formalism and establish the basic algebro-geometric setting including associated Burchnall-Chaundy curves, Baker-Akhiezer functions, trace formulas, and Dubrovin-type equations for Dirichlet and Neumann divisors. A detailed theta function representation of all algebro-geometric quasi-periodic solutions and related quantities of the Bsq hierarchy is provided. As an example Halphen's equation and its algebro-geometric (elliptic) solutions including their rational limit are discussed. Finally, we represent the diagonal Green's function within this formalism.

Typeset by $\mathcal{AMS}$-LaTeX .
Version: January 17, 2000

To Monika, Peter, Clemens, and Judith.

Contents

Preface

This work grew out of a planned project *Algebro-Geometric Solutions of the Gelfand-Dickey Hierarchy and Trace Formulas for First-Order Self-Adjoint Hamiltonian Systems* which unfortunately never could be realized.

The results achieved concerning the Boussinesq hierarchy are published in the following sequel of papers

(i) R. Dickson, F. Gesztesy, and K. Unterkofler, *A New Approach to the Boussinesq Hierarchy*, Math. Nachr. **198** (1999), 51–108.

(ii) R. Dickson, F. Gesztesy, and K. Unterkofler, *Algebro-geometric solutions of the Boussinesq hierarchy*, Rev. Math. Phys. **11** (1999), 823–879.

(iii) K. Unterkofler. *On the solutions of Halphen's equation*, preprint 1999, Diff. Integral Eqs. in print.

Acknowledgments.
I would like to thank my colleagues and friends, especially W. Bulla, R. Dickson, F. Gesztesy, B. Thaller, G. Teschl, S. Timischl, and R. Weikard for continuous help and encouragement during this project, furthermore the Department of Mathematics of the University of Missouri, Columbia, for the extraordinary hospitality during several stays.

Karl Unterkofler

Graz, Austria
December, 1999

Chapter 1

Introduction

The Boussinesq (Bsq) equation,

$$u_{tt} = u_{xx} + 3(u^2)_{xx} - u_{xxxx}, \tag{1.1}$$

was originally introduced in 1871 as a model for one-dimensional weakly nonlinear dispersive water waves propagating in both directions (cf. the recent discussion in [**77**]). It is customary to cast the equation in yet another form and instead write it as the system of equations

$$q_{0,t} + \frac{1}{6}q_{1,xxx} + \frac{2}{3}q_1 q_{1,x} = 0, \qquad q_{1,t} - 2q_{0,x} = 0. \tag{1.2}$$

Introducing

$$q_1(x,t) = -(6u(x, 3^{-1/2}t) + 1)/4, \tag{1.3}$$

equation (1.1) results upon eliminating q_0 (cf. also [**38**]).

The principal subject of this paper concerns algebro-geometric quasi-periodic solutions of the completely integrable hierarchy of Boussinesq equations, of which (1.2) is just the first of infinitely many members. In order to be able to give a more precise description of the concepts involved, we briefly recall some basic notation in connection with the Boussinesq hierarchy.

The Boussinesq hierarchy is defined in terms of Lax pairs (L_3, P_m) of differential expressions, where L_3 is a fixed one-dimensional third-order linear differential expression,

$$L_3 = \frac{d^3}{dx^3} + q_1\frac{d}{dx} + \frac{1}{2}q_{1,x} + q_0, \tag{1.4}$$

and P_m is a differential expression of order $m \not\equiv 0 \pmod 3$, such that the commutator of L_3 and P_m becomes a differential expression of order one. For the Boussinesq equation (1.2) itself, we have $m = 2$, that is,

$$P_2 = \frac{d^2}{dx^2} + \frac{2}{3}q_1, \tag{1.5}$$

and the resulting Lax commutator representation of the Boussinesq equation then reads

$$\mathrm{Bsq}_2(q_0, q_1) = \frac{d}{dt}L_3 - [P_2, L_3] = 0, \quad \text{that is,} \quad \begin{cases} q_{0,t} + \frac{1}{6}q_{1,xxx} + \frac{2}{3}q_1 q_{1,x} = 0, \\ q_{1,t} - 2q_{0,x} = 0. \end{cases} \tag{1.6}$$

A systematic, in fact, recursive approach to all differential expressions P_m will be reviewed in Section 3.1.

However, before turning to the contents of each section, it seems appropriate to review the existing literature on the subject and its relation to our approach. Despite a fair number of papers on the Boussinesq system, the current status of research has not yet reached the high level of the KdV hierarchy, or more generally, that of the AKNS hierarchy. From the perspective of completely integrable systems, the reasons for this discrepancy are easily traced back to the enormously increased complexity when making the step from the second-order operator L_2 associated with the KdV hierarchy to the third-order operator L_3 in connection with the Bsq hierarchy. On an algebro-geometrical level this difference amounts to hyperelliptic curves in the KdV (and AKNS) context as opposed to non-hyperelliptic ones that arise in the Bsq case.

The classical paper on the Bsq equation, or perhaps more appropriately, the nonlinear string equation, is due to Zakharov [**94**]. In particular, he introduced the basic Lax pair (L_3, P_2) and discussed the infinite set of polynomial integrals of motion. In many ways closest in spirit to our approach is the seminal paper by McKean [**72**] (see also [**71**]) describing spatially periodic solutions of the Bsq equation. In contrast to [**72**] though, we concentrate here on the algebro-geometric (i.e., finite-genus) case and make no assumptions of periodicity in order to describe all algebro-geometric quasi-periodic solutions. The application of inverse scattering techniques for the third-order differential expression L_3 to the initial value problem of the Bsq equation is discussed in great detail by Deift, Tomei, and Trubowitz [**18**] and Beals, Deift, and Tomei [**8**]. General existence theorems (local and global in time) for solutions of the Bsq equation can also be found, for instance, in Craig [**17**], Bona and Sachs [**10**], and Fang and Grillakis [**29**], and the references therein. In particular, [**8**], [**10**], [**17**], [**18**], [**65**], [**72**], and [**73**] further discuss and contrast the blow-up mechanism for solutions of the nonlinear string equation obtained by Kalantarov and Ladyzhenskaya [**59**]. Other special classes of solutions have been considered by a variety of authors. For instance, certain classes of rational Bsq solutions are treated by Airault [**4**], Airault, McKean, and Moser [**5**], Chudnovsky [**16**], and Latham and Previato [**64**]. In addition, the classical dressing method of Zhakarov and Shabat to construct particular classes of solutions for very general systems of integrable equations, as described, for instance, in [**95**], [**96**], [**97**], and [**98**], should be mentioned in this context. Moreover, certain algebro-geometric Bsq solutions, obtained as special solutions of the Kadomtsev-Petviashvili (KP) equation or by the reduction theory of Riemann theta functions, are briefly discussed by Dubrovin [**24**], Matveev and Smirnov [**66**], [**67**], [**68**], Previato [**81**], [**82**], Previato and Verdier [**84**], and Smirnov [**88**], [**89**]. The latter solutions appear as special cases of a general scheme of constructing algebro-geometric solutions of completely integrable systems developed by Krichever [**61**], [**62**], [**63**] and Dubrovin [**23**], [**25**] (see also [**9**], [**37**], [**76**], [**86**]).

Next we describe the content of this paper. Since the inevitable complexity of the Bsq formalism tends to cloud the simplicity of the basic ideas involved, we decided to include a corresponding treatment of the KdV hierarchy in Section 2.1–2.3, especially since the latter case is by far the most transparent one within the Gelfand-Dickey hierarchy. Following Al'ber [**6**], [**7**] (see also [**19**], Ch. 12, [**34**]) we describe a recursive approach to Lax pairs of the KdV hierarchy in Section 2.1 and establish its connection with the Burchnall-Chaundy theory [**13**], [**14**], [**15**] and hence with hyperelliptic curves branched at infinity. Combining the recursive formalism of Section 2.1 with a polynomial approach to represent positive divisors of degree n

on a hyperelliptic curve of genus n originally developed by Jacobi [**58**] and applied to the KdV case by Mumford [**75**], Section III.a.1 and McKean [**74**], a detailed analysis of the stationary KdV hierarchy is provided in Section 2.2. The corresponding time-dependent formalism of the KdV hierarchy is then developed in Section 2.3. Our presentation of Sections 2.1–2.3 follows the one in [**40**].

Our principal contribution to this subject is a unified framework that yields all algebro-geometric quasi-periodic solutions of the entire Boussines hierarchy at once.

In Section 3.1 we develop a recursive construction of the stationary Bsq hierarchy. The stationary Boussinesq hierarchy is then obtained by imposing the t-independent Lax commutator relations

$$[P_m, L_3] = 0, \quad m \not\equiv 0 \pmod 3)), \tag{1.7}$$

assuming q_0 and q_1 to be t-independent. From the differential expression P_m we construct two polynomials $S_m(z)$ and $T_m(z)$ in z, which are both x-independent. This leads immediately to the classical Burchnall-Chaundy polynomial (cf. [**13**], [**14**]), and hence to a (generally, non-hyperelliptic) curve $\mathcal{K}_{m-1}$ of arithmetic genus $m-1$, the central object in the analysis to follow.

The recursive approach of Section 3.1 is then combined with a fundamental polynomial approach (in the spirit of Jacobi's treatment of the hyperelliptic case in Section 2.2) to represent positive divisors of degree n on Bsq curves of genus n in order to analyze the stationary Bsq hierarchy in Section 3.2. Rather than studying the Baker-Akhiezer function ψ (i.e., the common eigenfunction ψ of the commuting operators L_3 and P_m) directly, our main object is a meromorphic function ϕ equal to the logarithmic x-derivative of ψ, such that ϕ satisfies a nonlinear second-order differential equation. Moreover, we describe Dubrovin-type equations for the analogs of Dirichlet and Neumann eigenvalues when compared to the KdV hierarchy.

Section 3.3 then presents the explicit theta function representations of the Baker-Akhiezer function, the meromorphic function ϕ, and in particular, that of the potentials q_1 and q_0 for the entire Boussinesq hierarchy (the latter being the analog of the celebrated Its-Matveev formula [**57**] in the KdV context).

Sections 3.4 and 3.5 then extend the analyses of Sections 3.2 and 3.3, respectively, to the time-dependent case. Each equation in the hierarchy is permitted to evolve in terms of an independent deformation (time) parameter t_r. As initial data we use a stationary solution of the mth equation of the Boussinesq hierarchy and then construct a time-dependent solution of the rth equation of the Boussinesq hierarchy. The Baker-Akhiezer function, the meromorphic function ϕ, the analogs of the Dubrovin equations, and the theta function representations of Section 3.3 are all extended to the time-dependent case.

Chapter 4 investigates Halphen's equation and provides a variety of explicit examples illustrating the Bsq formalisms.

Finally, in Chapter 5 we represent the diagonal Green's function within this formalism.

In Appendix A we provide an introduction to the theory of Riemann surfaces and their theta functions. Appendix B is a collection of results on trigonal Riemann surfaces associated with Bsq-type curves.

It should perhaps be noted at this point that our elementary algebraic approach to the Bsq hierarchy and its algebro-geometric solutions is in fact universally applicable to $1+1$-dimensional hierarchies of soliton equations such as the KdV hierarchy [**40**], the AKNS hierarchy [**39**], the combined sine-Gordon and mKdV hierarchy [**36**], and the Toda and Kac-van Moerbeke hierarchies [**12**] (see also [**37**]).

Finally, we mention that a combination of the Bsq formalism developed in this paper and the Picard-type techniques introduced in a recent explicit characterization of all elliptic solutions of the KdV hierarchy in [**45**] (see also [**44**]) are expected to yield a similar characterization of all elliptic solutions of the Bsq hierarchy, a topic that continues to attract considerable interest (see, e.g., [**66**], [**68**], [**81**], [**82**], [**88**]). Recently Weikard [**92**] (cf. [**91**]) proved an analogous theorem for the entire Gelfand-Dickii hierarchy for rational and simply periodic algebro-geometric potentials.

Chapter 2

The Recursive Approach to the Korteweg de Vries Hierarchy and Hyperelliptic Curves

2.1. The Recursive Approach to the KdV Hierarchy

Following the treatment in [**40**] we present in this section the recursive approach to Lax pairs of the KdV hierarchy and its connection with the Burchnall-Chaundy theory [**13**], [**14**], [**15**] and hence with hyperelliptic curves branched at infinity. Originally, this approach was advocated by Al'ber [**6**], [**7**] (see also [**19**], Ch. 12, [**34**], [**39**], [**41**]).

Suppose $q_0 \in C^\infty(\mathbb{R})$ (or q_0 meromorphic on $\mathbb{C}$) and introduce the second-order differential expression

$$L_2 = \frac{d^2}{d\,x^2} + q_0(x), \quad x \in \mathbb{R} \ \text{(or } \ \mathbb{C}). \tag{2.1}$$

In order to explicitly construct odd-order differential expressions $P_r, r \not\equiv 0 \pmod 2$ commuting with L_2, that will be used later to define the stationary KdV hierarchy, one proceeds as follows.

Pick $n \in \mathbb{N}_0 (= \mathbb{N} \cup \{0\})$ and define $\{f_\ell(x)\}_{0 \le \ell \le n+1}$ recursively by

$$\begin{aligned} f_0 &= 1, \\ 2 f_{\ell,x}(x) &= \frac{1}{2}\, f_{\ell-1,xxx}(x) + 2\, q_0(x)\, f_{\ell-1,x}(x) + q_{0,x}(x)\, f_{\ell-1}(x), \quad 1 \le \ell \le n+1. \end{aligned} \tag{2.2}$$

Explicitly, one computes

$$f_0 = 1, \quad f_1 = \frac{1}{2}\, q_0 + c_1, \quad f_2 = \frac{1}{8}\, q_{0,xx} + \frac{3}{8}\, q_0^2 + c_1 \,\frac{1}{2}\, q_0 + c_2, \quad \text{etc.}, \tag{2.3}$$

where $\{c_\ell\}_{1\le\ell\le n}$ are integration constants. Given (2.2), one defines the differential expression of order r by

$$P_r = \sum_{\ell=0}^{n} \Big(-\frac{1}{2}\, f_{n-\ell,x} + f_{n-\ell} \frac{d}{d\,x} \Big) L_2^{\ell} + \sum_{\ell=0}^{n} k_{r,\ell} L_2^{\ell}, \quad k_{r,\ell} \in \mathbb{C}, \quad 0 \le \ell \le n,$$
$$r = 2\,n+1, \quad n \in \mathbb{N}_0, \tag{2.4}$$

and verifies

$$[P_r, L_2] = 2 f_{n+1,x}, \quad r = 2\,n+1, \; n \in \mathbb{N}_0 \tag{2.5}$$

(where [. , .] denotes the commutator symbol). The pair (L_2, P_r) represents the celebrated Lax pair for the KdV hierarchy. Varying $n \in \mathbb{N}_0$, the stationary KdV hierarchy is then defined by the vanishing of the commutators of P_r and L_2 in (2.5), that is, by

$$[P_r, L_2] = 0, \quad r = 2\,n+1, \; n \in \mathbb{N}_0, \tag{2.6}$$

or equivalently, by

$$f_{n+1,x} = 0, \quad n \in \mathbb{N}_0. \tag{2.7}$$

Explicitly, one obtains for the first few equations of the stationary KdV hierarchy

$$\begin{aligned} &q_{0,x} = 0, \\ &\frac{1}{4}\, q_{0,xxx} + \frac{3}{2}\, q_0 q_{0,x} + c_1\, q_{0,x} = 0, \\ &\frac{1}{16}\, q_{0,xxxxx} + \frac{5}{8}\, q_0\, q_{0,xxx} + \frac{5}{4}\, q_{0,x}\, q_{0,xx} + \frac{15}{8}\, q_0^2\, q_{0,x} + c_1 \big(\frac{1}{4}\, q_{0,xxx} + \frac{3}{2}\, q_0\, q_{0,x}\big) \\ &\quad + c_2\, q_{0,x} = 0, \\ &\text{etc.} \end{aligned} \tag{2.8}$$

By definition, solutions $q_0(x)$ of any of the stationary KdV equations (2.8) are called **algebro-geometric finite-gap potentials** associated with the KdV hierarchy. If $f_{n+1,x} = 0$, one also calls q_0 a stationary n-gap solution.

Next, we introduce the polynomial F_r of degree n with respect to $z \in \mathbb{C}$,

$$F_r(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}(x) z^{\ell}, \quad f_0 = 1, \quad r = 2\,n+1. \tag{2.9}$$

Explicitly, the first few polynomials F_r read

$$\begin{aligned} &F_1 = 1, \\ &F_3 = z + \big(\frac{1}{2}\, q_0 + c_1\big), \\ &F_5 = z^2 + \big(\frac{1}{2}\, q_0 + c_1\big) z + \big(\frac{1}{8}\, q_{0,xx} + \frac{3}{8}\, q_0^2 + c_1\, \frac{1}{2}\, q_0 + c_2\big), \\ &\text{etc.} \end{aligned} \tag{2.10}$$

Given (2.9), (2.6) respectively, (2.7) becomes

$$\frac{1}{2}\,F_{r,xxx} - 2(z-q_0)\,F_{r,x} + q_{0,x}\,F_r = 0. \tag{2.11}$$

Multiplying (2.11) by F_r and integrating once results in

$$R_r(z) = -\frac{1}{2}\,F_{r,xx}\,F_r + \frac{1}{4}F_{r,x}^2 + (z-q_0)\,F_r^2, \tag{2.12}$$

where the integration constant $R_r(z)$ is seen to be a monic polynomial in z of degree $2\,n+1$. Thus we may write

$$R_r(z) = \prod_{m=0}^{2n}(z-E_m), \qquad \{E_m\}_{0\le m\le 2n}\subset\mathbb{C}. \tag{2.13}$$

Next, we consider the kernel (i.e., the formal null space in a purely algebraic sense) of $(L_2-z), z\in\mathbb{C}$,

$$(L_2-z)\psi = 0, \quad \psi=\psi(z,x), \quad z\in\mathbb{C} \tag{2.14}$$

and, taking into account (2.6), that is, $[P_r, L_2]=0$, compute the restriction of P_r to the $\ker(L_2-z)$. Using

$$\psi_{xx} = (z-q_0)\psi, \quad \psi_{xxx} = (z-q_0)\psi_x - q_{0,x}\psi, \quad \text{etc.}, \tag{2.15}$$

to eliminate higher-order derivatives of ψ, one obtains from (2.2), (2.4), (2.7), (2.9), and (2.11),

$$P_r\Big|_{\ker(L_2-z)} = \Big(F_r(z,x)\frac{d}{dx} + G_r(z,x)\Big)\Big|_{\ker(L_2-z)}, \tag{2.16}$$

where

$$G_r(z,x) = -\frac{1}{2}F_{r,x} + k_r(z), \tag{2.17}$$

and (cf. (2.4))

$$k_r(z) = \sum_{\ell=0}^{n} k_{r,\ell}z^\ell. \tag{2.18}$$

The construction of P_r in (2.4) and (2.16) should be contrasted with the one based on formal pseudo-differential expressions originally developed by Gel'fand-Dickey [**33**] and further refined by Adler [**3**] (see also [**19**], Ch. 1).

Still assuming $f_{n+1,x}=0$ as in (2.7), $[P_r, L_2]=0$ in (2.6) yields an algebraic relationship between P_r and L_2 by a celebrated result of Burchnall and Chaundy [**13**], [**14**], [**15**] (see also [**93**]). The following theorem gives a detailed account of this relationship.

Theorem 2.1. *Assume $f_{n+1,x}=0$, that is $[P_r, L_2]=0$ for some $r=2\,n+1$, $n\in\mathbb{N}_0$. Then the Burchnall-Chaundy polynomial $\mathcal{F}_{(r-1)/2}(L_2,P_r)$ of the pair (L_2,P_r) explicitly reads (cf. (2.13) and (2.18))*

$$\mathcal{F}_{(r-1)/2}(L_2,P_r) = \Big(P_r - k_r(L_2)\Big)^2 - R_r(L_2) = 0, \tag{2.19}$$

$$k_r(z) = \sum_{\ell=0}^{n} k_{r,\ell}z^\ell, \quad R_r(z) = \prod_{m=0}^{2n}(z-E_m), \quad z\in\mathbb{C}, \quad r=2n+1, \; n\in\mathbb{N}_0.$$

Proof. Let $\psi_j \in \ker(L_2 - z)$, $j = 1, 2$ be linearly independent. Since $[P_r, L_2] = 0$, one can represent P_r as a 2×2 matrix $\mathcal{P}_r(z)$ on $\ker(L_2 - z)$,

$$P_r \psi_j = \sum_{k=1}^{2} \mathcal{P}_{r,j,k} \psi_k, \tag{2.20}$$

$$\mathcal{P}_r \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \mathcal{P}_{r,1,1} & \mathcal{P}_{r,1,2} \\ \mathcal{P}_{r,2,1} & \mathcal{P}_{r,2,2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\mathcal{P}_{r,1,j} = \frac{W(P_r\psi_j, \psi_2)}{W(\psi_1, \psi_2)}, \quad \mathcal{P}_{r,2,j} = \frac{W(\psi_1, P_r\psi_j)}{W(\psi_1, \psi_2)}, \quad 1 \leq j \leq 2. \tag{2.21}$$

Using (2.11) and (2.15)–(2.18) one verifies

$$\operatorname{tr}(\mathcal{P}_r(z)) = 2\, k_r(z), \tag{2.22}$$

$$\det(\mathcal{P}_r(z)) = \frac{W(P_r\psi_1(z), P_r\psi_2(z))}{W(\psi_1(z), \psi_2(z))} = k_r(z)^2 - R_r(z). \tag{2.23}$$

(Here tr(.) and det(.) denote the trace and determinant, respectively and $W(f, g) = fg' - f'g$ denotes the Wronskian of f and g. The characteristic polynomial $\det(y - \mathcal{P}_r(z)) = 0$ of $\mathcal{P}_r(z)$ then yields

$$\mathcal{F}_{(r-1)/2}(z, y) = y^2 - y \operatorname{tr}(\mathcal{P}_r(z)) + \det(\mathcal{P}_r(z)) = \big(y - k_r(z)\big)^2 - R_r(z) = 0. \tag{2.24}$$

The result (2.19) then follows from the Cayley-Hamilton theorem, since $z \in \mathbb{C}$ is arbitrary. □

Remark 2.2. *Equation (2.24) naturally leads to the (possibly singular) hyperelliptic curve* $\mathcal{K}_{(r-1)/2}$,

$$\mathcal{K}_{(r-1)/2}: \quad \mathcal{F}_{(r-1)/2}(z, y) = \big(y - k_r(z)\big)^2 - R_r(z) = 0, \tag{2.25}$$

$$k_r(z) = \sum_{\ell=0}^{n} k_{r,\ell} z^\ell, \quad R_r(z) = \prod_{m=0}^{2n} (z - E_m), \quad r = 2n + 1, \ n \in \mathbb{N}_0$$

of (arithmetic) genus $n = (r-1)/2$. *In the nonsingular case, where* $E_m \neq E_{m'}$ *for* $m \neq m'$, *the Riemann theta function associated with (the one-point compactification of)* $\mathcal{K}_{(r-1)/2}$ *then yields an explicit expression for* $q_0(x)$ *originally derived by Its and Matveev* [**57**].

Finally, introducing a deformation parameter $t_r \in \mathbb{R}$ in q_0 (i.e., $q_0(x) \to q_0(x, t_r)$), the time-dependent KdV hierarchy is defined as the collection of evolution equations (varying $r \in 2\,\mathbb{N}_0 + 1$),

$$\frac{d}{dt_r} L_2(t_r) - [P_r(t_r), L_2(t_r)] = 0, \quad (x, t_r) \in \mathbb{R}^2, \quad r = 2\,n + 1, \quad n \in \mathbb{N}_0, \tag{2.26}$$

or equivalently, by

$$\mathrm{KdV}_r(q_0) = q_{0,t_r} - 2 f_{n+1,x} = 0, \quad (x, t_r) \in \mathbb{R}^2, \quad r = 2\,n + 1, \quad n \in \mathbb{N}_0, \tag{2.27}$$

that is, by

$$\mathrm{KdV}_r(q_0) = q_{0,t_r} - \frac{1}{2} F_{r,xxx} + 2\,(z - q_0)\, F_{r,x} - q_{0,x}\, F_r = 0,$$
$$(x, t_r) \in \mathbb{R}^2, \quad r = 2\,n + 1, \quad n \in \mathbb{N}_0. \tag{2.28}$$

Explicitly, one obtains for the first few equations in (2.27),

$$\mathrm{KdV}_1(q_0) = q_{0,t_1} - q_{0,x} = 0,$$

$$\begin{aligned} \mathrm{KdV}_3(q_0) &= q_{0,t_3} - \frac{1}{4}\, q_{0,xxx} - \frac{3}{2}\, q_0\, q_{0,x} - c_1\, q_{0,x} = 0, \\ \mathrm{KdV}_5(q_0) &= q_{0,t_5} - \frac{1}{16}\, q_{0,xxxxx} - \frac{5}{8}\, q_0\, q_{0,xxx} - \frac{5}{4}\, q_{0,x}\, q_{0,xx} - \frac{15}{8}\, q_0^2\, q_{0,x} \\ &\quad - c_1\, \big(\frac{1}{4}\, q_{0,xxx} + \frac{3}{2}\, q_0\, q_{0,x}\big) - c_2\, q_{0,x} = 0, \end{aligned} \tag{2.29}$$

etc.

Remark 2.3. *We chose to start by postulating the recursion relation (2.2) and then developed the whole formalism based on (2.2), (2.4)–(2.6). Alternatively, one could have started from*

$$(L_2 - z)\psi(P) = 0, \quad (P_r - y(P))\psi(P) = 0, \quad P = (z, y(P)) \in \mathcal{K}_{(r-1)/2}\backslash\{P_\infty\} \tag{2.30}$$

and obtained the recursion relation (2.2) and the remaining stationary results of this section as a consequence of (2.9) and (2.16). Similarly, starting with

$$(L_2 - z)\psi(P, t_r) = 0, \quad (\frac{\partial}{\partial t_r} - P_r)\psi(P, t_r) = 0, \quad t_r \in \mathbb{R}, \tag{2.31}$$

one infers the time-dependent results (2.26)–(2.29).

2.2. The Stationary KdV Formalism

In this section we continue our discussion of the KdV hierarchy and focus our attention on the stationary case. Following [**40**] we outline the connections between the polynomial approach described in Section 2.1 and a fundamental meromorphic function $\phi(P, x)$ defined on the hyperelliptic curve $\mathcal{K}_{(r-1)/2}$ in (2.25). Moreover, we discuss in some detail the associated stationary Baker-Akhiezer function $\psi(P, x, x_0)$, the common eigenfunction of L_2 and P_r (we recall that $[P_r, L_2] = 0$), and associated positive (Dirichlet and Neumann) divisors of degree $(r-1)/2$ on $\mathcal{K}_{(r-1)/2}$. The latter topic was originally developed by Jacobi [**58**] and applied to the KdV case by Mumford [**75**], Section III.a.1 and McKean [**74**].

We recall the hyperelliptic curve $\mathcal{K}_{(r-1)/2}$ in (2.25),

$$\begin{gathered} \mathcal{K}_{(r-1)/2}: \quad \mathcal{F}_{(r-1)/2}(z, y) = \big(y - k_r(z)\big)^2 - R_r(z) = 0, \\ k_r(z) = \sum_{\ell=0}^{n} k_{r,\ell} z^\ell, \quad R_r(z) = \prod_{m=0}^{2n} (z - E_m), \end{gathered} \tag{2.32}$$

where $r \in 2\,\mathbb{N}_0 + 1$ will be fixed throughout this section and denote its one-point compactification (joining the branch point P_∞) by the same symbol $\mathcal{K}_{(r-1)/2}$. (In the following $\mathcal{K}_{(r-1)/2}$ will always denote the compactified curve.) Thus $\mathcal{K}_{(r-1)/2}$ becomes a (possibly singular) two-sheeted hyperelliptic Riemann surface of arithmetic genus $(r-1)/2$ in a standard manner. We now introduce a bit more notation in this context. Points P on $\mathcal{K}_{(r-1)/2}$ are represented as pairs $P = (z, y(P))$ satisfying (2.32) together with $P_\infty = (\infty, \infty)$, the point at infinity. The complex structure on $\mathcal{K}_{(r-1)/2}$ is defined in the usual way by introducing local coordinates $\zeta_{P_0}: \ P \to (z - z_0)$ near points $P_0 \in \mathcal{K}_{(r-1)/2}$ which are neither branch nor singular points of $\mathcal{K}_{(r-1)/2}$, $\zeta_{P_\infty}: \ P \to 1/z^{1/2}$ near the branch point $P_\infty \in \mathcal{K}_{(r-1)/2}$ (with an appropriate determination of the branch of $z^{1/2}$) and similarly at branch and/or singular points of $\mathcal{K}_{(r-1)/2}$. The holomorphic sheet exchange map (involution) $*$ is defined by

$$*: \begin{cases} \mathcal{K}_{(r-1)/2} \to \mathcal{K}_{(r-1)/2} \\ P = (z, y_j(z)) \to P^* = (z, y_{j+1 \ (\mathrm{mod}\ 2)}(z)), \quad j = 1, 2 \end{cases}, \tag{2.33}$$

where $y_j(z),\ j = 1, 2$ denote the two branches of $y(P)$ satisfying $\mathcal{F}_{(r-1)/2}(z, y) = 0$, that is,

$$(y - y_1(z))\,(y - y_2(z)) = (y - k_r(z))^2 - R_r(z) = 0. \tag{2.34}$$

Finally, positive divisors on $\mathcal{K}_{(r-1)/2}$ of degree $n = (r-1)/2$ are denoted by

$$\mathcal{D}_{P_1,\dots,P_n}: \begin{cases} \mathcal{K}_{(r-1)/2} \to \mathbb{N}_0 \\ P \to \mathcal{D}_{P_1,\dots,P_n}(P) = \begin{cases} m \text{ if } P \text{ occurs } m \text{ times in } \{P_1,\dots,P_n\} \\ 0 \text{ if } P \notin \{P_1,\dots,P_n\} \end{cases} \end{cases}. \tag{2.35}$$

Given these preliminaries, let $\psi(P, x, x_0)$ denote the common normalized (cf. (2.39)) eigenfunction of L_2 and P_r, whose existence is guaranteed by the commutativity of L_2 and P_r (cf., e.g., [**13**], [**14**]), that is, by

$$[P_r, L_2] = 0, \qquad r = 2\,n + 1 \tag{2.36}$$

for a given $n \in \mathbb{N}_0$, or equivalently, by the requirement,

$$f_{n+1,x} = 0. \tag{2.37}$$

Explicitly, this yields

$$L_2\psi(P, x, x_0) = z\,\psi(P, x, x_0), \quad P_r\psi(P, x, x_0) = y(P)\,\psi(P, x, x_0), \tag{2.38}$$
$$P = (z, y(P)) \in \mathcal{K}_{(r-1)/2} \setminus \{P_\infty\}, \quad x \in \mathbb{R}$$

for some fixed $x_0 \in \mathbb{R}$ with the assumed normalization,

$$\psi(P, x_0, x_0) = 1, \qquad P \in \mathcal{K}_{(r-1)/2} \setminus \{P_\infty\}. \tag{2.39}$$

$\psi(P, x, x_0)$ is called the stationary Baker-Akhiezer (BA) function of the KdV hierarchy. Closely related to $\psi(P, x, x_0)$ is the following meromorphic function $\phi(P, x)$ on $\mathcal{K}_{(r-1)/2}$ defined by

$$\phi(P, x) = \frac{\psi_x(P, x, x_0)}{\psi(P, x, x_0)}, \quad P \in \mathcal{K}_{(r-1)/2},\ x \in \mathbb{R}, \tag{2.40}$$

such that

$$\psi(P, x, x_0) = \exp\Big(\int_{x_0}^{x} d\,x' \phi(P, x')\Big), \qquad P \in \mathcal{K}_{(r-1)/2} \setminus \{P_\infty\}. \tag{2.41}$$

Since $\phi(P, x)$ is a fundamental object for the stationary KdV hierarchy we next seek its connection with the recursion formalism of Section 2.1. Recalling (2.16) and (2.17), one infers

$$P_r\psi = F_r\psi_x + \Big(-\frac{1}{2}F_{r,x} + k_r\Big)\psi = y\,\psi \tag{2.42}$$

and

$$\big(P_r\psi\big)_x = \Big(\frac{1}{2}F_{r,x} + k_r\Big)\psi_x + \Big((z - q_0)F_r - \frac{1}{2}F_{r,xx}\Big)\psi = y\,\psi_x \tag{2.43}$$

using (2.15). Thus

$$\phi = \frac{\psi_x}{\psi} = \frac{y - k_r + \frac{1}{2}F_{r,x}}{F_r} = \frac{(z - q_0)F_r - \frac{1}{2}F_{r,xx}}{y - k_r - \frac{1}{2}F_{r,x}}. \tag{2.44}$$

Introducing

$$D_n(z,x) = F_r(z,x), \tag{2.45}$$

$$N_{n+1}(z,x) = (z-q_0)F_r(z,x) - \frac{1}{2}F_{r,xx}(z,x) \tag{2.46}$$

then yields

$$\phi(P,x) = \frac{y(P) - k_r(z) + \frac{1}{2}D_{n,x}(z,x)}{D_n(z,x)} \tag{2.47}$$

$$= \frac{N_{n+1}(z,x)}{y(P) - k_r(z) - \frac{1}{2}D_{n,x}(z,x)}, \quad P = (z, y(P)) \in \mathcal{K}_{(r-1)/2} \tag{2.48}$$

and

$$D_n(z,x)\, N_{n+1}(z,x) = \big(y(P) - k_r(z)\big)^2 - \frac{1}{4}D_{n,x}(z,x)^2. \tag{2.49}$$

In order to motivate our introduction of the basic quantity $\phi(P,x)$ we started with the common eigenfunction $\psi(P,x,x_0)$ of L_2 and P_r. However, given (2.12) and the definitions (2.45), (2.46), we could have defined $\phi(P,x)$ as in (2.47) and then verified that $\psi(P,x,x_0)$ in (2.41) satisfies (2.38) and (2.39). Since by (2.9) D_n and N_{n+1} are monic polynomials with respect to z of degree n and $n+1$ respectively, we may write

$$D_n(z,x) = \prod_{j=1}^{n}(z - \mu_j(x)), \tag{2.50}$$

$$N_{n+1}(z,x) = \prod_{\ell=0}^{n}(z - \nu_\ell(x)). \tag{2.51}$$

Defining

$$\hat{\mu}_j(x) = \big(\mu_j(x), y(\hat{\mu}_j(x))\big) = \big(\mu_j(x), k_r(\mu_j(x)) + \frac{1}{2}D_{n,x}(\mu_j(x),x)\big) \in \mathcal{K}_{(r-1)/2}, \quad 1 \le j \le n, \quad x \in \mathbb{R}, \tag{2.52}$$

$$\hat{\nu}_\ell(x) = \big(\nu_\ell(x), y(\hat{\nu}_\ell(x))\big) = \big(\nu_\ell(x), k_r(\nu_\ell(x)) - \frac{1}{2}D_{n,x}(\nu_\ell(x),x)\big) \in \mathcal{K}_{(r-1)/2}, \quad 0 \le \ell \le n, \quad x \in \mathbb{R}, \tag{2.53}$$

one infers from (2.47) and (2.48) that the divisor $\big(\phi(P,x)\big)$ of $\phi(P,x)$ is given by

$$\big(\phi(P,x)\big) = \mathcal{D}_{\hat{\nu}_0(x),\dots,\hat{\nu}_n(x)}(P) - \mathcal{D}_{P_\infty,\hat{\mu}_1(x),\dots,\hat{\mu}_n(x)}(P). \tag{2.54}$$

Here we used our convention (2.35) and the additive notation for divisors. Equivalently, $\hat{\nu}_0(x),\dots,\hat{\nu}_n(x)$ are the $n+1$ zeros of $\phi(P,x)$ and $P_\infty, \hat{\mu}_1(x),\dots,\hat{\mu}_n(x)$ its $n+1$ poles.

Further properties of $\phi(P,x)$ and $\psi(P,x,x_0)$ are summarized in

Lemma 2.4. *Assume (2.36)–(2.40),* $P = (z,y(P)) \in \mathcal{K}_{(r-1)/2} \setminus \{P_\infty\}$, $r = 2n+1$, *and let* $(z,x,x_0) \in \mathbb{C} \times \mathbb{R}^2$. *Then*

(*i*). $\phi(P,x)$ *satisfies the Riccati-type equation*

$$\phi_x(P,x) + \phi(P,x)^2 = z - q_0(x). \tag{2.55}$$

(*ii*). $$\phi(P,x)\,\phi(P^*,x) = -\frac{N_{n+1}(z,x)}{D_n(z,x)}. \tag{2.56}$$

$$(iii).\ \phi(P,x)+\phi(P^*,x)=\frac{D_{n,x}(z,x)}{D_n(z,x)}. \tag{2.57}$$

$$(iv).\ \phi(P,x)-\phi(P^*,x)=\frac{2\,(y(P)-k_r(z))}{D_n(z,x)}, \tag{2.58}$$

$$\big(y(P)-k_r(z)\big)\phi(P,x)+\big(y(P^*)-k_r(z)\big)\phi(P^*,x)=\frac{2\,R_r(z)}{D_n(z,x)}. \tag{2.59}$$

$$(v).\ \psi(P,x,x_0)\,\psi(P^*,x,x_0)=\frac{D_n(z,x)}{D_n(z,x_0)}. \tag{2.60}$$

$$(vi).\ \psi_x(P,x,x_0)\psi_x(P^*,x,x_0)=-\frac{N_{n+1}(z,x)}{D_n(z,x_0)}. \tag{2.61}$$

$$(vii).\ \psi(P,x,x_0)=\left(\frac{D_n(z,x)}{D_n(z,x_0)}\right)^{1/2}\exp\Big\{\big(y(P)-k_r(z)\big)\int_{x_0}^{x}d\,x'D_n(z,x')^{-1}\Big\}. \tag{2.62}$$

$$(viii).\ N_{n+1,x}(z,x)=-(z-q_0(x))D_{n,x}(z,x). \tag{2.63}$$

Proof. (2.55) follows from $\phi=\psi_x/\psi$ and $\psi_{xx}=(z-q_0)\psi$. (2.56)–(2.59) follow from (2.47), (2.49), and

$$y(P)+y(P^*)=2\,k_r(z),\qquad y(P)\,y(P^*)=k_r(z)^2-R_r(z). \tag{2.64}$$

(2.60) follows from (2.62) and (2.56) and (2.61) from (2.60) and (2.56). In order to prove (2.62) it suffices to insert (2.48) into (2.40). (2.63) finally follows by differentiating (2.49) with respect to x (using (2.12) and (2.45)) and checking the resulting equation at the $n+1$ zeros $\nu_\ell(x)$ of $N_{n+1}(z,x)$. □

A comparison of (2.50), (2.51) and (2.60), (2.61) reveals that the divisors $\mathcal{D}_{P_\infty,\hat\mu_1(x),\dots,\hat\mu_n(x)}$ and $\mathcal{D}_{\hat\nu_0(x),\dots,\hat\nu_n(x)}$ in (2.54) are the Dirichlet and Neumann divisors associated with $L_2=\frac{d}{dx^2}+q_0(x)$ (see [**40**] for further spectral interpretations in this context). In particular, (2.56), (2.60), and (2.61) clarify the role played by D_n and N_{n+1}. Up to normalizations, D_n represents the product of the two branches of ψ and N_{n+1} the product of the two branches of ψ_x, their zeros represent Dirichlet and Neumann eigenvalues of L_2 with the corresponding boundary conditions imposed at the point $x\in\mathbb{R}$.

The reader puzzled by our definition (2.45) might compare with (3.65) in the Bsq case where F_r and D_n considerably differ from each other but the analogs of (2.56), (2.60), and (2.61) remain valid as can be seen from (3.90), (3.93), and (3.94). Using the hyperelliptic curve (2.32) we could have replaced $y-k_r(z)$ by $R_r(z)^{1/2}$ in (2.44), (2.47), (2.48) and (2.49). However, a quick look at (3.82) reveals that the polynomial behavior of the numerator and denominator of $\phi(P,x)$ with respect to y in (2.44), (2.47), and (2.48) is the key in generalizing this formalism from the KdV to the Bsq case.

Returning to $D_n(z,x)$ and $N_{n+1}(z,x)$ we note that (2.2), (2.9), (2.45), and (2.46) yield

$$\begin{aligned} &D_0=1,\\ &D_1=z+\frac{1}{2}\,q_0+c_1,\\ &D_2=z^2+\big(\frac{1}{2}\,q_0+c_1\big)z+\frac{1}{8}\,q_{0,xx}+\frac{3}{8}\,q_0^2+c_1\frac{1}{2}\,q_0+c_2,\\ &\text{etc.}\end{aligned} \tag{2.65}$$

and

$$\begin{aligned}
N_1 &= z - q_0, \\
N_2 &= z^2 + \big(-\frac{1}{2}\, q_0 + c_1\big) z - \frac{1}{4}\, q_{0,xx} - \frac{1}{2}\, q_0^2 - c_1\, q_0, \qquad (2.66) \\
N_3 &= z^3 + \big(-\frac{1}{2}\, q_0 + c_1\big) z^2 + \big(-\frac{1}{8}\, q_{0,xx} - \frac{1}{8}\, q_0^2 - c_1 \frac{1}{2}\, q_0 + c_2\big) z \\
&\quad - \frac{1}{16}\, q_{0,xxxx} - \frac{3}{8}\, q_0^3 - \frac{3}{8}\, q_{0,x}^2 - \frac{1}{2}\, q_0 q_{0,xx} - c_1 \frac{1}{4}\, q_{0,xx} - c_1 \frac{1}{2}\, q_0^2 - c_2 q_0,
\end{aligned}$$

etc.

Concerning the dynamics of the zeros $\mu_j(x)$ and $\nu_\ell(x)$ of $D_n(z,x)$ and $N_{n+1}(z,x)$ one obtains the following equations first derived by Dubrovin [**22**] in the Dirichlet case.

Lemma 2.5. *Assume (2.37), (2.50), (2.51) and let* $x \in \mathbb{R}$. *Then*

$$(i).\quad \mu_{j,x}(x) = \frac{-2\,\big(y(\hat{\mu}_j(x)) - k_r(\mu_j(x))\big)}{\prod_{\substack{k=1\\k\neq j}}^{n} \big(\mu_j(x) - \mu_k(x)\big)}, \qquad 1 \leq j \leq n. \tag{2.67}$$

$$(ii).\quad \nu_{\ell,x}(x) = \frac{-2\,\big(\nu_\ell(x) - q_0(x)\big)\big(y(\hat{\nu}_j(x)) - k_r(\nu_j(x))\big)}{\prod_{\substack{m=0\\m\neq \ell}}^{n} \big(\nu_\ell(x) - \nu_m(x)\big)}, \qquad 0 \leq \ell \leq n. \tag{2.68}$$

Proof. (2.67) is clear from (2.50) and (2.52), and (2.68) follows from (2.51), (2.53) and (2.63). □

We conclude this section with some hints concerning trace formulas for the KdV invariants in terms of Dirichlet and Neumann data.

Lemma 2.6. *Assume (2.37) and let* $x \in \mathbb{R}$. *Then*

$$\begin{aligned}
(i).\quad & \frac{1}{2}\, q_0(x) + c_1 = -\sum_{j_1=1}^{n} \mu_{j_1}(x), \\
& \frac{1}{8}\, q_{0,xx}(x) + \frac{3}{8}\, q_0(x)^2 + c_1 \frac{1}{2}\, q_0(x) + c_2 = \sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{n} \mu_{j_1}(x)\, \mu_{j_2}(x), \qquad (2.69)
\end{aligned}$$

etc.

$$\begin{aligned}
(ii).\quad & \frac{1}{2}\, q_0(x) - c_1 = \sum_{\ell_1=0}^{n} \nu_{\ell_1}(x), \\
& \frac{1}{8}\, q_{0,xx}(x) + \frac{1}{8}\, q_0(x)^2 + c_1 \frac{1}{2}\, q_0(x) - c_2 = -\sum_{\substack{\ell_1,\ell_2=0\\ \ell_1<\ell_2}}^{n} \nu_{\ell_1}(x)\, \nu_{\ell_2}(x), \qquad (2.70)
\end{aligned}$$

etc.

Here

$$c_1 = -\frac{1}{2}\sum_{m_1=0}^{2n} E_{m_1}, \qquad c_2 = \frac{1}{2}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{2n} E_{m_1}E_{m_2} - \frac{1}{8}\Big(\sum_{m_1=0}^{2n} E_{m_1}\Big)^2, \tag{2.71}$$

etc.

Proof. (2.69) and (2.71) follow by comparison of powers of z substituting (2.50) into (2.9) (taking into account (2.3)) and (2.12) (taking into account(2.13)). (2.70) is proven similarly using (2.32), (2.49)–(2.51), and the fact that $D_{n,x}(z,x)^2 = O(z^{2n-2})$ as $z \to \infty$. □

For a systematic approach to trace formulas based on a second-order nonlinear differential equation satisfied by the diagonal Green's function of L_2 in the Dirichlet case (2.69) and an analogous treatment of the Neumann case (2.70), see [**40**]. (The latter approach goes far beyond the special algebro-geometric situation presented in this section.)

2.3. The Time-Dependent KdV Formalism

In our final KdV section we indicate how to generalize the polynomial approach of Sections 2.1 and 2.2 to the time-dependent KdV hierarchy. Again we lean on the material presented in [**40**].

Our starting point is a stationary n-gap solution $q_0^{(0)}(x)$ associated with $\mathcal{K}_n$ satisfying

$$\mathrm{KdV}_{2n+1}(q_0^{(0)}) = -2f_{n+1,x} = 0, \quad x \in \mathbb{R} \tag{2.72}$$

for some fixed $n \in \mathbb{N}_0$ and a given set of integration constants $\{c_\ell\}_{1\le \ell\le n}$. Our aim is to construct the r-th KdV flow

$$\mathrm{KdV}_r(q_0) = 0, \quad q_0(x,t_{0,r}) = q_0^{(0)}(x), \quad x \in \mathbb{R} \tag{2.73}$$

for some fixed $r \in 2\mathbb{N}_0 + 1$ and $t_{0,r} \in \mathbb{R}$. In terms of Lax pairs this amounts to solving

$$\frac{d}{dt_r}L_2(t_r) - [\tilde{P}_r(t_r), L_2(t_r)] = 0, \quad t_r \in \mathbb{R}, \tag{2.74}$$

$$[P_{2n+1}(t_{0,r}), L_2(t_{0,r})] = 0. \tag{2.75}$$

As a consequence one obtains

$$[P_{2n+1}(t_r), L_2(t_r)] = 0, \qquad t_r \in \mathbb{R}, \tag{2.76}$$

$$\begin{aligned}\Big(P_{2n+1}(t_r) - k_{2n+1}(L_2(t_r))\Big)^2 &= R_{2n+1}(L_2(t_r))\\ &= \prod_{m=0}^{2n}\big(L_2(t_r) - E_m\big), \qquad t_r \in \mathbb{R}\end{aligned} \tag{2.77}$$

since the KdV flows are isospectral deformations of $L_2(t_{0,r})$.

We emphasize that the integration constants $\{\tilde{c}_\ell\}$ in $\tilde{P}_r$ and $\{c_\ell\}$ in P_{2n+1} are independent of each other (even if $r = 2n+1$). Hence we shall employ the notation $\tilde{P}_r, \tilde{k}_r, \tilde{F}_r, \tilde{G}_r$, etc. in order to distinguish them from $P_{2n+1}, k_{2n+1}, F_{2n+1}, G_{2n+1}$, etc. In addition, we followed a more elaborate notation inspired by Hirota's τ-function approach and indicated the individual r-th KdV flow by a separate time variable $t_r \in \mathbb{R}$. (The latter notation suggests considering all KdV flows simultaneously by introducing $\underline{t} = (t_1, t_3, t_5, \dots)$.)

Instead of working directly with (2.74) and (2.76), we find it more convenient to take the following equations as our point of departure,

$$q_{0,t_r} = \frac{1}{2}\tilde{F}_{r,xxx} - 2(z-q_0)\tilde{F}_{r,x} + q_{0,x}\tilde{F}_r, \quad (x,t_r)\in\mathbb{R}^2, \tag{2.78}$$

$$-\frac{1}{2}F_{2n+1}F_{2n+1,xx} + \frac{1}{4}F_{2n+1,x}^2 + (z-q_0)F_{2n+1}^2 = R_{2n+1}, \quad (x,t_r)\in\mathbb{R}^2, \tag{2.79}$$

where (cf. (2.9))

$$F_{2n+1}(z,x,t_r) = \sum_{\ell=0}^{n} f_{n-\ell}(x,t_r)z^\ell,$$

$$F_{2n+1}(z,x,t_{0,r}) = F_{2n+1}^{(0)}(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}^{(0)}(x)z^\ell \tag{2.80}$$

for fixed $t_{0,r}\in\mathbb{R}$, $n\in\mathbb{N}_0$, $r\in 2\,\mathbb{N}_0+1$. Here $f_\ell(x,t_r)$ and $f_\ell^{(0)}(x)$ are defined as in (2.2) with $q_0(x)$ replaced by $q_0(x,t_r)$ and $q_0^{(0)}(x)$, respectively.

In analogy to (2.45), (2.46), (2.50), and (2.51), we introduce

$$D_n(z,x,t_r) = F_{2n+1}(z,x,t_r) = \prod_{j=1}^{n}(z-\mu_j(x,t_r)), \tag{2.81}$$

$$\begin{aligned} N_{n+1}(z,x,t_r) &= \big(z-q_0(x,t_r)\big)F_{2n+1}(z,x,t_r) - \frac{1}{2}F_{2n+1,xx}(z,x,t_r) \\ &= \prod_{\ell=0}^{n}(z-\nu_\ell(x,t_r)), \end{aligned} \tag{2.82}$$

such that

$$D_n(z,x,t_r)\,N_{n+1}(z,x,t_r) = R_{2n+1}(z) - \frac{1}{4}D_{n,x}(z,x,t_r)^2. \tag{2.83}$$

Hence we can define, in analogy to (2.47) and (2.48), the following meromorphic function $\phi(P,x,t_r)$ on $\mathcal{K}_n$ the fundamental ingredient for the construction of algebro-geometric solutions of the time-dependent KdV hierarchy,

$$\phi(P,x,t_r) = \frac{y(P) - k_{2n+1}(z) + \frac{1}{2}D_{n,x}(z,x,t_r)}{D_n(z,x,t_r)} \tag{2.84}$$

$$= \frac{N_{n+1}(z,x,t_r)}{y(P) - k_{2n+1}(z) - \frac{1}{2}D_{n,x}(z,x,t_r)}, \quad P=(z,y(P))\in\mathcal{K}_n \tag{2.85}$$

As in (2.52) and (2.53) one introduces Dirichlet and Neumann data by

$$\begin{aligned} \hat{\mu}_j(x,t_r) &= \big(\mu_j(x,t_r), y(\hat{\mu}_j(x,t_r))\big) \\ &= \big(\mu_j(x,t_r), k_{2n+1}(\mu_j(x,t_r)) + \frac{1}{2}D_{n,x}(\mu_j(x,t_r),x,t_r)\big)\in\mathcal{K}_n, \\ &\qquad 1\le j\le n, \quad (x,t_r)\in\mathbb{R}^2, \end{aligned} \tag{2.86}$$

$$\begin{aligned} \hat{\nu}_\ell(x,t_r) &= \big(\nu_\ell(x,t_r), y(\hat{\nu}_\ell(x,t_r))\big) \\ &= \big(\nu_\ell(x,t_r), k_{2n+1}(\nu_\ell(x,t_r)) - \frac{1}{2}D_{n,x}(\nu_\ell(x,t_r),x,t_r)\big)\in\mathcal{K}_n, \\ &\qquad 0\le \ell\le n, \quad (x,t_r)\in\mathbb{R}^2, \end{aligned} \tag{2.87}$$

and infers that the divisor $\big(\phi(P,x,t_r)\big)$ of $\phi(P,x,t_r)$ is given by

$$\big(\phi(P,x,t_r)\big) = \mathcal{D}_{\hat{\nu}_0(x,t_r),\dots,\hat{\nu}_n(x,t_r)}(P) - \mathcal{D}_{P_\infty,\hat{\mu}_1(x,t_r),\dots,\hat{\mu}_n(x,t_r)}(P). \tag{2.88}$$

Next we define the time-dependent BA-function $\psi(P,x,x_0,t_r,t_{0,r})$ by

$$\psi(P,x,x_0,t_r,t_{0,r}) = \exp\Big\{\int_{x_0}^{x} d\,x'\phi(P,x',t_r) + \int_{t_{0,r}}^{t_r} d\,s\big(\tilde{F}_r(z,x_0,s)\phi(P,x_0,s) \\ -\frac{1}{2}\tilde{F}_{r,x}(z,x_0,s) + \tilde{k}_r(z)\big)\Big\}, \quad P\in\mathcal{K}_n\setminus\{P_\infty\}, \quad (x,t_r)\in\mathbb{R}^2, \tag{2.89}$$

with fixed $(x_0,t_{0,r})\in\mathbb{R}^2$. The following lemma records some properties of $\phi(P,x,t_r)$ and $\psi(P,x,x_0,t_r,t_{0,r})$ (see [**40**] for the original result).

Lemma 2.7. *Assume (2.78)–(2.82), $P=(z,y(P))\in\mathcal{K}_n\setminus\{P_\infty\}$, and let $(z,x,x_0,t_r,t_{0,r})\in\mathbb{C}\times\mathbb{R}^4$. Then*

(*i*). *$\phi(P,x,t_r)$ satisfies*

$$\phi_x(P,x,t_r) + \phi(P,x,t_r)^2 = z - q_0(x,t_r), \tag{2.90}$$

$$\phi_{t_r}(P,x,t_r) = \partial_x\big(\tilde{F}_r(z,x,t_r)\phi(P,x,t_r) - \frac{1}{2}\tilde{F}_{r,x}(z,x,t_r) + \tilde{k}_r(z)\big). \tag{2.91}$$

(*ii*). *$\psi(P,x,x_0,t_r,t_{0,r})$ satisfies*

$$\psi_{xx}(P,x,x_0,t_r,t_{0,r}) + (q_0(x,t_r)-z)\psi(P,x,x_0,t_r,t_{0,r}) = 0, \tag{2.92}$$

$$\psi_{t_r}(P,x,x_0,t_r,t_{0,r}) = \big(\tilde{F}_r(z,x,t_r)\phi(P,x,t_r) - \frac{1}{2}\tilde{F}_{r,x}(z,x,t_r) \\ + \tilde{k}_r(z)\big)\psi(P,x,x_0,t_r,t_{0,r}) \tag{2.93}$$

(*i.e.*, $(L_2-z)\psi=0$, $(P_{2n+1}-y)\psi=0$, $\psi_{t_r}=\tilde{P}_r\psi$).

(*iii*). $$\phi(P,x,t_r)\,\phi(P^*,x,t_r) = -\frac{N_{n+1}(z,x,t_r)}{D_n(z,x,t_r)}. \tag{2.94}$$

(*iv*). $$\phi(P,x,t_r) + \phi(P^*,x,t_r) = \frac{D_{n,x}(z,x,t_r)}{D_n(z,x,t_r)}. \tag{2.95}$$

(*v*). $$\phi(P,x,t_r) - \phi(P^*,x,t_r) = \frac{2\,(y(P)-k_{2n+1}(z))}{D_n(z,x,t_r)}, \tag{2.96}$$

$$\big(y(P)-k_{2n+1}(z)\big)\phi(P,x,t_r) + \big(y(P^*)-k_{2n+1}(z)\big)\phi(P^*,x,t_r) = \frac{2\,R_{2n+1}(z)}{D_n(z,x,t_r)}. \tag{2.97}$$

Proof. (i). (2.90) follows from (2.79), and (2.84). In order to prove (2.91) one first derives from (2.78), (2.79) and (2.84) that

$$(\partial_x + 2\,\phi)\big(\phi_{t_r} - \partial_x\big(\tilde{F}_r\phi - \frac{1}{2}\tilde{F}_{r,x} + \tilde{k}_r(z)\big)\big) = 0.$$

Thus

$$\phi_{t_r} - \partial_x\big(\tilde{F}_r\phi - \frac{1}{2}\tilde{F}_{r,x} + \tilde{k}_r(z)\big) = Ce^{-\int^x d\,x'2\phi}, \tag{2.98}$$

where C is independent of x (but may depend on P and t_r). The high-energy behavior $\phi(P,x,t_r) \underset{|z|\to\infty}{=} O(|z|^{1/2})$ (cf. (2.84)) then proves $C=0$ since the left-hand side of (2.98) is

meromorphic on $\mathcal{K}_n$ (and hence especially near P_∞).
(ii). (2.92) is clear from (2.89) ($\phi = \psi_x/\psi$) and (2.90). (2.93) follows from (2.89) and (2.91).
(iii)–(v) follow as in Lemma 2.4 (ii)–(iv). □

Next we introduce

$$\tilde{N}_{r+1}(z,x,t_r) = (z - q_0(x,t_r))\tilde{F}_r(z,x,t_r) - \frac{1}{2}\tilde{F}_{r,xx}(z,x,t_r) \tag{2.99}$$

and note that by (2.78),

$$\tilde{N}_{r+1,x}(z,x,t_r) = -q_0(x,t_r) - (z - q_0(x,t_r))\tilde{F}_{r,x}(z,x,t_r). \tag{2.100}$$

In analogy to (2.63) one also obtains

$$N_{n+1,x}(z,x,t_r) = -(z - q_0(x,t_r))D_{n,x}(z,x,t_r). \tag{2.101}$$

We recall (cf. [**40**]),

Lemma 2.8. *Assume (2.78)–(2.82) and let $(z,x,t_r) \in \mathbb{C} \times \mathbb{R}^2$. Then*

$$(i).\ D_{n,t_r}(z,x,t_r) = \tilde{F}_r(z,x,t_r)D_{n,x}(z,x,t_r) - \tilde{F}_{r,x}(z,x,t_r)D_n(z,x,t_r). \tag{2.102}$$

$$(ii).\ D_{n,x\,t_r}(z,x,t_r) = 2\big(\tilde{N}_{r+1}(z,x,t_r)D_n(z,x,t_r) - N_{n+1}(z,x,t_r)\tilde{F}_r(z,x,t_r)\big). \tag{2.103}$$

$$(iii).\ N_{n+1,t_r}(z,x,t_r) = \tilde{F}_{r,x}(z,x,t_r)N_{n+1}(z,x,t_r) - D_{n,x}(z,x,t_r)\tilde{N}_{r+1}(z,x,t_r). \tag{2.104}$$

Proof. In order to prove (2.102) consider $\phi_{t_r}(P) - \phi_{t_r}(P^*)$ and combine (2.84) and (2.91). (2.103) follows from (2.102) using (2.82). (2.104) is a consequence of (2.82), (2.102), and (2.103). □

The remaining analogs of Lemma 2.4 (v)–(vii) then read (cf. again [**40**])

Lemma 2.9. *Assume (2.78)–(2.82), $P = (z, y(P)) \in \mathcal{K}_n \setminus \{P_\infty\}$, and let $(z,x,x_0,t_r,t_{0,r}) \in \mathbb{C} \times \mathbb{R}^4$. Then*

$$(i).\ \psi(P,x,x_0,t_r,t_{0,r})\psi(P^*,x,x_0,t_r,t_{0,r}) = e^{2\,\tilde{k}_r(z)(t_r - t_{0,r})}\frac{D_n(z,x,t_r)}{D_n(z,x_0,t_{0,r})}. \tag{2.105}$$

$$(ii).\ \psi_x(P,x,x_0,t_r,t_{0,r})\psi_x(P^*,x,x_0,t_r,t_{0,r}) = -e^{2\,\tilde{k}_r(z)(t_r - t_{0,r})}\frac{N_{n+1}(z,x,t_r)}{D_n(z,x_0,t_{0,r})}. \tag{2.106}$$

$$\begin{aligned}(iii).\ \psi(P,x,x_0,t_r,t_{0,r}) &= \left(\frac{D_n(z,x,t_r)}{D_n(z,x_0,t_{0,r})}\right)^{1/2} \exp\bigg\{\tilde{k}_r(z)(t_r - t_{0,r}) \\ &\quad + \big(y(P) - k_{2n+1}(z)\big)\Big(\int_{x_0}^{x} dx' \frac{1}{D_n(z,x',t_r)} + \int_{t_{0,r}}^{t_r} ds\,\frac{\tilde{F}_r(z,x_0,s)}{D_n(z,x_0,s)}\Big)\bigg\}.\end{aligned} \tag{2.107}$$

Proof. (2.105) follows from (2.89), (2.95) and (2.102). (2.106) is clear from $\psi_x(P)\,\psi_x(P^*) = \phi(P)\,\phi(P^*)\,\psi(P)\,\psi(P^*)$, (2.94), and (2.105). (2.107) finally is a consequence of (2.89), (2.95), (2.96), and (2.102) by splitting up $\phi(P) = \frac{1}{2}(\phi(P) + \phi(P^*)) + \frac{1}{2}(\phi(P) - \phi(P^*))$ in (2.89). □

The dynamics of the zeros $\mu_j(x,t_r)$ and $\nu_\ell(x,t_r)$ of $D_n(z,x,t_r)$ and $N_{n+1}(z,x,t_r)$, in analogy to Lemma 2.5, are then described by Dubrovin's equations as follows.

Lemma 2.10. *Assume (2.78)–(2.82) and let* $(x,t_r)\in\mathbb{R}^2$*. Then*

$$(i).\quad \mu_{j,x}(x,t_r)=\frac{-2\big(y(\hat\mu_j(x,t_r))-k_{2n+1}(\mu_j(x,t_r))\big)}{\prod_{\substack{k=1\\k\neq j}}^{n}\big(\mu_j(x,t_r)-\mu_k(x,t_r)\big)},\qquad 1\le j\le n, \tag{2.108}$$

$$\mu_{j,t_r}(x,t_r)=\frac{-2\,\tilde F_r(\mu_j(x,t_r),x,t_r)\big(y(\hat\mu_j(x,t_r))-k_{2n+1}(\mu_j(x,t_r))\big)}{\prod_{\substack{k=1\\k\neq j}}^{n}\big(\mu_j(x,t_r)-\mu_k(x,t_r)\big)},\quad 1\le j\le n. \tag{2.109}$$

$$(ii).\quad \nu_{\ell,x}(x,t_r)=\frac{-2\big(\nu_\ell(x,t_r)-q_0(x,t_r)\big)\big(y(\hat\nu_\ell(x,t_r))-k_{2n+1}(\nu_\ell(x,t_r))\big)}{\prod_{\substack{m=0\\m\neq \ell}}^{n}\big(\nu_\ell(x,t_r)-\nu_m(x,t_r)\big)},\quad 0\le\ell\le n, \tag{2.110}$$

$$\nu_{\ell,t_r}(x,t_r)=\frac{-2\,\tilde N_r(\nu_\ell(x,t_r),x,t_r)\big(y(\hat\nu_\ell(x,t_r))-k_{2n+1}(\nu_\ell(x,t_r))\big)}{\prod_{\substack{m=0\\m\neq \ell}}^{n}\big(\nu_\ell(x,t_r)-\nu_m(x,t_r)\big)},\qquad 0\le\ell\le n. \tag{2.111}$$

Proof. (2.108) and (2.110) are completely analogous to (2.67) and (2.68). (2.109) (respectively, (2.111)) follows from (2.81), (2.86), and (2.102) (respectively, (2.82), (2.87), and (2.104)). □

The initial condition

$$q_0(x,t_{0,r})=q_0^{(0)}(x),\quad x\in\mathbb{R} \tag{2.112}$$

in (2.73) is taken care of by

$$\hat\mu_j(x,t_{0,r})=\hat\mu_j^{(0)}(x),\quad 1\le j\le n,\quad x\in\mathbb{R} \tag{2.113}$$

(cf. (2.80) and (2.81)). There is, of course, an analogous condition

$$\hat\nu_\ell(x,t_{0,r})=\hat\nu_\ell^{(0)}(x),\quad 0\le\ell\le n,\quad x\in\mathbb{R}. \tag{2.114}$$

Finally, the trace relations in Lemma 2.6 extend in a one-to-one manner to the present time-dependent setting by substituting,

$$\begin{aligned}&q_0(x)\to q_0(x,t_r),\\ &\mu_j(x)\to\mu_j(x,t_r),\ \ 1\le j\le n,\qquad \nu_\ell(x)\to\nu_\ell(x,t_r),\ \ 0\le\ell\le n,\end{aligned} \tag{2.115}$$

keeping $\{c_\ell\}_{1\le\ell\le n}$ as in (2.71) since $\mathcal{K}_n$ is t_r-independent.

Chapter 3

The Recursive Approach to the Boussinesq Hierarchy and Algebraic Curves

3.1. The Recursive Approach to the Bsq Hierarchy

In analogy to the KdV case in Section 2.1 we now develop the recursive approach to the Bsq hierarchy. These results are new.

Suppose q_0, q_1 are meromorphic on $\mathbb{C}$ and introduce the third-order differential expression

$$L_3 = \frac{d^3}{dx^3} + q_1 \frac{d}{dx} + \frac{1}{2}\, q_{1,x} + q_0, \quad x \in \mathbb{C}. \tag{3.1}$$

(For computational reasons we found L_3 as in (3.1) more convenient to work with than its alternative $\tilde{L}_3 = \frac{d^3}{dx^3} + q_1 \frac{d}{dx} + q_0$.)

For each fixed $m \in \mathbb{N}_0$ $(= \mathbb{N} \cup \{0\})$ with $m \not\equiv 0 \pmod 3$) we write

$$m = 3n + \varepsilon, \quad \varepsilon \in \{1, 2\}, \tag{3.2}$$

and then construct two distinct differential expressions of order $3n+1$ and $3n+2$, respectively, denoted by P_m, where $m = 3n+1$ or $m = 3n+2$. In order for these differential expressions P_m to commute with L_3, one proceeds as follows.

Pick $n \in \mathbb{N}_0$, $\varepsilon \in \{1, 2\}$, and define the sequences $\{f_\ell^{(\varepsilon)}(x)\}_{\ell=0,\dots,n+1}$ and $\{g_\ell^{(\varepsilon)}(x)\}_{\ell=0,\dots,n+1}$ recursively by

$$(f_0^{(\varepsilon)}, g_0^{(\varepsilon)}) = (c_0^{(\varepsilon)}, d_0^{(\varepsilon)}) = \begin{cases} (0,1) & \text{for } \varepsilon = 1, \\ (1, d_0^{(2)}) & \text{for } \varepsilon = 2, \end{cases} \qquad d_0^{(2)} \in \mathbb{C},$$

$$3 f_{\ell,x}^{(\varepsilon)} = 2 g_{\ell-1,xxx}^{(\varepsilon)} + 2 q_1 g_{\ell-1,x}^{(\varepsilon)} + q_{1,x} g_{\ell-1}^{(\varepsilon)} + 3 q_0 f_{\ell-1,x}^{(\varepsilon)} + 2 q_{0,x} f_{\ell-1}^{(\varepsilon)}, \tag{3.3}$$

$$3g^{(\varepsilon)}_{\ell,x} = 3q_0 g^{(\varepsilon)}_{\ell-1,x} + q_{0,x} g^{(\varepsilon)}_{\ell-1} - \frac{1}{6} f^{(\varepsilon)}_{\ell-1,xxxxx} - \frac{5}{6} q_1 f^{(\varepsilon)}_{\ell-1,xxx} - \frac{5}{4} q_{1,x} f^{(\varepsilon)}_{\ell-1,xx}$$
$$- \big(\frac{3}{4} q_{1,xx} + \frac{2}{3} q_1^2\big) f^{(\varepsilon)}_{\ell-1,x} - \big(\frac{1}{6} q_{1,xxx} + \frac{2}{3} q_1 q_{1,x}\big) f^{(\varepsilon)}_{\ell-1}, \quad \ell = 1, \dots, n+1.$$

However, as most of the ensuing discussion can be made for both cases simultaneously, we write

$$f_\ell = f^{(\varepsilon)}_\ell, \qquad g_\ell = g^{(\varepsilon)}_\ell, \tag{3.4}$$

and only make the distinction explicit when necessary.

Explicitly, one computes

(i) Let $m \equiv 1 \pmod 3$ (i.e., $\varepsilon = 1$):

$$\begin{aligned}
f^{(1)}_0 &= 0, \qquad g^{(1)}_0 = 1,\\
3f^{(1)}_1 &= q_1 + 3c^{(1)}_1, \qquad 3g^{(1)}_1 = q_0 + 3d^{(1)}_1,\\
3f^{(1)}_2 &= \frac{2}{3} q_{0,xx} + \frac{4}{3} q_0 q_1 + c^{(1)}_1 2q_0 + d^{(1)}_1 q_1 + 3c^{(1)}_2,\\
3g^{(1)}_2 &= -\frac{1}{18} q_{1,xxxx} - \frac{1}{6} q_{1,x}^2 - \frac{4}{27} q_1^3 - \frac{1}{3} q_1 q_{1,xx} + \frac{2}{3} q_0^2\\
&\quad + c^{(1)}_1 \big(-\frac{1}{6} q_{1,xx} - \frac{1}{3} q_1^2\big) + d^{(1)}_1 q_0 + 3d^{(1)}_2,\\
3f^{(1)}_3 &= -\frac{1}{27} q_{1,xxxxxx} - \frac{7}{27} q_1 q_{1,xxxx} - \frac{35}{54} q_{1,x} q_{1,xxx} - \frac{49}{108} q_{1,xx}^2 - \frac{14}{27} q_1^2 q_{1,xx}\\
&\quad - \frac{35}{54} q_1 q_{1,x}^2 - \frac{7}{81} q_1^4 + \frac{14}{9} q_0 q_{0,xx} + \frac{7}{9} q_{0,x}^2 + \frac{14}{9} q_0^2 q_1\\
&\quad + c^{(1)}_1 \big(-\frac{1}{9} q_{1,xxxx} - \frac{5}{9} q_1 q_{1,xx} - \frac{5}{12} q_{1,x}^2 - \frac{5}{27} q_1^3 + \frac{5}{3} q_0^2\big)\\
&\quad + d^{(1)}_1 \big(\frac{2}{3} q_{0,xx} + \frac{4}{3} q_0 q_1\big) + c^{(1)}_2 2q_0 + d^{(1)}_2 q_1 + 3c^{(1)}_3,\\
3g^{(1)}_3 &= \frac{14}{27} q_0^3 - \frac{28}{81} q_0 q_1^3 - \frac{7}{9} q_{0,x} q_1 q_{1,x} - \frac{7}{18} q_0 q_{1,x}^2 - \frac{14}{27} q_1^2 q_{0,xx}\\
&\quad - \frac{7}{9} q_0 q_1 q_{1,xx} - \frac{14}{27} q_{0,xx} q_{1,xx} - \frac{7}{18} q_{1,x} q_{0,xxx} - \frac{7}{27} q_{0,x} q_{1,xxx}\\
&\quad - \frac{7}{27} q_1 q_{0,xxxx} - \frac{7}{54} q_0 q_{1,xxxx} - \frac{1}{27} q_{0,xxxxxx}\\
&\quad + c^{(1)}_1 \big(-\frac{1}{9} q_{0,xxxx} - \frac{5}{18} q_0 q_{1,xx} - \frac{5}{9} q_1 q_{0,xx} - \frac{5}{18} q_{1,x} q_{0,x} - \frac{5}{9} q_0 q_1^2\big)\\
&\quad + d^{(1)}_1 \big(-\frac{1}{18} q_{1,xxxx} - \frac{1}{6} q_{1,x}^2 - \frac{4}{27} q_1^3 - \frac{1}{3} q_1 q_{1,xx} + \frac{2}{3} q_0^2\big)\\
&\quad + c^{(1)}_2 \big(-\frac{1}{6} q_{1,xx} - \frac{1}{3} q_1^2\big) + d^{(1)}_2 q_0 + 3d^{(1)}_3,
\end{aligned} \tag{3.5}$$

etc.

(ii) Let $m \equiv 2 \pmod 3$ (i.e., $\varepsilon = 2$):

$$\begin{aligned}
f^{(2)}_0 &= 1, \qquad g^{(2)}_0 = d^{(2)}_0,\\
3f^{(2)}_1 &= 2q_0 + d^{(2)}_0 q_1 + 3c^{(2)}_1, \qquad 3g^{(2)}_1 = -\frac{1}{6} q_{1,xx} - \frac{1}{3} q_1^2 + d^{(2)}_0 q_0 + 3d^{(2)}_1,
\end{aligned}$$

$$
\begin{aligned}
3f_2^{(2)} &= \big(-\frac{1}{9}\, q_{1,xxxx} - \frac{5}{9}\, q_1 q_{1,xx} - \frac{5}{27}\, q_1^3 - \frac{5}{12}\, q_{1,x}^2 + \frac{5}{3}\, q_0^2 \big) \\
&\quad + d_0^{(2)} \big(\frac{2}{3}\, q_{0,xx} + \frac{4}{3}\, q_0 q_1 \big) + c_1^{(2)}\, 2q_0 + d_1^{(2)}\, q_1 + 3c_2^{(2)}, \\
3g_2^{(2)} &= \big(-\frac{1}{9}\, q_{0,xxxx} - \frac{5}{9}\, q_1^2 q_0 - \frac{5}{18}\, q_0 q_{1,xx} - \frac{5}{9}\, q_1 q_{0,xx} - \frac{5}{18}\, q_{0,x} q_{1,x} \big) \\
&\quad + d_0^{(2)} \big(-\frac{1}{18}\, q_{1,xxxx} - \frac{1}{6}\, q_{1,x}^2 - \frac{4}{27}\, q_1^3 - \frac{1}{3}\, q_1 q_{1,xx} + \frac{2}{3}\, q_0^2 \big) \\
&\quad + c_1^{(2)} \big(-\frac{1}{6}\, q_{1,xx} - \frac{1}{3}\, q_1^2 \big) + d_1^{(2)}\, q_0 + 3d_2^{(2)}, \qquad (3.6) \\
3f_3^{(2)} &= \frac{40}{27}\, q_0^3 - \frac{40}{81}\, q_0 q_1^3 - \frac{10}{9}\, q_0 q_{1,x}^2 - \frac{20}{27}\, q_1^2 q_{0,xx} - \frac{40}{27}\, q_0 q_1 q_{1,xx} \\
&\quad - \frac{26}{27}\, q_{0,xx} q_{1,xx} - \frac{40}{27}\, q_1 q_{1,x} q_{0,x} - \frac{8}{9}\, q_{1,x} q_{0,xxx} - \frac{14}{27}\, q_{0,x} q_{1,xxx} \\
&\quad - \frac{4}{9}\, q_1 q_{0,xxxx} - \frac{8}{27}\, q_0 q_{1,xxxx} - \frac{2}{27}\, q_{0,xxxxxx} \\
&\quad + d_0^{(2)} \big(-\frac{1}{27}\, q_{1,xxxxxx} - \frac{7}{24}\, q_1 q_{1,xxxx} - \frac{35}{54}\, q_{1,x} q_{1,xxx} - \frac{49}{108}\, q_{1,xx}^2 \\
&\quad - \frac{14}{27}\, q_1^2 q_{1,xx} - \frac{35}{54}\, q_1 q_{1,x}^2 - \frac{7}{81}\, q_1^4 + \frac{14}{9}\, q_0 q_{0,xx} + \frac{7}{9}\, q_{0,x}^2 + \frac{14}{9}\, q_0^2 q_1 \big) \\
&\quad + c_1^{(2)} \big(-\frac{1}{9}\, q_{1,xxxx} - \frac{5}{9}\, q_1 q_{1,xx} - \frac{5}{12}\, q_{1,x}^2 - \frac{5}{27}\, q_1^3 + \frac{5}{3}\, q_0^2 \big) \\
&\quad + d_1^{(2)} \big(\frac{2}{3}\, q_{0,xx} + \frac{4}{3}\, q_0 q_1 \big) + c_2^{(2)}\, 2q_0 + d_2^{(2)}\, q_1 + 3c_3^{(2)}, \\
3g_3^{(2)} &= \frac{8}{243}\, q_1^5 - \frac{20}{27}\, q_0^2 q_1^2 - \frac{20}{27}\, q_{0,x}^2 q_1 - \frac{20}{27}\, q_0 q_{0,x} q_{1,x} + \frac{35}{81}\, q_1^2 q_{1,x}^2 \\
&\quad - \frac{40}{27}\, q_0 q_1 q_{0,xx} - \frac{16}{27}\, q_{0,xx}^2 + \frac{70}{243}\, q_1^3 q_{1,xx} + \frac{11}{18}\, q_{1,x}^2 q_{1,xx} \\
&\quad + \frac{17}{27}\, q_1 q_{1,xx}^2 - \frac{2}{3}\, q_{0,x} q_{0,xxx} + \frac{68}{81}\, q_1 q_{1,x} q_{1,xxx} + \frac{7}{27}\, q_{1,xxx}^2 \\
&\quad - \frac{10}{27}\, q_0^2 q_{1,xx} - \frac{8}{27}\, q_0 q_{0,xxxx} + \frac{17}{81}\, q_1^2 q_{1,xxxx} + \frac{67}{162}\, q_{1,xx} q_{1,xxxx} \\
&\quad + \frac{5}{27}\, q_{1,x} q_{1,xxxxx} + \frac{5}{81}\, q_1 q_{1,xxxxxx} + \frac{1}{162}\, q_{1,xxxxxxxx} \\
&\quad + d_0^{(2)} \big(\frac{14}{27}\, q_0^3 - \frac{28}{81}\, q_0 q_1^3 - \frac{7}{9}\, q_{0,x} q_1 q_{1,x} - \frac{7}{18}\, q_0 q_{1,x}^2 - \frac{14}{27}\, q_1^2 q_{0,xx} \\
&\quad - \frac{7}{9}\, q_0 q_1 q_{1,xx} - \frac{14}{27}\, q_{0,xx} q_{1,xx} - \frac{7}{18}\, q_{1,x} q_{0,xxx} - \frac{7}{27}\, q_{0,x} q_{1,xxx} \\
&\quad - \frac{7}{27}\, q_1 q_{0,xxxx} - \frac{7}{54}\, q_0 q_{1,xxxx} - \frac{1}{27}\, q_{0,xxxxxx} \big) \\
&\quad + c_1^{(2)} \big(-\frac{1}{9}\, q_{0,xxxx} - \frac{5}{18}\, q_0 q_{1,xx} - \frac{5}{9}\, q_1 q_{0,xx} - \frac{5}{18}\, q_{1,x} q_{0,x} - \frac{5}{9}\, q_0 q_1^2 \big) \\
&\quad + d_1^{(2)} \big(-\frac{1}{18}\, q_{1,xxxx} - \frac{1}{6}\, q_{1,x}^2 - \frac{4}{27}\, q_1^3 - \frac{1}{3}\, q_1 q_{1,xx} + \frac{2}{3}\, q_0^2 \big) \\
&\quad + c_2^{(2)} \big(-\frac{1}{6} q_{1,xx} - \frac{1}{3}\, q_1^2 \big) + d_2^{(2)} q_0 + 3d_3^{(2)},
\end{aligned}
$$

etc.,

where $\{c_\ell^{(\varepsilon)}\}_{\ell=1,\dots,n}$, $\{d_\ell^{(\varepsilon)}\}_{\ell=0,\dots,n}$ are integration constants, which arise when solving (3.3). It is convenient to introduce the homogeneous case where all free integration constants vanish. We introduce

$$\hat{f}_\ell^{(\varepsilon)} = f_\ell^{(\varepsilon)} \,|_{c_p^{(\varepsilon)}=d_p^{(\varepsilon)}=0,\, p=1,\dots,\ell}, \qquad \hat{g}_\ell^{(\varepsilon)} = g_\ell^{(\varepsilon)} \,|_{c_p^{(\varepsilon)}=d_p^{(\varepsilon)}=0,\, p=1,\dots,\ell} \tag{3.7}$$

and use (cf. (3.3))

$$c_0^{(1)} = 0, \quad c_0^{(2)} = 1, \quad d_0^{(1)} = 1, \quad d_0^{(2)} = 0. \tag{3.8}$$

We do not list these functions explicitly, however, this notation allows us to write

$$f_\ell^{(\varepsilon)} = \sum_{p=0}^{\ell} (d_p^{(\varepsilon)} \hat{f}_{\ell-p}^{(1)} + c_p^{(\varepsilon)} \hat{f}_{\ell-p}^{(2)}), \qquad g_\ell^{(\varepsilon)} = \sum_{p=0}^{\ell} (d_p^{(\varepsilon)} \hat{g}_{\ell-p}^{(1)} + c_p^{(\varepsilon)} \hat{g}_{\ell-p}^{(2)}). \tag{3.9}$$

Given (3.3) one defines the differential expression P_m of order m by

$$\begin{aligned} P_m = \sum_{\ell=0}^{n} \Big(f_{n-\ell}^{(\varepsilon)} \frac{d^2}{dx^2} + \big(g_{n-\ell}^{(\varepsilon)} - \frac{1}{2} f_{n-\ell,x}^{(\varepsilon)}\big) \frac{d}{dx} \\ + \big(\frac{1}{6} f_{n-\ell,xx}^{(\varepsilon)} - g_{n-\ell,x}^{(\varepsilon)} + \frac{2}{3} q_1 f_{n-\ell}^{(\varepsilon)}\big)\Big) L_3^\ell + \sum_{\ell=0}^{n} k_{m,\ell} L_3^\ell, \\ k_{m,\ell} \in \mathbb{C}, \quad \ell = 0,\dots,n, \quad m = 3n+\varepsilon,\, \varepsilon \in \{1,2\},\, n \in \mathbb{N}_0, \end{aligned} \tag{3.10}$$

and verifies that

$$\begin{aligned} [P_m, L_3] = 3\, f_{n+1,x}^{(\varepsilon)} \frac{d}{dx} + \frac{3}{2} f_{n+1,xx}^{(\varepsilon)} + 3\, g_{n+1,x}^{(\varepsilon)}, \\ m = 3n+\varepsilon,\, \varepsilon \in \{1,2\},\, n \in \mathbb{N}_0 \end{aligned} \tag{3.11}$$

(where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator symbol). The pair (L_3, P_m) represents the Lax pair for the Bsq hierarchy. Varying $n \in \mathbb{N}_0$ and $\varepsilon \in \{1,2\}$, the stationary Bsq hierarchy is then defined by the vanishing of the commutator of P_m and L_3 in (3.11), that is, by

$$[P_m, L_3] = 0, \qquad m = 3n+\varepsilon,\, \varepsilon \in \{1,2\},\, n \in \mathbb{N}_0, \tag{3.12}$$

or equivalently, by

$$f_{n+1,x}^{(\varepsilon)} = 0, \quad g_{n+1,x}^{(\varepsilon)} = 0, \qquad \varepsilon \in \{1,2\},\, n \in \mathbb{N}_0. \tag{3.13}$$

Explicitly, one obtains for the first few equations of the stationary Boussinesq hierarchy,

$$\begin{aligned} & m = 1 \text{ (i.e., } n = 0 \text{ and } \varepsilon = 1) : \\ & \quad q_{0,x} = 0, \quad q_{1,x} = 0. \\ & m = 2 \text{ (i.e., } n = 0 \text{ and } \varepsilon = 2) : \\ & \quad -\frac{1}{6} q_{1,xxx} - \frac{2}{3} q_1 q_{1,x} + d_0^{(2)} q_{0,x} = 0, \quad 2\, q_{0,x} + d_0^{(2)} q_{1,x} = 0. \\ & m = 4 \text{ (i.e., } n = 1 \text{ and } \varepsilon = 1) : \\ & \quad -\frac{1}{18} q_{1,xxxxx} - \frac{1}{3} q_1 q_{1,xxx} - \frac{2}{3} q_{1,x} q_{1,xx} - \frac{4}{9} q_1^2 q_{1,x} + \frac{4}{3} q_0 q_{0,x} \\ & \quad + c_1^{(1)} \big(-\frac{1}{6} q_{1,xxx} - \frac{2}{3} q_1 q_{1,x} \big) + d_1^{(1)} q_{0,x} = 0, \\ & \quad \frac{2}{3} q_{0,xxx} + \frac{4}{3} q_1 q_{0,x} + \frac{4}{3} q_{1,x} q_0 + c_1^{(1)} 2 q_{0,x} + d_1^{(1)} q_{1,x} = 0, \end{aligned} \tag{3.14}$$

$m = 5$ (i.e., $n = 1$ and $\varepsilon = 2$) :

$$\frac{1}{9}\,q_{0,xxxxx} + \frac{5}{18}\,q_0 q_{1,xxx} + \frac{5}{9}\,q_1 q_{0,xxx} + \frac{5}{9}\,q_{1,xx} q_{0,x} + \frac{5}{6}\,q_{1,x} q_{0,xx} + \frac{5}{9}\,q_1^2 q_{0,x} + \frac{10}{9}\,q_0 q_1 q_{1,x} + d_0^{(2)}\big(\frac{1}{18}\,q_{1,xxxxx} + \frac{1}{3}\,q_1 q_{1,xxx} + \frac{2}{3}\,q_{1,x} q_{1,xx} + \frac{4}{9}\,q_1^2 q_{1,x} - \frac{4}{3}\,q_0 q_{0,x}\big) + c_1^{(2)}\,\big(\frac{1}{6}\,q_{1,xxx} + \frac{2}{3}\,q_1 q_{1,x}\big) - d_1^{(2)} q_{0,x} = 0,$$

$$-\frac{1}{9}\,q_{1,xxxxx} - \frac{5}{9}\,q_1 q_{1,xxx} - \frac{25}{18}\,q_{1,x} q_{1,xx} - \frac{5}{9}\,q_1^2 q_{1,x} + \frac{10}{3}\,q_0 q_{0,x} + d_0^{(2)}\,\big(\frac{2}{3}\,q_{0,xxx} + \frac{4}{3}\,q_1 q_{0,x} + \frac{4}{3}\,q_{1,x} q_0\big) + c_1^{(2)}\,2q_{0,x} + d_1^{(2)}\,q_{1,x} = 0.$$

etc.

By definition, solutions (q_0, q_1) of any of the stationary Bsq equations (3.14) are called **stationary algebro-geometric Bsq solutions** or simply **algebro-geometric Bsq potentials**.

Next, we introduce two polynomials F_m and G_m, both of degree at most n with respect to the variable $z \in \mathbb{C}$,

$$F_m(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}^{(\varepsilon)}(x) z^\ell, \tag{3.15}$$

$$G_m(z,x) = \sum_{\ell=0}^{n} g_{n-\ell}^{(\varepsilon)}(x) z^\ell, \quad m = 3n+\varepsilon,\, \varepsilon \in \{1,2\},\, n \in \mathbb{N}_0. \tag{3.16}$$

In terms of homogeneous quantities we define (cf. (3.7) and (3.8))

$$\widehat{F}_\ell = F_\ell \,\big|_{c_p^{(\varepsilon)} = d_p^{(\varepsilon)} = 0,\, p=1,\dots,n}, \qquad \widehat{G}_\ell = G_\ell \,\big|_{c_p^{(\varepsilon)} = d_p^{(\varepsilon)} = 0,\, p=1,\dots,n}\,. \tag{3.17}$$

We may then write

$$F_m = \sum_{j=0}^{n} (c_{n-j}^{(\varepsilon)} \widehat{F}_{3j+2} + d_{n-j}^{(\varepsilon)} \widehat{F}_{3j+1}), \qquad G_m = \sum_{j=0}^{n} (c_{n-j}^{(\varepsilon)} \widehat{G}_{3j+2} + d_{n-j}^{(\varepsilon)} \widehat{G}_{3j+1}). \tag{3.18}$$

Explicitly, the first few polynomials F_m, G_m read

$$\begin{aligned}
&F_1 = 0, \quad G_1 = 1,\\
&F_2 = 1, \quad G_2 = d_0^{(2)},\\
&F_4 = \frac{1}{3}\,q_1 + c_1^{(1)}, \quad G_4 = z + \frac{1}{3}\,q_0 + d_1^{(1)},\\
&F_5 = z + \frac{2}{3}\,q_0 + d_0^{(2)}\,\frac{1}{3}\,q_1 + c_1^{(2)}, \quad G_5 = d_0^{(2)} z - \frac{1}{18}\,q_{1,xx} - \frac{1}{9}\,q_1^2 + d_0^{(2)}\,\frac{1}{3}\,q_0 + d_1^{(2)},
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
F_7 &= z\,\big(\frac{1}{3}\,q_1 + c_1^{(1)}\big) + \frac{2}{9}\,q_{0,xx} + \frac{4}{9}\,q_0 q_1 + c_1^{(1)}\,\frac{2}{3}\,q_0 + d_1^{(1)}\,\frac{1}{3}\,q_1 + c_2^{(1)},\\
G_7 &= z^2 + z\,\big(\frac{1}{3}\,q_0 + d_1^{(1)}\big) + \frac{1}{3}\,\big(-\frac{1}{18}\,q_{1,xxxx} - \frac{1}{6}\,q_{1,x}^2 - \frac{4}{27} q_1^3\\
&\quad - \frac{1}{3}\,q_1 q_{1,xx} + \frac{2}{3}\,q_0^2\big) + c_1^{(1)}\,\frac{1}{3}\,\big(-\frac{1}{6} q_{1,xx} - \frac{1}{3}\,q_1^2\big) + d_1^{(1)}\,\frac{1}{3}\,q_0 + d_2^{(1)},
\end{aligned}$$

$$
\begin{aligned}
F_8 &= z^2 + z\big(\frac{2}{3}\,q_0 + d_0^{(2)}\,\frac{1}{3}\,q_1 + c_1^{(2)}\big)\\
&\quad + \frac{1}{3}\,\big(-\frac{1}{9}\,q_{1,xxxx} - \frac{5}{9}\,q_1q_{1,xx} - \frac{5}{27}\,q_1^3 - \frac{5}{12}\,q_{1,x}^2 + \frac{5}{3}\,q_0^2\big)\\
&\quad + d_0^{(2)}\frac{1}{3}\,\big(\frac{2}{3}\,q_{0,xx} + \frac{4}{3}\,q_0q_1\big) + c_1^{(2)}\,\frac{2}{3}\,q_0 + d_1^{(2)}\,\frac{1}{3}\,q_1 + c_2^{(2)},\\
G_8 &= d_0^{(2)}z^2 + z\big(-\frac{1}{18}\,q_{1,xx} - \frac{1}{9}\,q_1^2 + d_0^{(2)}\,\frac{1}{3}\,q_0 + d_1^{(2)}\big)\\
&\quad + \frac{1}{3}\,\big(-\frac{1}{9}\,q_{0,xxxx} - \frac{5}{9}\,q_1^2q_0 - \frac{5}{18}\,q_0q_{1,xx} - \frac{5}{9}\,q_1q_{0,xx} - \frac{5}{18}\,q_{0,x}q_{1,x}\big)\\
&\quad + d_0^{(2)}\,\frac{1}{3}\,\big(-\frac{1}{18}\,q_{1,xxxx} - \frac{1}{6}\,q_{1,x}^2 - \frac{4}{27}\,q_1^3 - \frac{1}{3}\,q_1q_{1,xx} + \frac{2}{3}\,q_0^2\big)\\
&\quad + c_1^{(2)}\,\frac{1}{3}\,\big(-\frac{1}{6}\,q_{1,xx} - \frac{1}{3}\,q_1^2\big) + d_1^{(2)}\,\frac{1}{3}\,q_0 + d_2^{(2)},
\end{aligned} \tag{3.20}
$$

etc.

Given (3.15) and (3.16), (3.12) (or equivalently, (3.13)) becomes

$$
2\,G_{m,xxx} + 2\,q_1G_{m,x} + q_{1,x}G_m - 3\,(z-q_0)F_{m,x} + 2\,q_{0,x}F_m \quad = 0, \tag{3.21}
$$

$$
\begin{aligned}
&\frac{1}{6}\,F_{m,xxxxx} + \frac{5}{6}\,q_1F_{m,xxx} + \frac{5}{4}\,q_{1,x}F_{m,xx} + \big(\frac{3}{4}\,q_{1,xx} + \frac{2}{3}\,q_1^2\big)F_{m,x}\\
&\quad +\big(\frac{1}{6}\,q_{1,xxx} + \frac{2}{3}\,q_1q_{1,x}\big)F_m + 3(z-q_0)G_{m,x} - q_{0,x}G_m \quad = 0.
\end{aligned} \tag{3.22}
$$

Multiplying (3.21) by G_m and (3.22) by F_m and taking the difference one can integrate the resulting expression to get

$$
\begin{aligned}
S_m(z) &= -\frac{1}{6}\,F_{m,xxxx}F_m + \frac{1}{6}\,F_{m,xxx}F_{m,x} - \frac{1}{12}\,F_{m,xx}^2 - \frac{5}{6}\,q_1F_{m,xx}F_m - \frac{5}{12}\,q_{1,x}F_{m,x}F_m\\
&\quad + \frac{5}{12}\,q_1F_{m,x}^2 - \frac{1}{3}\,\big(\frac{1}{2}\,q_{1,xx} + q_1^2\big)F_m^2 + 2\,G_{m,xx}G_m - G_{m,x}^2 + q_1G_m^2 - 3(z-q_0)F_mG_m,
\end{aligned} \tag{3.23}
$$

where the integration constant $S_m(z)$ is a polynomial in z of degree at most $2n-1+\varepsilon$, $m = 3n+\varepsilon$, $\varepsilon \in \{1,2\}$, $n \in \mathbb{N}_0$,

$$
S_m(z) = \sum_{p=0}^{2\,n-1+\varepsilon} s_{m,p}z^p, \quad m = 3n+\varepsilon,\, \varepsilon \in \{1,2\},\, n \in \mathbb{N}_0. \tag{3.24}
$$

Similarly, multiplying (3.22) by $\big(\frac{1}{3}\,F_{m,xx}F_m - \frac{1}{4}\,F_{m,x}^2 + \frac{1}{3}\,q_1F_m^2 + G_m^2\big)$ and (3.21) by $\big(\frac{2}{3}\,q_1F_mG_m - (z-q_0)F_m^2\big)$ and summing one can integrate the resulting expression to get the second integration constant $T_m(z)$,

$$
\begin{aligned}
T_m(z) \; &= \frac{1}{18}\,F_{m,xxxx}F_{m,xx}F_m - \frac{1}{24}\,F_{m,xxxx}F_{m,x}^2\\
&\quad + \frac{1}{36}\,F_{m,xxx}F_{m,xx}F_{m,x} - \frac{1}{108}\,F_{m,xx}^3 - \frac{1}{36}F_mF_{m,xxx}^2 + \frac{1}{18}\,q_1F_{m,xxxx}F_m^2\\
&\quad - \frac{1}{18}\,q_{1,x}F_{m,xxx}F_m^2 - \frac{1}{9}\,q_1F_{m,xxx}F_{m,x}F_m + \frac{1}{18}\,q_{1,xx}F_{m,xx}F_m^2\\
&\quad + \frac{2}{9}\,q_{1,x}F_{m,xx}F_{m,x}F_m - \frac{7}{72}\,q_1F_{m,xx}F_{m,x}^2 + \frac{7}{36}\,q_1F_{m,xx}^2F_m
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{5}{18}\, q_1^2 F_{m,xx} F_m^2 - \frac{1}{24}\, q_{1,xx} F_{m,x}^2 F_m - \frac{7}{48}\, q_{1,x} F_{m,x}^3 + \frac{1}{12}\, q_{1,x} q_1 F_{m,x} F_m^2 \\
&- \frac{1}{6}\, q_1^2 F_{m,x}^2 F_m + \big(\frac{2}{27}\, q_1^3 - \frac{1}{36}\, q_{1,x}^2 + \frac{1}{18}\, q_{1,xx} q_1 + (z-q_0)^2\big) F_m^3 \\
&+ (z-q_0) G_m^3 + \frac{1}{6}\, F_{m,xxxx} G_m^2 - \frac{1}{3}\, F_{m,xxx} G_{m,x} G_m + F_m G_{m,xx}^2 \\
&+ \frac{1}{3}\, F_{m,xx}\big(G_{m,x}^2 + G_{m,xx} G_m\big) - F_{m,x} G_{m,xx} G_{m,x} - q_1 (z-q_0) F_m^2 G_m \\
&+ \frac{2}{3}\, q_1^2 F_m G_m^2 + \frac{5}{6}\, q_1 F_{m,xx} G_m^2 - \frac{4}{3}\, q_1 F_{m,x} G_{m,x} G_m + \frac{7}{12}\, q_{1,x} F_{m,x} G_m^2 \\
&+ \frac{1}{3}\, q_1 F_m G_{m,x}^2 + \frac{4}{3}\, q_1 F_m G_{m,xx} G_m + \frac{1}{6}\, q_{1,xx} F_m G_m^2 - \frac{1}{3}\, q_{1,x} F_m G_{m,x} G_m \\
&+ (z-q_0) F_{m,x} F_m G_{m,x} - \frac{1}{4}\, (z-q_0) F_{m,x}^2 G_m - 2(z-q_0) F_m^2 G_{m,xx},
\end{aligned}
\tag{3.25}
$$

where the integration constant $T_m(z)$ is a monic polynomial of degree m,

$$
T_m(z) = z^m + \sum_{q=0}^{m-1} t_{m,q} z^q, \quad m = 3n + \varepsilon,\ \varepsilon \in \{1,2\},\ n \in \mathbb{N}_0. \tag{3.26}
$$

Next, we consider the algebraic kernel of $(L_3 - z)$, $z \in \mathbb{C}$ (i.e., the formal nullspace in a purely algebraic sense),

$$
\ker(L_3 - z) = \{\psi : \mathbb{C} \to \mathbb{C} \cup \{\infty\} \text{ meromorphic } \mid (L_3 - z)\psi = 0\}, \quad z \in \mathbb{C}. \tag{3.27}
$$

Taking into account (3.12), that is, $[P_m, L_3] = 0$, computing the restriction of P_m to $\ker(L_3\text{-}z)$, and using

$$
\psi_{xxx} = -q_1 \psi_x + \big(z - \frac{1}{2} q_{1,x} - q_0\big)\psi, \quad \text{etc.}, \tag{3.28}
$$

to eliminate higher-order derivatives of ψ, one obtains from (3.3), (3.10), (3.13), (3.15), (3.16), (3.21), and (3.22)

$$
P_m\Big|_{\ker(L_3 - z)} = \Big(F_m \frac{d^2}{dx^2} + \big(G_m - \frac{1}{2} F_{m,x}\big)\frac{d}{dx} + H_m\Big)\Big|_{\ker(L_3 - z)}. \tag{3.29}
$$

Here

$$
H_m(z,x) = \frac{1}{6}\, F_{m,xx}(z,x) + \frac{2}{3}\, q_1(x) F_m(z,x) - G_{m,x}(z,x) + k_m(z) \tag{3.30}
$$

and (cf. (3.10))

$$
k_m(z) = \sum_{\ell=0}^{n} k_{m,\ell} z^\ell \tag{3.31}
$$

is an integration constant. The presence of this constant $k_m(z)$ in (3.30), and hence in (3.29), corresponds to adding an arbitrary polynomial in L_3 to the non-trivial part of the differential expression P_m (cf. (3.10)). This polynomial in L_3 obviously commutes with L_3, and without loss of generality we henceforth choose to suppress its presence by setting $k_m(z) = 0$.

Again the reader might want to contrast our construction of P_m in (3.10) and (3.29) with the one based on formal pseudo-differential expressions in [**3**], [**19**], Ch. 1, and [**33**].

Still assuming $f^{(\varepsilon)}_{n+1,x} = g^{(\varepsilon)}_{n+1,x} = 0$ as in (3.13), $[P_m, L_3] = 0$ in (3.10) yields an algebraic relationship between P_m and L_3 by appealing to a result of Burchnall and Chaundy [**13**], [**14**] (see also [**32**], [**49**], [**83**], [**93**]). In fact, one can prove

Theorem 3.1. *Assume* $f^{(\varepsilon)}_{n+1,x} = g^{(\varepsilon)}_{n+1,x} = 0$, *that is,* $[P_m, L_3] = 0$, $m = 3n+\varepsilon$, $\varepsilon \in \{1,2\}$, $n \in \mathbb{N}_0$. *Then the Burchnall-Chaundy polynomial* $\mathcal{F}_{m-1}(L_3, P_m)$ *of the pair* (L_3, P_m) *explicitly reads* (*cf.* (3.24) *and* (3.26))

$$\begin{gathered}\mathcal{F}_{m-1}(L_3, P_m) = P_m^3 + P_m\, S_m(L_3) - T_m(L_3) = 0,\\ S_m(z) = \sum_{p=0}^{2n-1+\varepsilon} s_{m,p} z^p, \quad T_m(z) = z^m + \sum_{q=0}^{m-1} t_{m,q} z^q, \\ m = 3n+\varepsilon,\ \varepsilon \in \{1,2\},\ n \in \mathbb{N}_0.\end{gathered} \tag{3.32}$$

Proof. Let $\psi_j(x) \in \ker(L_3 - z)$, $j = 1,2,3$ be linearly independent. Since $[P_m, L_3] = 0$ one can represent P_m as a 3×3 matrix $\mathcal{P}_m$ on $\ker(L_3 - z)$,

$$P_m \psi_j = \sum_{k=1}^{3} \mathcal{P}_{m,j,k} \psi_k, \tag{3.33}$$

$$\mathcal{P}_m \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \mathcal{P}_{m,1,1} & \mathcal{P}_{m,1,2} & \mathcal{P}_{m,1,3} \\ \mathcal{P}_{m,2,1} & \mathcal{P}_{m,2,2} & \mathcal{P}_{m,2,3} \\ \mathcal{P}_{m,3,1} & \mathcal{P}_{m,3,2} & \mathcal{P}_{m,3,3} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{3.34}$$

$$\begin{aligned}\mathcal{P}_{m,1,j}(z) &= \frac{W(P_m\psi_j, \psi_2, \psi_3)}{W(\psi_1,\psi_2,\psi_3)}, \quad \mathcal{P}_{m,2,j}(z) = \frac{W(\psi_1, P_m\psi_j, \psi_3)}{W(\psi_1,\psi_2,\psi_3)},\\ \mathcal{P}_{m,3,j}(z) &= \frac{W(\psi_1, \psi_2, P_m\psi_j)}{W(\psi_1,\psi_2,\psi_3)}, \qquad 1 \le j \le 3.\end{aligned}$$

(Here $W(f,g,h)$ denotes the Wronski determinant of f, g and h.) Using (3.21), (3.22) and (3.28)–(3.31) one verifies

$$\operatorname{tr}(\mathcal{P}_m(z)) = 3\, k_m(z), \tag{3.35}$$

$$\begin{aligned}\mathrm{M}_1(\mathcal{P}_m(z)) &= \mathcal{P}_{m,1,1}\mathcal{P}_{m,2,2} + \mathcal{P}_{m,1,1}\mathcal{P}_{m,3,3} + \mathcal{P}_{m,2,2}\mathcal{P}_{m,3,3}\\ &\quad - \mathcal{P}_{m,2,3}\mathcal{P}_{m,3,2} - \mathcal{P}_{m,3,1}\mathcal{P}_{m,1,3} - \mathcal{P}_{m,1,2}\mathcal{P}_{m,2,1}\\ &= \frac{W(P_m\psi_1, P_m\psi_2, \psi_3) + W(\psi_1, P_m\psi_2, P_m\psi_3) + W(P_m\psi_1, \psi_2, P_m\psi_3)}{W(\psi_1,\psi_2,\psi_3)}\\ &= 3\, k_m(z)^2 + S_m(z),\end{aligned} \tag{3.36}$$

$$\det(\mathcal{P}_m(z)) = \frac{W(P_m\psi_1, P_m\psi_2, P_m\psi_3)}{W(\psi_1,\psi_2,\psi_3)} = k_m(z)^3 + k_m(z) S_m(z) + T_m(z). \tag{3.37}$$

The characteristic polynomial $\det(y - \mathcal{P}_m(z)) = 0$ of $\mathcal{P}_m(z)$ then yields

$$\begin{aligned}\mathcal{F}_{m-1}(z,y) &= y^3 - y^2 \operatorname{tr}(\mathcal{P}_m(z)) + y\, \mathrm{M}_1(\mathcal{P}_m(z)) - \det(\mathcal{P}_m(z))\\ &= \big(y - k_m(z)\big)^3 + \big(y - k_m(z)\big) S_m(z) - T_m(z) = 0.\end{aligned} \tag{3.38}$$

Since $z \in \mathbb{C}$ is arbitrary, the result (3.32) then follows from the Cayley-Hamilton theorem (as in the proof of Theorem 2.1). □

Remark 3.2. *Equation (3.38) naturally leads to the plane algebraic curve* $\mathcal{K}_{m-1}$,

$$\mathcal{K}_{m-1} : \mathcal{F}_{m-1}(z, y) = (y - k_m(z))^3 + (y - k_m(z))\, S_m(z) - T_m(z) = 0, \tag{3.39}$$
$$k_m(z) = \sum_{\ell=0}^{n} k_{m,\ell} z^\ell, \quad S_m(z) = \sum_{p=0}^{2n+1-s} s_{m,p} z^p, \quad 0 \le s \le 2n+1,$$
$$T_m(z) = \sum_{q=0}^{m} t_{m,q} z^q, \quad t_{m,m} = 1, \quad m = 3n+1 \text{ or } m = 3n+2, \quad n \in \mathbb{N}_0$$

of (arithmetic) genus $m-1$. *For* $m \ge 4$ *these curves are non-hyperelliptic.*

Examples illustrating this formalism can be found in Chapter 4.

Finally, introducing a deformation parameter $t_m \in \mathbb{C}$ into the pair (q_0, q_1) (i.e., $q_\ell(x) \to q_\ell(x, t_m)$, $\ell = 0, 1$), the time-dependent Bsq hierarchy is defined as a collection of evolution equations (varying $m = 3n + \varepsilon$, $\varepsilon \in \{1, 2\}$, $n \in \mathbb{N}_0$)

$$\frac{d}{d\,t_m} L_3(t_m) - [P_m(t_m), L_3(t_m)] = 0,$$
$$(x, t_m) \in \mathbb{C}^2,\ m = 3n + \varepsilon,\ \varepsilon \in \{1, 2\},\ n \in \mathbb{N}_0, \tag{3.40}$$

or equivalently, by

$$\mathrm{Bsq}_m(q_0, q_1) = \begin{cases} q_{0,t_m} - 3\, g^{(\varepsilon)}_{n+1,x} = 0, \\ q_{1,t_m} - 3\, f^{(\varepsilon)}_{n+1,x} = 0, \end{cases}$$
$$(x, t_m) \in \mathbb{C}^2,\ m = 3n + \varepsilon,\ \varepsilon \in \{1, 2\},\ n \in \mathbb{N}_0, \tag{3.41}$$

that is, by

$$\mathrm{Bsq}_m(q_0, q_1) = \begin{cases} q_{0,t_m} + \frac{1}{6} F_{m,xxxxx} + \frac{5}{6} q_1 F_{m,xxx} + \frac{5}{4} q_{1,x} F_{m,xx} + (\frac{3}{4} q_{1,xx} + \frac{2}{3} q_1^2) F_{m,x} \\ \quad + (\frac{1}{6} q_{1,xxx} + \frac{2}{3} q_1 q_{1,x}) F_m + 3(z - q_0) G_{m,x} - q_{0,x} G_m = 0, \\ q_{1,t_m} - 2 G_{m,xxx} - 2 q_1 G_{m,x} - q_{1,x} G_m + 3(z - q_0) F_{m,x} - 2 q_{0,x} F_m = 0, \end{cases}$$
$$(x, t_m) \in \mathbb{C}^2,\ m = 3n + \varepsilon,\ \varepsilon \in \{1, 2\},\ n \in \mathbb{N}_0. \tag{3.42}$$

Explicitly, one obtains for the first few equations in (3.41),

$$\mathrm{Bsq}_1(q_0, q_1) = \begin{cases} q_{0,t_1} - q_{0,x} = 0, \\ q_{1,t_1} - q_{1,x} = 0, \end{cases}$$

$$\mathrm{Bsq}_2(q_0, q_1) = \begin{cases} q_{0,t_2} + \frac{1}{6} q_{1,xxx} + \frac{2}{3} q_1 q_{1,x} - d_0^{(2)} q_{0,x} = 0, \\ q_{1,t_2} - 2\, q_{0,x} - d_0^{(2)} q_{1,x} = 0, \end{cases}$$
$$\tag{3.43}$$

$$\mathrm{Bsq}_4(q_0,q_1) = \begin{cases} q_{0,t_4} + \tfrac{1}{18}\, q_{1,xxxxx} + \tfrac{1}{3}\, q_1 q_{1,xxx} + \tfrac{2}{3}\, q_{1,x} q_{1,xx} + \tfrac{4}{9}\, q_1^2 q_{1,x} \\[2mm] \qquad -\tfrac{4}{3}\, q_0 q_{0,x} + c_1^{(1)}\big(\tfrac{1}{6}\, q_{1,xxx} + \tfrac{2}{3}\, q_1 q_{1,x}\big) - d_1^{(1)} q_{0,x} = 0, \\[2mm] q_{1,t_4} - \tfrac{2}{3}\, q_{0,xxx} - \tfrac{4}{3}\, q_1 q_{0,x} - \tfrac{4}{3}\, q_{1,x} q_0 - c_1^{(1)} 2 q_{0,x} - d_1^{(1)} q_{1,x} = 0, \end{cases}$$

$$\mathrm{Bsq}_5(q_0,q_1) = \begin{cases} q_{0,t_5} + \tfrac{1}{9}\, q_{0,xxxxx} + \tfrac{5}{18}\, q_0 q_{1,xxx} + \tfrac{5}{9}\, q_1 q_{0,xxx} + \tfrac{5}{9}\, q_{1,xx} q_{0,x} \\[2mm] \qquad + \tfrac{5}{6}\, q_{1,x} q_{0,xx} + \tfrac{5}{9}\, q_1^2 q_{0,x} + \tfrac{10}{9}\, q_0 q_1 q_{1,x} + d_0^{(2)}\big(\tfrac{1}{18}\, q_{1,xxxxx} \\[2mm] \qquad + \tfrac{1}{3}\, q_1 q_{1,xxx} + \tfrac{2}{3}\, q_{1,x} q_{1,xx} + \tfrac{4}{9}\, q_1^2 q_{1,x} - \tfrac{4}{3}\, q_0 q_{0,x}\big) \\[2mm] \qquad + c_1^{(2)}\, \big(\tfrac{1}{6}\, q_{1,xxx} + \tfrac{2}{3}\, q_1 q_{1,x}\big) - d_1^{(2)} q_{0,x} = 0 \\[2mm] q_{1,t_5} + \tfrac{1}{9}\, q_{1,xxxxx} + \tfrac{5}{9}\, q_1 q_{1,xxx} + \tfrac{25}{18}\, q_{1,x} q_{1,xx} + \tfrac{5}{9}\, q_1^2 q_{1,x} \\[2mm] \qquad - \tfrac{10}{3}\, q_0 q_{0,x} - d_0^{(2)}\, \big(\tfrac{2}{3}\, q_{0,xxx} + \tfrac{4}{3}\, q_1 q_{0,x} + \tfrac{4}{3}\, q_{1,x} q_0\big) \\[2mm] \qquad - c_1^{(2)}\, 2 q_{0,x} - d_1^{(2)}\, q_{1,x} = 0 \end{cases},$$

etc.

Remark 3.3. *Due to our choice of L_3 in (3.1) (as opposed to $\tilde{L}_3$ mentioned immediately after (3.1)) our* Bsq_2 *system in (3.43) differs slightly from the standard* Bsq *system discussed, for instance, in* [**10**], [**17**], [**38**], *and* [**94**]. *In fact, the simple transformation (put $d_0^{(2)} = 0$ for simplicity),*

$$q_0 \to \tilde{q}_0 = q_0 + \frac{1}{2}\, q_{1,x}, \quad q_1 \to \tilde{q}_1 = q_1 \tag{3.44}$$

transforms Bsq_2 *into*

$$\mathrm{Bsq}_2(\tilde{q}_0,\tilde{q}_1) = \begin{cases} \tilde{q}_{0,t_2} - \tilde{q}_{0,xx} + \tfrac{2}{3}\, (\tilde{q}_{1,xxx} + \tilde{q}_1 \tilde{q}_{1,x}) = 0 \\[2mm] \tilde{q}_{1,t_2} - 2\, \tilde{q}_{0,x} + \tilde{q}_{1,xx} = 0 \end{cases}, \tag{3.45}$$

which in turn transforms into the nonlinear string equation

$$u_{tt} = b u_{xx} + a(u^2)_{xx} - \frac{1}{3}\, u_{xxxx}, \tag{3.46}$$

where

$$q_1(x,t) = -\frac{1}{4}\big(6\, a u(x,t) + 3\, b\big), \quad t = t_2, \tag{3.47}$$

with $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$ arbitrary constants. Moreover, we should emphasize that our Bsq_2 *system in (3.43) or (3.45) differs from the Kaup-Boussinesq system (see, e.g.,* [**85**] *and the references therein), whose algebro-geometric quasi-periodic solutions can be derived from an associated hyperelliptic curve (not branched at infinity)* [**69**], [**87**] *as opposed to the non-hyperelliptic case typical in our paper (for genus larger than 2).*

Remark 3.4. *As in Section 2.1 (cf. Remark 2.3) we decided to start by postulating the recursion relation (3.3) as the point of departure for developing our formalism. Alternatively, we could have started with*

$$(L_3 - z)\psi(P) = 0, \quad (P_m - y(P))\psi(P) = 0, \quad P = (z, y(P)) \in \mathcal{K}_{m-1} \setminus \{P_\infty\} \tag{3.48}$$

in the stationary case, respectively by

$$(L_3 - z)\psi(P, t_m) = 0, \quad (\frac{\partial}{\partial t_m} - P_m)\psi(P, t_m) = 0, \quad t_m \in \mathbb{R} \tag{3.49}$$

in the time-dependent case. This then yields (3.3) as a consequence of (3.15), (3.16), and (3.29) and analogously one infers (3.40)–(3.43).

3.2. The Stationary Boussinesq Formalism

We continue our study of the Bsq hierarchy and focus, in particular, on the stationary case. Our main strategy will be to develop the Bsq material in close analogy to the KdV discussion in Chapter 2 and establish the connections between the polynomial approach described in Section 2.1 and a fundamental meromorphic function $\phi(P, x)$ defined on the Boussinesq curve $\mathcal{K}_{m-1}$ in (3.39). Moreover, we discuss in some detail the associated stationary Baker-Akhiezer function $\psi(P, x, x_0)$, the common eigenfunction of L_3 and P_m, and associated positive divisors of degree $m - 1$ on $\mathcal{K}_{m-1}$. The latter topic was originally developed by Jacobi [**58**] in the case of hyperelliptic curves and applied to the KdV case by Mumford [**75**], Section III.a.1 and McKean [**74**].

Before we enter any further details we should perhaps stress one important point. In spite of the considerable complexity of the formulas displayed at various places in Sections 2.1–2.2, the basic underlying formalism is a recursive one as described in depth in [**20**]. Consequently, the majority of our formalism can be generated using symbolic calculation programs (such as *Mathematica* or *Maple*).

We recall the Bsq curve $\mathcal{K}_{m-1}$ in (3.39)

$$\begin{aligned} \mathcal{K}_{m-1} :\ & \mathcal{F}_{m-1}(z, y) = y^3 + y\, S_m(z) - T_m(z) = 0, \\ & S_m(z) = \sum_{p=0}^{2n-1+\varepsilon} s_{m,p} z^p, \quad T_m(z) = z^m + \sum_{q=0}^{m-1} t_{m,q} z^q, \\ & m = 3n + \varepsilon,\ \varepsilon \in \{1, 2\},\ n \in \mathbb{N}_0, \end{aligned} \tag{3.50}$$

(where $m = 3n + \varepsilon$, $\varepsilon \in \{1, 2\}$, $n \in \mathbb{N}_0$ will be fixed throughout this section) and denote its compactification (adding the branch point P_∞) by the same symbol $\mathcal{K}_{m-1}$. (In the following $\mathcal{K}_{m-1}$ will always denote the compactified curve.) Thus $\mathcal{K}_{m-1}$ becomes a (possibly singular) three-sheeted Riemann surface of arithmetic genus $m-1$ in a standard manner. We will need a bit more notation in this context. Points P on $\mathcal{K}_{m-1}$ are represented as pairs $P = (z, y)$ satisfying (3.50) together with P_∞, the point at infinity. The complex structure on $\mathcal{K}_{m-1}$ is defined in the usual way by introducing local coordinates $\zeta_{P_0} : P \to (z - z_0)$ near points $P_0 \in \mathcal{K}_{m-1}$ which are neither branch nor singular points of $\mathcal{K}_{m-1}$, $\zeta_{P_\infty} : P \to z^{-1/3}$ near the branch point $P_\infty \in \mathcal{K}_{m-1}$ (with an appropriate determination of the branch of $z^{1/3}$)

and similarly at branch and/or singular points of $\mathcal{K}_{m-1}$. The holomorphic map $*$, changing sheets, is defined by

$$* : \begin{cases} \mathcal{K}_{m-1} \to \mathcal{K}_{m-1}, \\ P = (z, y_j(z)) \to P^* = (z, y_{j+1 (\mathrm{mod}\ 3)})(z)), \quad j = 1, 2, 3, \end{cases} \tag{3.51}$$

$$P^{**} := (P^*)^*, \qquad \text{etc.}, \tag{3.52}$$

where $y_j(z),\ j = 1, 2, 3$ denote the three branches of $y(P)$ satisfying $\mathcal{F}_{m-1}(z, y) = 0$. Finally, positive divisors on $\mathcal{K}_{m-1}$ of degree $m-1$ are denoted by

$$\mathcal{D}_{P_1,\dots,P_{m-1}} : \begin{cases} \mathcal{K}_{m-1} \to \mathbb{N}_0, \\ P \to \mathcal{D}_{P_1,\dots,P_{m-1}}(P) = \begin{cases} k \text{ if } P \text{ occurs } k \\ \qquad \text{times in } \{P_1, \dots, P_{m-1}\}, \\ 0 \text{ if } P \notin \{P_1, \dots, P_{m-1}\}. \end{cases} \end{cases} \tag{3.53}$$

Specific details on curves of Bsq-type (i.e., trigonal curves with a triple point at P_∞) as defined in (3.50) can be found in Appendix B.

Given these preliminaries, let $\psi(P, x, x_0)$ denote the common normalized eigenfunction of L_3 and P_m, whose existence is guaranteed by the commutativity of L_3 and P_m (cf., e.g., [**13**], [**14**]), that is, by

$$[P_m, L_3] = 0, \quad m = 3n + \varepsilon \tag{3.54}$$

for a given $\varepsilon \in \{1, 2\}$, and $n \in \mathbb{N}_0$, or equivalently, by the requirement

$$f^{(\varepsilon)}_{n+1,x} = 0, \qquad g^{(\varepsilon)}_{n+1,x} = 0. \tag{3.55}$$

Explicitly, this yields

$$L_3\psi(P, x, x_0) = z(P)\,\psi(P, x, x_0), \quad P_m\psi(P, x, x_0) = y(P)\,\psi(P, x, x_0), \tag{3.56}$$
$$P = (z, y) \in \mathcal{K}_{m-1}\backslash\{P_\infty\}, \quad x \in \mathbb{C}.$$

Assuming the normalization,

$$\psi(P, x_0, x_0) = 1, \qquad P \in \mathcal{K}_{m-1}\backslash\{P_\infty\} \tag{3.57}$$

for some fixed $x_0 \in \mathbb{C}$, $\psi(P, x, x_0)$ is called the stationary Baker-Akhiezer function for the Bsq hierarchy. Closely related to $\psi(P, x, x_0)$ is the following meromorphic function $\phi(P, x)$ on $\mathcal{K}_{m-1}$ defined by

$$\phi(P, x) = \frac{\psi_x(P, x, x_0)}{\psi(P, x, x_0)}, \quad P \in \mathcal{K}_{m-1},\ x \in \mathbb{C}, \tag{3.58}$$

such that

$$\psi(P, x, x_0) = \exp\left(\int_{x_0}^{x} d\,x'\phi(P, x')\right), \qquad P \in \mathcal{K}_{m-1}\backslash\{P_\infty\}. \tag{3.59}$$

Since $\phi(P, x)$ is a fundamental object for the stationary Bsq hierarchy, we next intend to establish its connection with the recursion formalism of Section 3.1. In pursuit of this connection, it is necessary to define a variety of further polynomials A_m, B_m, C_m, D_{m-1}, E_m, J_m, and N_m with respect to $z \in \mathbb{C}$,

$$A_m(z, x) = -G_m(z, x)^2 - \frac{1}{3}\,q_1(x)\,F_m(z, x)^2 + \frac{1}{4}\,F_{m,x}(z, x)^2 - \frac{1}{3}\,F_m(z, x)\,F_{m,xx}(z, x), \tag{3.60}$$

$$
\begin{aligned}
B_m(z,x) = {} & (z-q_0(x))\big(-2\,F_m(z,x)^2\,G_m(z,x) + \frac{1}{2}\,F_m(z,x)^2\,F_{m,x}(z,x)\big) \\
& - G_m(z,x)^2\,G_{m,x}(z,x) + \frac{1}{4}\,F_{m,x}(z,x)^2\,G_{m,x}(z,x) \\
& - \frac{1}{6}\,q_{1,x}(x)\,F_m(z,x)^2\,G_m(z,x) - \frac{1}{2}\,q_{1,x}(x)\,F_m(z,x)^2\,F_{m,x}(z,x) \\
& + \frac{1}{6}\,G_m(z,x)^2\,F_{m,xx}(z,x) - \frac{11}{18}\,q_1(x)\,F_m(z,x)^2\,F_{m,xx}(z,x) \\
& - \frac{1}{24}\,F_{m,x}(z,x)^2\,F_{m,xx}(z,x) + \frac{1}{36}\,F_m(z,x)\,F_{m,xx}(z,x)^2 \\
& + \frac{2}{3}\,q_1(x)\,F_m(z,x)\,G_m(z,x)^2 - \frac{2}{9}\,q_1(x)^2\,F_m(z,x)^3 \\
& - \frac{2}{3}\,q_1(x)F_m(z,x)\,G_m(z,x)\,F_{m,x}(z,x) + \frac{1}{6}\,q_1(x)F_m(z,x)\,F_{m,x}(z,x)^2 \\
& + F_m(z,x)\,G_m(z,x)\,G_{xx}(z,x) - \frac{1}{2}\,F_m(z,x)\,F_{m,x}(z,x)\,G_{m,xx}(z,x) \\
& - \frac{1}{6}\,q_{1,xx}(x)\,F_m(z,x)^3 - \frac{1}{6}\,F_m(z,x)\,G_m(z,x)\,F_{m,xxx}(z,x) \\
& + \frac{1}{12}\,F_m(z,x)\,F_{m,x}(z,x)\,F_{m,xxx}(z,x) - \frac{1}{6}\,F_m(z,x)^2\,F_{m,xxxx}(z,x) \\
& - F_m(z,x)\,G_{m,x}(z,x)^2, \qquad (3.61)
\end{aligned}
$$

$$
\begin{aligned}
C_m(z,x) = {} & (z-q_0(x))F_m(z,x)^2 - \frac{2}{3}\,q_1(x)F_m(z,x)G_m(z,x) + \frac{1}{6}\,q_{1,x}(x)F_m(z,x)^2 \\
& + G_m(z,x)G_{m,x}(z,x) + \frac{1}{2}\,F_{m,x}(z,x)G_{m,x}(z,x) - \frac{1}{12}\,F_{m,x}(z,x)F_{m,xx}(z,x) \\
& - \frac{1}{6}\,G_m(z,x)F_{m,xx}(z,x) - F_m(z,x)G_{m,xx}(z,x) \\
& + \frac{1}{3}\,q_1(x)F_m(z,x)F_{m,x}(z,x) + \frac{1}{6}\,F_m(z,x)F_{m,xxx}(z,x), \qquad (3.62)
\end{aligned}
$$

$$
\begin{aligned}
E_m(z,x) = {} & (z-q_0)\Big(2F_m(z,x)G_m(z,x)^2 + \frac{1}{3}\,q_1(x)F_m(z,x)^3 \\
& + F_m(z,x)F_{m,x}(z,x)G_m(z,x) + \frac{1}{3}\,F_m(z,x)^2F_{m,xx}(z,x)\Big) \\
& + \frac{1}{6}\,q_1(x)F_m(z,x)F_{m,x}(z,x)G_{m,x}(z,x) - \frac{1}{9}\,q_1(x)^2F_m(z,x)^2G_m(z,x) \\
& - \frac{1}{2}\,q_1(x)F_{m,x}(z,x)G_m(z,x)^2 + \frac{1}{6}\,q_1(x)^2F_m(z,x)^2F_{m,x}(z,x) \\
& - \frac{5}{12}\,q_1(x)F_{m,x}(z,x)^2G_m(z,x) - \frac{5}{24}\,q_1(x)F_{m,x}(z,x)^3 \\
& + \frac{1}{3}\,q_1(x)F_m(z,x)G_m(z,x)G_{m,x}(z,x) - q_1(x)G_m(z,x)^3 \\
& + \frac{1}{18}\,q_1(x)q_{1,x}(x)F_m(z,x)^3 - \frac{1}{6}\,q_{1,x}(x)F_m(z,x)F_{m,x}(z,x)G_m(z,x) \\
& - \frac{1}{12}\,q_{1,x}(x)F_m(z,x)F_{m,x}(z,x)^2 - \frac{1}{18}\,q_1(x)F_m(z,x)F_{m,xx}(z,x)G_m(z,x) \\
& + \frac{7}{36}\,q_1(x)F_m(z,x)F_{m,x}(z,x)F_{m,xx}(z,x) + \frac{1}{3}\,F_{m,xx}(z,x)G_m(z,x)G_{m,x}(z,x)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{6} F_{m,x}(z,x) F_{m,xx}(z,x) G_{m,x}(z,x) + \frac{1}{18} q_{1,x}(x) F_m(z,x)^2 F_{m,xx}(z,x) \\
&+ \frac{1}{18} F_{m,xx}(z,x)^2 G_m(z,x) + \frac{1}{36} F_{m,x}(z,x) F_{m,xx}(z,x)^2 \\
&- 2\, G_m(z,x)^2 G_{m,xx}(z,x) - \frac{1}{3} q_1(x) F_m(z,x)^2 G_{m,xx}(z,x) \\
&- F_{m,x}(z,x) G_m(z,x) G_{m,xx}(z,x) - \frac{1}{3} F_m(z,x) F_{m,xx}(z,x) G_{m,xx}(z,x) \\
&+ \frac{1}{18} q_1(x) F_m(z,x)^2 F_{m,xxx}(z,x) - \frac{1}{6} F_{m,x}(z,x) F_{m,xxx}(z,x) G_m(z,x) \\
&- \frac{1}{12} F_{m,x}(z,x)^2 F_{m,xxx}(z,x) + \frac{1}{18} F_m(z,x) F_{m,xx}(z,x) F_{m,xxx}(z,x), \qquad (3.63)
\end{aligned}
$$

$$
J_m(z,x) = H_{m,x}(z,x) + \left(z - q_0(x) - \frac{1}{2} q_{1,x}(x)\right) F_m(z,x), \qquad (3.64)
$$

$$
\begin{aligned}
D_{m-1}(z,x) = \epsilon(r) \Big(&(z - q_0(x) - \frac{1}{6} q_{1,x}) F_m(z,x)^3 - G_m(z,x)^3 + \frac{1}{4} G_m(z,x)\, F_{m,x}(z,x)^2 \\
&- q_1(x)\, F_m(z,x)^2\, G_m(z,x) + \frac{1}{2} G_m(z,x)^2\, F_{m,x}(z,x) - \frac{1}{8} F_{m,x}(z,x)^3 \\
&- \frac{1}{6} q_1(x)\, F_m(z,x)^2\, F_x(z,x) - F_m(z,x)\, G_m(z,x)\, G_{m,x}(z,x) \\
&+ \frac{1}{2} F_m(z,x)\, F_{m,x}(z,x)\, G_{m,x}(z,x) - \frac{1}{2} F_m(z,x)\, G_m(z,x)\, F_{m,xx}(z,x) \\
&+ \frac{1}{4} F_m(z,x)\, F_{m,x}(z,x)\, F_{m,xx}(z,x) - F_m(z,x)^2\, G_{m,xx}(z,x) \\
&- \frac{1}{6} F_m(z,x)^2\, F_{m,xxx}(z,x) \Big), \qquad (3.65)
\end{aligned}
$$

$$
\begin{aligned}
N_m(z,x) = \epsilon(r) \frac{1}{144} \Big(&144\, (z - q_0(x))^2\, F_m(z,x)^3 - 144\, (z - q_0(x))\, G_m(z,x)^2\, F_{m,x}(z,x) \\
&- 144\, (z - q_0(x))\, q_1(x)\, F_m(z,x)^2\, G_m(z,x) - 144\, (z - q_0(x))\, G_m(z,x)^3 \\
&+ 120\, (z - q_0(x))\, q_1(x)\, F_m(z,x)^2\, F_{m,x}(z,x) - 24\, q_{1,xx}\, F_m(z,x)\, G_m(z,x)^2 \\
&- 36\, (z - q_0(x))\, G_m(z,x)\, F_{m,x}(z,x)^2 + 16\, q_1(x)^2\, F_m(z,x)\, F_{m,x}(z,x)^2 \\
&+ 288\, (z - q_0(x))\, F_m(z,x)\, G_m(z,x)\, G_{m,x}(z,x) - 144\, q_1(x)\, G_m(z,x)^2\, G_{m,x}(z,x) \\
&+ 48\, q_1(x)\, G_m(z,x)\, F_{m,x}(z,x)\, G_{m,x}(z,x) + 60\, q_1(x)\, F_{m,x}(z,x)^2\, G_{m,x}(z,x) \\
&+ 48\, (z - q_0(x))\, q_{1,x}\, F_m(z,x)^3 - 24\, q_1(x)\, q_{1,x}\, F_m(z,x)^2\, G_m(z,x) \\
&- 84\, q_{1,x}\, G_m(z,x)^2\, F_{m,x}(z,x) + 20\, q_1(x)\, q_{1,x}\, F_m(z,x)^2\, F_{m,x}(z,x) \\
&- 84\, q_{1,x}\, G_m(z,x)\, F_{m,x}(z,x)^2 + 48\, (z - q_0(x))\, F_m(z,x)^2\, F_{m,xxx}(z,x) \\
&+ 48\, q_{1,x}\, F_m(z,x)\, G_m(z,x)\, G_{m,x}(z,x) + 24\, q_{1,x}\, F_m(z,x)\, F_{m,x}(z,x)\, G_{m,x}(z,x) \\
&+ 20\, q_1(x)\, F_m(z,x)\, F_{m,x}(z,x)\, F_{m,xxx}(z,x) - 96\, q_1(x)\, G_m(z,x)^2\, F_{m,xx}(z,x) \\
&- 96\, q_1(x)\, G_m(z,x)\, F_{m,x}(z,x)\, F_{m,xx}(z,x) - 24\, q_1(x)\, F_{m,x}(z,x)^2\, F_{m,xx}(z,x) \\
&- 288\, (z - q_0)\, F_m(z,x)^2\, G_{m,xx}(z,x) + 144\, q_1(x)\, F_m(z,x)\, G_m(z,x)\, G_{m,xx}(z,x) \\
&- 120\, q_1(x)\, F_m(z,x)\, F_{m,x}(z,x)\, G_{m,xx}(z,x) - 288\, G_m(z,x)\, G_{m,x}(z,x)\, G_{m,xx}(z,x)
\end{aligned}
$$

$$
\begin{aligned}
&- 144\, F_{m,x}(z,x)\, G_{m,x}(z,x)\, G_{m,xx}(z,x) - 48\; q_{1,x} F_m(z,x)^2\, G_{m,xx}(z,x) \\
&+ 144\, F_m(z,x)\, G_{m,xx}(z,x)^2 - 96\, q_1(x)^2\, F_m(z,x)\, G_m(z,x)\, F_{m,x}(z,x) \\
&- 24\, q_{1,xx}\, F_m(z,x)\, G_m(z,x)\, F_{m,x}(z,x) - 6\, q_{1,xx}\, F_m(z,x)\, F_{m,x}(z,x)^2 \\
&- 21\, q_{1,x}\, F_{m,x}(z,x)^3 - 24\, q_1(x)\, F_m(z,x)\, G_m(z,x)\, F_{m,xxx}(z,x) \\
&- 6\, F_{m,x}(z,x)^2\, F_{m,xxxx}(z,x) + 48\, G_m(z,x)\, G_{m,x}(z,x)\, F_{m,xxx}(z,x) \\
&+ 24\, F_{m,x}(z,x)\, G_{m,x}(z,x)\, F_{m,xxx}(z,x) + 8\, q_{1,x}\, F_m(z,x)^2\, F_{m,xxx}(z,x) \\
&- 48\, F_m(z,x)\, G_{m,xx}(z,x)\, F_{m,xxx}(z,x) + 4\, F_m(z,x)\, F_{m,xxx}(z,x)^2 \\
&- 24\, G_m(z,x)^2\, F_{m,xxxx}(z,x) - 24\, G_m(z,x)\, F_{m,x}(z,x)\, F_{m,xxxx}(z,x) \\
&+ 144\, (z - q_0(x))\, F_m(z,x)\, F_{m,x}(z,x)\, G_{m,x}(z,x) + 4\, q_{1,x}^2\, F_m(z,x)^3 \Big),
\end{aligned} \tag{3.66}
$$

where

$$
\varepsilon(m) = \begin{cases} 1 & \text{for } m \equiv 2 \pmod 3, \\ -1 & \text{for } m \equiv 1 \pmod 3). \end{cases} \tag{3.67}
$$

The quantities $A_m, \dots, N_m$ in (3.60)–(3.66) are of course not independent of each other. There exist various interrelationships between them and S_m, T_m (cf. (3.50)), some of which are summarized below.

Lemma 3.5. *Let $(z,x) \in \mathbb{C} \times \mathbb{R}$. Then*

$$
(i).\ A_m\, C_m - B_m(G_m + \frac{1}{2}\, F_{m,x}) + F_m\, E_m + S_m\, F_m\, (G_m + \frac{1}{2}\, F_{m,x}) = 0. \tag{3.68}
$$

$$
(ii).\ B_m\, C_m + A_m\, E_m + S_m\, \big(A_m\, (G_m + \frac{1}{2}\, F_{m,x}) - F_m\, C_m\big) - T_m\, F_m\, (G_m + \frac{1}{2}\, F_{m,x}) = 0. \tag{3.69}
$$

$$
(iii).\ C_r = F_r\, J_r - (G_r + \frac{1}{2} F_{m,x})(H_r - k_r). \tag{3.70}
$$

$$
(iv).\ B_m = \frac{2}{3}\, S_m\, F_m + \frac{1}{3}\, \epsilon(r)\, D_{m-1,x}. \tag{3.71}
$$

$$
(v).\ \epsilon(r)\, (G_m + \frac{1}{2}\, F_{m,x})\, D_{m-1} = F_m\, B_m - A_m^2 - S_m\, F_m^2. \tag{3.72}
$$

$$
(vi).\ \epsilon(r)\, C_m\, D_{m-1} = T_m\, F_m^2 - A_m B_m. \tag{3.73}
$$

$$
(vii).\ D_{m-1} N_m = B_m\, E_m - T_m(A_m\, (G_m + \frac{1}{2}\, F_{m,x}) - F_m\, C_m). \tag{3.74}
$$

$$
(viii).\ \epsilon(r)\, A_m\, N_m = T_m\, (G_m + \frac{1}{2}\, F_{m,x})^2 - C_m E_m. \tag{3.75}
$$

$$
(ix).\ \epsilon(r)\, F_m\, N_m = C_m^2 + E_m\, (G_m + \frac{1}{2}\, F_{m,x}) + S_m (G_m + \frac{1}{2}\, F_{m,x})^2. \tag{3.76}
$$

$$
(x).\ N_{m,x}\big(G_m + \frac{1}{2}\, F_{m,x}\big) = N_m\big(q_1\, F_m + F_{m,xx}\big) - \epsilon(r)\, J_m\, \big(2\, \big(G_m + \frac{1}{2}\, F_{m,x}\big)\, S_m + 3\, E_m\big). \tag{3.77}
$$

Proof. This is a straightforward (but tedious) consequence of (3.23), (3.25), (3.60)–(3.66). □

Next we derive a first formula for $\phi(P,x)$. By (3.30) and (3.56) one infers

$$P_m\psi = F_m\psi_{xx} + (G_m - \frac{1}{2}F_{m,x})\psi_x + H_m\psi = y\psi \tag{3.78}$$

and hence

$$\begin{aligned}\big(P_m\psi\big)_x &= F_{m,x}\psi_{xx} + F_m\psi_{xxx} + (G_{m,x} - \frac{1}{2}F_{m,xx})\psi_x + (G_m - \frac{1}{2}F_{m,x})\psi_{xx} \\ &\quad + H_{m,x}\psi + H_m\psi_x \\ &= F_{m,x}\psi_{xx} + (z - q_0 - \frac{1}{2}q_{1,x})F_m\psi - q_1F_m\psi_x + (G_{m,x} - \frac{1}{2}F_{m,xx})\psi_x \\ &\quad + (G_m - \frac{1}{2}F_{m,x})\psi_{xx} + H_{m,x}\psi + H_m\psi_x = y\psi_x. \end{aligned}\tag{3.79}$$

Using (3.78) in (3.79) in order to eliminate ψ_{xx} in terms of $\phi = \psi_x/\psi$ one infers

$$\phi = \frac{(G_m + \frac{1}{2}F_{m,x})(y - H_m) + (z - q_0 - \frac{1}{2}q_{1,x})F_m^2 + H_{m,x}F_m}{(y - H_m)F_m - (G_{m,x} - \frac{1}{2}F_{m,xx})F_m + q_1F_m^2 + G_m^2 - \frac{1}{4}F_{m,x}^2}. \tag{3.80}$$

In fact, (3.80) is just one of three expressions one can derive linking $\phi(P,x)$ and F_m, G_m.

Lemma 3.6. *Let $P = (z,y) \in \mathcal{K}_{m-1}$ and $(z,x) \in \mathbb{C}^2$. Then*

$$\phi(P,x) = \frac{(G_m(z,x) + 2^{-1}F_{m,x}(z,x))y(P) + C_m(z,x)}{F_m(z,x)y(P) - A_m(z,x)} \tag{3.81}$$

$$= \frac{F_m(z,x)y(P)^2 + A_m(z,x)y(P) + B_m(z,x)}{\varepsilon(m)D_{m-1}(z,x)} \tag{3.82}$$

$$= \frac{-\varepsilon(m)N_m(z,x)}{(G_m(z,x) + 2^{-1}F_{m,x}(z,x))y(P)^2 - C_m(z,x)y(P) - E_m(z,x)}. \tag{3.83}$$

Proof. (3.81) follows from (3.30), (3.60), (3.62), and (3.80). (3.82) is a consequence of (3.50), (3.72), (3.73) and (3.81). Similarly, (3.83) follows from (3.50), (3.75), (3.76), and (3.81). □

By inspection of (3.15) and (3.16) one infers that D_{m-1} and N_m are monic polynomials with respect to z of degree $m-1$ and m, respectively. Hence we may write

$$D_{m-1}(z,x) = \prod_{j=1}^{m-1}(z - \mu_j(x)), \tag{3.84}$$

$$N_m(z,x) = \prod_{\ell=0}^{m-1}(z - \nu_\ell(x)). \tag{3.85}$$

Defining

$$\hat{\mu}_j(x) = \big(\mu_j(x), \frac{A_m(\mu_j(x),x)}{F_m(\mu_j(x),x)}\big) \in \mathcal{K}_{m-1}, \quad j = 1,\dots,m-1, \quad x \in \mathbb{C}, \tag{3.86}$$

$$\hat{\nu}_\ell(x) = \big(\nu_\ell(x), -\frac{C_m(\nu_\ell(x),x)}{G_m(\nu_\ell(x),x) + \frac{1}{2}F_{m,x}(\nu_\ell(x),x)}\big) \in \mathcal{K}_{m-1}, \quad \ell = 0,\dots,m-1, \quad x \in \mathbb{C}, \tag{3.87}$$

one infers from (3.81) that the divisor $\big(\phi(P,x)\big)$ of $\phi(P,x)$ is given by (cf. (3.53))

$$\big(\phi(P,x)\big) = \mathcal{D}_{\hat{\nu}_0(x),\dots,\hat{\nu}_{m-1}(x)}(P) - \mathcal{D}_{P_\infty,\hat{\mu}_1(x),\dots,\hat{\mu}_{m-1}(x)}(P). \tag{3.88}$$

That is, $\hat{\nu}_0(x),\dots,\hat{\nu}_{m-1}(x)$ are the m zeros of $\phi(P,x)$ and $P_\infty, \hat{\mu}_1(x),\dots,\hat{\mu}_{m-1}(x)$ its m poles.

Further properties of $\phi(P,x)$ and $\psi(P,x,x_0)$ are summarized in

Theorem 3.7. *Assume* (3.54)–(3.58), $P=(z,y)\in\mathcal{K}_{m-1}\backslash\{P_\infty\}$, *and let* $(z,x,x_0)\in\mathbb{C}^3$. *Then*

(*i*) $\phi(P,x)$ *satisfies the second-order equation*

$$\phi_{xx}(P,x) + 3\,\phi_x(P,x)\phi(P,x) + \phi(P,x)^3 + q_1(x)\,\phi(P,x) = z - q_0(x) - \frac{1}{2}\,q_{1,x}(x). \tag{3.89}$$

$$(ii)\ \ \phi(P,x)\,\phi(P^*,x)\,\phi(P^{**},x) = \frac{N_m(z,x)}{D_{m-1}(z,x)}. \tag{3.90}$$

$$(iii)\ \ \phi(P,x) + \phi(P^*,x) + \phi(P^{**},x) = \frac{D_{m-1,x}(z,x)}{D_{m-1}(z,x)}. \tag{3.91}$$

$$(iv)\ \ y(P)\,\phi(P,x) + y(P^*)\,\phi(P^*,x) + y(P^{**})\,\phi(P^{**},x) = \frac{3\,T_m(z)\,F_m(z,x) - 2\,S_m(z)\,A_m(z,x)}{\varepsilon(m)D_{m-1}(z,x)}. \tag{3.92}$$

$$(v)\ \ \psi(P,x,x_0)\,\psi(P^*,x,x_0)\,\psi(P^{**},x,x_0) = \frac{D_{m-1}(z,x)}{D_{m-1}(z,x_0)}. \tag{3.93}$$

$$(vi)\ \ \psi_x(P,x,x_0)\,\psi_x(P^*,x,x_0)\,\psi_x(P^{**},x,x_0) = \frac{N_m(z,x)}{D_{m-1}(z,x_0)}. \tag{3.94}$$

$$(vii)\ \ \psi(P,x,x_0) = \bigg(\frac{D_{m-1}(z,x)}{D_{m-1}(z,x_0)}\bigg)^{1/3} \exp\bigg(\int_{x_0}^x d\,x'\varepsilon(m)D_{m-1}(z,x')^{-1} \times \Big(F_m(z,x')\,y(P)^2 + A_m(z,x')\,y(P) + \frac{2}{3}\,F_m(z,x')\,S_m(z)\Big)\bigg). \tag{3.95}$$

Proof. (3.89) is clear from $\phi=\psi_x/\psi$ and $\psi_{xxx}+q_1\psi_x+(q_0+\frac{1}{2}\,q_{1,x}-z)\psi=0$. (3.91) is a consequence of (3.71), (3.82), and

$$y(P)+y(P^*)+y(P^{**}) = 3\,k_m(z), \tag{3.96}$$

$$(y(P)-k_m(z))^2 + (y(P^*)-k_m(z))^2 + (y(P^{**})-k_m(z))^2 = -2\,S_m(z). \tag{3.97}$$

Similarly, (3.92) follows from (3.50), (3.82), (3.96), and (3.97). In order to prove (3.90) one employs (3.81), (3.96), (3.97), and

$$(y(P)-k_m(z))\,(y(P^*)-k_m(z))\,(y(P^{**})-k_m(z)) = T_m(z), \tag{3.98}$$

$$(y(P)-k_m(z))\,(y(P^*)-k_m(z)) + (y(P^*)-k_m(z))\,(y(P^{**})-k_m(z)) + (y(P^{**})-k_m(z))\,(y(P)-k_m(z)) = S_m(z) \tag{3.99}$$

to get

$$\phi(P,x)\,\phi(P^*,x)\,\phi(P^{**},x) = \frac{T_m\,(G_m+\frac{1}{2}\,F_{m,x})^3 + S_m\,C_m\,(G_m+\frac{1}{2}\,F_{m,x})^2 + C_m^3}{T_m\,F_m^3 - S_m\,F_m^2\,A_m - A_m^3}. \tag{3.100}$$

Using (3.65) and (3.66) one verifies that the numerator in (3.100) factors into $D^*_{m-1}\,N_m$ and the denominator into $D^*_{m-1}\,D_{m-1}$, where D^*_{m-1} is defined by

$$\begin{aligned}
D^*_{m-1}(z,x) = \epsilon(r)\Big(&(z-q_0(x)+\frac{1}{6}\,q_{1,x}\,)F_m(z,x)^3 - G_m(z,x)^3 + \frac{1}{4}\,G_m(z,x)\,F_{m,x}(z,x)^2\\
&- q_1(x)\,F_m(z,x)^2\,G_m(z,x) - \frac{1}{2}\,G_m(z,x)^2\,F_{m,x}(z,x) + \frac{1}{8}\,F_{m,x}(z,x)^3\\
&+ \frac{1}{6}\,q_1(x)\,F_m(z,x)^2\,F_x(z,x) + F_m(z,x)\,G_m(z,x)\,G_{m,x}(z,x)\\
&+ \frac{1}{2}\,F_m(z,x)\,F_{m,x}(z,x)\,G_{m,x}(z,x) - \frac{1}{2}\,F_m(z,x)\,G_m(z,x)\,F_{m,xx}(z,x)\\
&- \frac{1}{4}\,F_m(z,x)\,F_{m,x}(z,x)\,F_{m,xx}(z,x) - F_m(z,x)^2\,G_{m,xx}(z,x)\\
&+ \frac{1}{6}\,F_m(z,x)^2\,F_{m,xxx}(z,x)\Big).
\end{aligned} \tag{3.101}$$

(3.93) immediately follows from (3.90) and (3.95) and (3.94) from (3.93) and (3.90). It remains to prove (3.95). The latter directly follows after inserting (3.82) into (3.59) and then replacing B_m according to (3.71). □

Thus, up to normalizations, D_{m-1} represents the product of the three branches of ψ and N_m the product of the three branches of ψ_x, their zeros represent the analogs of Dirichlet and Neumann eigenvalues of L_3 with the corresponding boundary conditions imposed at the point $x \in \mathbb{C}$ when compared to the KdV Lax expression L_2.

Returning to $D_{m-1}(z,x)$ and $N_m(z,x)$ for a moment, we note that (3.3), (3.15), (3.16), (3.65), and (3.66) yield

$$\begin{aligned}
D_0 &= 1,\\
D_1 &= z - q_0(x) - 6^{-1}\,q_{1,x}(x) - d_0^{(2)}\,q_1(x) - (d_0^{(2)})^3, \qquad (3.102)\\
D_3 &= \frac{1}{648}\Big(648\,z^3 + z^2\,\big(648\,q_0(x) - 108\,q_{1,x}(x)\big) + z\,\big(216\,q_0(x)^2 + 48\,q_1(x)^3\\
&\quad + 72\,q_1(x)\,q_{0,x}(x) - 72\,q_0(x)\,q_{1,x}(x) - 18\,q_{1,x}(x)^2 + 36\,q_1(x)\,q_{1,xx}(x)\big)\\
&\quad + 24\,q_0(x)^3 + 48\,q_0(x)\,q_1(x)^3 + 24\,q_0(x)\,q_1(x)\,q_{0,x}(x) - 12\,q_0(x)^2\,q_{1,x}(x)\\
&\quad + 8\,q_1(x)^3\,q_{1,x}(x) - 12\,q_1(x)\,q_{0,x}(x)\,q_{1,x}(x) - 6\,q_0(x)\,q_{1,x}(x)^2 + 3\,q_{1,x}(x)^3\\
&\quad + 24\,q_1(x)^2\,q_{0,xx}(x) + 12\,q_0(x)\,q_1(x)\,q_{1,xx}(x) - 6\,q_1(x)\,q_{1,x}(x)\,q_{1,xx}(x)\\
&\quad + 4\,q_1(x)^2\,q_{1,xxx}(x) + 648\,{d_1^{(1)}}^3 + 1944\,d_1^{(1)}\,z^2 + 648\,{c_1^{(1)}}^3\,q_0(x) + 648\,{d_1^{(1)}}^2\,q_0(x)\\
&\quad + 216\,d_1^{(1)}\,q_0(x)^2 + 648\,{c_1^{(1)}}^2\,d_1^{(1)}\,q_1(x) + 864\,{c_1^{(1)}}^2\,q_0(x)\,q_1(x) + 432\,c_1^{(1)}\,d_1^{(1)}\,q_1(x)^2\\
&\quad + 360\,c_1^{(1)}\,q_0(x)\,q_1(x)^2 + 72\,d_1^{(1)}\,q_1(x)^3 + 216\,c_1^{(1)}\,d_1^{(1)}\,q_{0,x}(x) + 72\,c_1^{(1)}\,q_0(x)\,q_{0,x}(x)\\
&\quad + 72\,d_1^{(1)}\,q_1(x)\,q_{0,x}(x) + 108\,{c_1^{(1)}}^3\,q_{1,x}(x) - 108\,{d_1^{(1)}}^2\,q_{1,x}(x) - 72\,d_1^{(1)}\,q_0(x)\,q_{1,x}(x)\\
&\quad + 144\,{c_1^{(1)}}^2\,q_1(x)\,q_{1,x}(x) + 60\,c_1^{(1)}\,q_1(x)^2\,q_{1,x}(x) - 36\,c_1^{(1)}\,q_{0,x}(x)\,q_{1,x}(x)\\
&\quad - 18\,d_1^{(1)}\,q_{1,x}(x)^2 + 216\,{c_1^{(1)}}^2\,q_{0,xx}(x) + 144\,c_1^{(1)}\,q_1(x)\,q_{0,xx}(x) + 108\,c_1^{(1)}\,d_1^{(1)}\,q_{1,xx}(x)
\end{aligned}$$

$$
\begin{aligned}
&+ 36\, c_1^{(1)}\, q_0(x)\, q_{1,xx}(x) + 36\, d_1^{(1)}\, q_1(x)\, q_{1,xx}(x) - 18\, c_1^{(1)}\, q_{1,x}(x)\, q_{1,xx}(x) \\
&+ z\, \big(- 648\, {c_1^{(1)}}^3 + 1944\, {d_1^{(1)}}^2 + 1296\, d_1^{(1)}\, q_0(x) + 216\, c_1^{(1)}\, q_1(x)^2 + 216\, c_1^{(1)}\, q_{0,x}(x) \\
&- 216\, d_1^{(1)}\, q_{1,x}(x) + 108\, c_1^{(1)}\, q_{1,xx}(x)\big) + 36\, {c_1^{(1)}}^2\, q_{1,xxx}(x) + 24\, c_1^{(1)}\, q_1(x)\, q_{1,xxx}(x)\Big), \\
&\text{etc.,} \qquad (3.103)
\end{aligned}
$$

and

$$
\begin{aligned}
N_1 &= z - q_0(x), \\
N_2 &= \big(z - q_0(x) + 6^{-1}\, q_{1,x}(x)\big)^2 - d_0^{(2)}\big((z - q_0(x))q_1(x) - 6^{-1}\, q_1(x)q_{1,x}(x)\big) \\
&\quad - 6^{-1}\, (d_0^{(2)})^2 q_{1,xx}(x) - (d_0^{(2)})^3 (z - q_0(x)), \qquad (3.104) \\
N_4 &= \Big(3888\, z^4 + 1296\, z^3\, q_{1,x}(x) + z^2\, \big(288\, q_1(x)^3 - 2592\, q_0(x)^2 + 432\, q_1(x)\, q_{0,x}(x) \\
&\quad - 432\, q_0(x)\, q_{1,x}(x) + 864\, q_{1,x}(x)^2 + 1080\, q_1(x)\, q_{1,xx}(x) + 216\, q_{1,xxxx}(x)\big) \\
&\quad + z\, \big(1440\, q_0(x)\, q_1(x)\, q_{0,x}(x) - 1152\, q_0(x)^3 - 720\, q_0(x)^2\, q_{1,x}(x) \\
&\quad + 192\, q_1(x)^3\, q_{1,x}(x) - 432\, q_1(x)\, q_{0,x}(x)\, q_{1,x}(x) + 432\, q_0(x)\, q_{1,x}(x)^2 + 252\, q_{1,x}(x)^3 \\
&\quad - 144\, q_1(x)^2\, q_{0,xx}(x) + 864\, q_{0,x}(x)\, q_{0,xx}(x) + 720\, q_0(x)\, q_1(x)\, q_{1,xx}(x) \\
&\quad + 360\, q_1(x)\, q_{1,x}(x)\, q_{1,xx}(x) + 24\, q_1(x)^2\, q_{1,xxx}(x) - 144\, q_{0,x}(x)\, q_{1,xxx}(x) \\
&\quad + 144\, q_0(x)\, q_{1,xxxx}(x) + 72\, q_{1,x}(x)\, q_{1,xxxx}(x)\big) - 144\, q_0(x)^4 - 288\, q_0(x)^2\, q_1(x)^3 \\
&\quad + 432\, q_0(x)^2\, q_1(x)\, q_{0,x}(x) - 144\, q_0(x)^3\, q_{1,x}(x) + 288\, q_0(x)\, q_1(x)^3\, q_{1,x}(x) \\
&\quad + 48\, q_0(x)\, q_1(x)\, q_{0,x}(x)\, q_{1,x}(x) + 48\, q_0(x)^2\, q_{1,x}(x)^2 - 40\, q_1(x)^3\, q_{1,x}(x)^2 \\
&\quad - 84\, q_1(x)\, q_{0,x}(x)\, q_{1,x}(x)^2 + 84\, q_0(x)\, q_{1,x}(x)^3 - 432\, q_0(x)\, q_1(x)^2\, q_{0,xx}(x) \\
&\quad + 21\, q_{1,x}(x)^4 + 288\, q_0(x)\, q_{0,x}(x)\, q_{0,xx}(x) + 168\, q_1(x)^2\, q_{1,x}(x)\, q_{0,xx}(x) \\
&\quad + 144\, q_{0,x}(x)\, q_{1,x}(x)\, q_{0,xx}(x) - 144\, q_1(x)\, q_{0,xx}(x)^2 + 120\, q_0(x)^2\, q_1(x)\, q_{1,xx}(x) \\
&\quad + 120\, q_0(x)\, q_1(x)\, q_{1,x}(x)\, q_{1,xx}(x) + 30\, q_1(x)\, q_{1,x}(x)^2\, q_{1,xx}(x) \\
&\quad + 72\, q_0(x)\, q_1(x)^2\, q_{1,xxx}(x) - 48\, q_0(x)\, q_{0,x}(x)\, q_{1,xxx}(x) - 28\, q_1(x)^2\, q_{1,x}(x)\, q_{1,xxx}(x) \\
&\quad - 24\, q_{0,x}(x)\, q_{1,x}(x)\, q_{1,xxx}(x) + 48\, q_1(x)\, q_{0,xx}(x)\, q_{1,xxx}(x) - 4\, q_1(x)\, q_{1,xxx}(x)^2 \\
&\quad + 24\, q_0(x)^2\, q_{1,xxxx}(x) + 24\, q_0(x)\, q_{1,x}(x)\, q_{1,xxxx}(x) + 6\, q_{1,x}(x)^2\, q_{1,xxxx}(x)\Big)\frac{1}{3888} \\
&\quad + \Big(1944\, d_1^{(1)}\, z^3 + z^2\, \big(1944\, {d_1^{(1)}}^2 - 648\, {c_1^{(1)}}^3 - 648\, d_1^{(1)}\, q_0(x) + 216\, c_1^{(1)}\, q_1(x)^2 \\
&\quad - 432\, c_1^{(1)}\, q_{0,x}(x) + 432\, d_1^{(1)}\, q_{1,x}(x) + 108\, c_1^{(1)}\, q_{1,xx}(x)\big) + z\, \big(648\, {d_1^{(1)}}^3 + 1296\, {c_1^{(1)}}^3\, q_0(x) \\
&\quad - 1296\, {d_1^{(1)}}^2\, q_0(x) - 1080\, d_1^{(1)}\, q_0(x)^2 + 648\, {c_1^{(1)}}^2\, d_1^{(1)}\, q_1(x) + 864\, {c_1^{(1)}}^2\, q_0(x)\, q_1(x) \\
&\quad + 432\, c_1^{(1)}\, d_1^{(1)}\, q_1(x)^2 + 144\, c_1^{(1)}\, q_0(x)\, q_1(x)^2 + 72\, d_1^{(1)}\, q_1(x)^3 - 432\, c_1^{(1)}\, d_1^{(1)}\, q_{0,x}(x) \\
&\quad + 288\, c_1^{(1)}\, q_0(x)\, q_{0,x}(x) + 288\, d_1^{(1)}\, q_1(x)\, q_{0,x}(x) - 216\, {c_1^{(1)}}^3\, q_{1,x}(x) + 216\, {d_1^{(1)}}^2\, q_{1,x}(x) \\
&\quad - 288\, d_1^{(1)}\, q_0(x)\, q_{1,x}(x) - 288\, {c_1^{(1)}}^2\, q_1(x)\, q_{1,x}(x) + 24\, c_1^{(1)}\, q_1(x)^2\, q_{1,x}(x)
\end{aligned}
$$

$$
\begin{aligned}
&- 144\, c_1^{(1)}\, q_{0,x}(x)\, q_{1,x}(x) + 270\, d_1^{(1)}\, q_{1,x}(x)^2 + 432\, {c_1^{(1)}}^2\, q_{0,xx}(x) + 72\, c_1^{(1)}\, q_1(x)\, q_{0,xx}(x)\\
&+ 216\, c_1^{(1)}\, d_1^{(1)}\, q_{1,xx}(x) + 72\, c_1^{(1)}\, q_0(x)\, q_{1,xx}(x) + 360\, d_1^{(1)}\, q_1(x)\, q_{1,xx}(x)\\
&+ 36\, c_1^{(1)}\, q_{1,x}(x)\, q_{1,xx}(x) - 72\, {c_1^{(1)}}^2\, q_{1,xxx}(x) - 12\, c_1^{(1)}\, q_1(x)\, q_{1,xxx}(x) + 72\, d_1^{(1)}\, q_{1,xxxx}(x)\big)\\
&- 648\, {d_1^{(1)}}^3\, q_0(x) - 648\, {c_1^{(1)}}^3\, q_0(x)^2 - 648\, {d_1^{(1)}}^2\, q_0(x)^2 - 648\, {c_1^{(1)}}^2\, d_1^{(1)}\, q_0(x)\, q_1(x)\\
&- 216\, d_1^{(1)}\, q_0(x)^3 - 864\, {c_1^{(1)}}^2\, q_0(x)^2\, q_1(x) - 432\, c_1^{(1)}\, d_1^{(1)}\, q_0(x)\, q_1(x)^2 - 360\, c_1^{(1)}\, q_0(x)^2\, q_1(x)^2\\
&- 72\, d_1^{(1)}\, q_0(x)\, q_1(x)^3 + 432\, c_1^{(1)}\, d_1^{(1)}\, q_0(x)\, q_{0,x}(x) + 144\, c_1^{(1)}\, q_0(x)^2\, q_{0,x}(x)\\
&+ 216\, {d_1^{(1)}}^2\, q_1(x)\, q_{0,x}(x) + 288\, d_1^{(1)}\, q_0(x)\, q_1(x)\, q_{0,x}(x) + 216\, {c_1^{(1)}}^3\, q_0(x)\, q_{1,x}(x)\\
&- 216\, {d_1^{(1)}}^2\, q_0(x)\, q_{1,x}(x) - 144\, d_1^{(1)}\, q_0(x)^2\, q_{1,x}(x) + 108\, {c_1^{(1)}}^2\, d_1^{(1)}\, q_1(x)\, q_{1,x}(x)\\
&+ 432\, {c_1^{(1)}}^2\, q_0(x)\, q_1(x)\, q_{1,x}(x) + 216\, c_1^{(1)}\, d_1^{(1)}\, q_1(x)^2\, q_{1,x}(x) + 264\, c_1^{(1)}\, q_0(x)\, q_1(x)^2\, q_{1,x}(x)\\
&+ 60\, d_1^{(1)}\, q_1(x)^3\, q_{1,x}(x) - 72\, c_1^{(1)}\, d_1^{(1)}\, q_{0,x}(x)\, q_{1,x}(x) + 48\, c_1^{(1)}\, q_0(x)\, q_{0,x}(x)\, q_{1,x}(x)\\
&- 48\, d_1^{(1)}\, q_1(x)\, q_{0,x}(x)\, q_{1,x}(x) - 18\, {c_1^{(1)}}^3\, q_{1,x}(x)^2 + 126\, {d_1^{(1)}}^2\, q_{1,x}(x)^2 + 66\, d_1^{(1)}\, q_0(x)\, q_{1,x}(x)^2\\
&- 48\, {c_1^{(1)}}^2\, q_1(x)\, q_{1,x}(x)^2 - 34\, c_1^{(1)}\, q_1(x)^2\, q_{1,x}(x)^2 - 12\, c_1^{(1)}\, q_{0,x}(x)\, q_{1,x}(x)^2 + 42\, d_1^{(1)}\, q_{1,x}(x)^3\\
&- 432\, {c_1^{(1)}}^2\, q_0(x)\, q_{0,xx}(x) - 216\, c_1^{(1)}\, d_1^{(1)}\, q_1(x)\, q_{0,xx}(x) - 360\, c_1^{(1)}\, q_0(x)\, q_1(x)\, q_{0,xx}(x)\\
&- 72\, d_1^{(1)}\, q_1(x)^2\, q_{0,xx}(x) + 144\, d_1^{(1)}\, q_{0,x}(x)\, q_{0,xx}(x) + 72\, {c_1^{(1)}}^2\, q_{1,x}(x)\, q_{0,xx}(x)\\
&+ 108\, c_1^{(1)}\, q_1(x)\, q_{1,x}(x)\, q_{0,xx}(x) - 72\, c_1^{(1)}\, q_{0,xx}(x)^2 + 108\, c_1^{(1)}\, {d_1^{(1)}}^2\, q_{1,xx}(x)\\
&+ 72\, c_1^{(1)}\, d_1^{(1)}\, q_0(x)\, q_{1,xx}(x) + 12\, c_1^{(1)}\, q_0(x)^2\, q_{1,xx}(x) + 180\, {d_1^{(1)}}^2\, q_1(x)\, q_{1,xx}(x)\\
&+ 120\, d_1^{(1)}\, q_0(x)\, q_1(x)\, q_{1,xx}(x) + 36\, c_1^{(1)}\, d_1^{(1)}\, q_{1,x}(x)\, q_{1,xx}(x) + 12\, c_1^{(1)}\, q_0(x)\, q_{1,x}(x)\, q_{1,xx}(x)\\
&+ 60\, d_1^{(1)}\, q_1(x)\, q_{1,x}(x)\, q_{1,xx}(x) + 3\, c_1^{(1)}\, q_{1,x}(x)^2\, q_{1,xx}(x) + 72\, {c_1^{(1)}}^2\, q_0(x)\, q_{1,xxx}(x)\\
&+ 36\, c_1^{(1)}\, d_1^{(1)}\, q_1(x)\, q_{1,xxx}(x) + 60\, c_1^{(1)}\, q_0(x)\, q_1(x)\, q_{1,xxx}(x) + 12\, d_1^{(1)}\, q_1(x)^2\, q_{1,xxx}(x)\\
&- 24\, d_1^{(1)}\, q_{0,x}(x)\, q_{1,xxx}(x) - 12\, {c_1^{(1)}}^2\, q_{1,x}(x)\, q_{1,xxx}(x) - 18\, c_1^{(1)}\, q_1(x)\, q_{1,x}(x)\, q_{1,xxx}(x)\\
&+ 24\, c_1^{(1)}\, q_{0,xx}(x)\, q_{1,xxx}(x) - 2\, c_1^{(1)}\, q_{1,xxx}(x)^2 + 36\, {d_1^{(1)}}^2\, q_{1,xxxx}(x)\\
&+ 24\, d_1^{(1)}\, q_0(x)\, q_{1,xxxx}(x) + 12\, d_1^{(1)}\, q_{1,x}(x)\, q_{1,xxxx}(x)\Big)\frac{1}{648},
\end{aligned}
$$

etc.

Concerning the dynamics of the zeros $\mu_j(x)$ and $\nu_\ell(x)$ of $D_{m-1}(z,x)$ and $N_m(z,x)$ one obtains the following Dubrovin-type equations.

Lemma 3.8. *Suppose the curve $\mathcal{K}_{m-1}$ is nonsingular and assume* (3.55) *to hold.*
(i) Suppose the zeros $\{\mu_j(x)\}_{j=1,\dots,m-1}$ of $D_{m-1}(\,\cdot\,,x)$ remain distinct in Ω_μ, where $\Omega_\mu \subseteq \mathbb{C}$

is open and connected. Then $\{\mu_j(x)\}_{j=1,\dots,m-1}$ *satisfy the system of differential equations*

$$\mu_{j,x}(x) = \frac{-\varepsilon(m)\, F_m(\mu_j(x),x)\big(3y(\hat{\mu}_j(x))^2 + S_m(\mu_j(x))\big)}{\prod\limits_{\substack{k=1\\k\neq j}}^{m-1}\big(\mu_j(x)-\mu_k(x)\big)}, \quad j=1,\dots,m-1, \tag{3.105}$$

with initial conditions

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,m-1} \subset \mathcal{K}_{m-1}, \tag{3.106}$$

for some fixed $x_0 \in \Omega_\mu$. *The initial value problem* (3.105), (3.106) *has a unique solution* $\{\hat{\mu}_j(x)\}_{j=1,\dots,m-1} \subset \mathcal{K}_{m-1}$ *satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_{m-1}), \quad j=1,\dots,m-1. \tag{3.107}$$

(ii) Suppose the zeros $\{\nu_\ell(x)\}_{\ell=0,\dots,m-1}$ *of* $N_m(\,\cdot\,,x)$ *remain distinct in* Ω_ν, *where* $\Omega_\nu \subseteq \mathbb{C}$ *is open and connected. Then* $\{\nu_\ell(x)\}_{\ell=0,\dots,m-1}$ *satisfy the system of differential equations*

$$\nu_{\ell,x}(x) = \frac{-\varepsilon(m)\, J_m(\nu_\ell(x),x)\big(3y(\hat{\nu}_\ell(x))^2 + S_m(\nu_\ell(x))\big)}{\prod\limits_{\substack{k=0\\k\neq \ell}}^{m-1}\big(\nu_\ell(x)-\nu_k(x)\big)}, \quad \ell=0,\dots,m-1, \tag{3.108}$$

with initial conditions

$$\{\hat{\nu}_\ell(x_0)\}_{\ell=0,\dots,m-1} \subset \mathcal{K}_{m-1}, \tag{3.109}$$

for some fixed $x_0 \in \Omega_\nu$. *The initial value problem* (3.108), (3.109) *has a unique solution* $\{\hat{\nu}_\ell(x)\}_{\ell=0,\dots,m-1} \subset \mathcal{K}_{m-1}$ *satisfying*

$$\hat{\nu}_\ell \in C^\infty(\Omega_\nu, \mathcal{K}_{m-1}), \quad \ell=0,\dots,m-1. \tag{3.110}$$

Proof. Combining (3.71), (3.72), (3.81) and (3.86) yields

$$\epsilon(m) D_{m-1,x}(\mu_j(x),x) = F_m(\mu_j(x),x)\big[3\left(y(\hat{\mu}_j(x)) - k_m(\mu_j(x))\right)^2 + S_m(\mu_j(x))\big] \tag{3.111}$$

which in turn implies (3.105) using (3.84). Similarly, combining (3.76), (3.77), (3.81) and (3.87) yields

$$\epsilon(m)\, N_{m,x}(\nu_\ell(x),x) = J_m(\nu_\ell(x),x)\big[3\left(y(\hat{\nu}_\ell(x)) - k_m(\nu_\ell(x))\right)^2 + S_m(\nu_\ell(x))\big] \tag{3.112}$$

implying (3.108) by means of (3.85). □

We emphasize that $2\,(y-k_m)$ in (2.67) and (2.68) and $3\,(y-k_m)^2+S_m$ in (3.105) and (3.108) is precisely the y-derivative of the Burchnall-Chaundy polynomial, that is, $\frac{\partial}{\partial y}\mathcal{F}_m(z,y)$.

We conclude this section with a few trace formulas for the Bsq invariants in terms of $\mu_j(x)$ and $\nu_\ell(x)$ analogous to the KdV case in Lemma 2.6.

Lemma 3.9. *Assume (3.55) and let* $x \in \mathbb{R}$. *Then*

(i). For $m=2$:

$$\frac{1}{6}\, q_{1,x}(x) + q_0(x) + d_0^{(2)}\, q_1(x) + {d_0^{(2)}}^3 = \mu_1(x) \tag{3.113}$$

and for $m>2$:

$m \equiv 1 \pmod 3$:

$$\frac{1}{6}\, q_{1,x}(x) - q_0(x) - 3\, d_1^{(1)} = \sum_{j_1=1}^{m-1} \mu_{j_1}(x),$$

$$\frac{1}{18}\, q_{1,xxxx}(x) + \frac{1}{9}\, q_{0,xxx}(x) + \frac{5}{18}\, q_1(x) q_{1,xx}(x) + \frac{7}{36}\, q_{1,x}(x)^2 + \frac{1}{3}\, q_0(x) q_{1,x}(x) + \frac{1}{9}\, q_1(x) q_{0,x}(x) + \frac{2}{27}\, q_1(x)^3 - q_0(x)^2 + \frac{1}{2}\, d_1^{(1)} q_{1,x}(x) - 3\, d_1^{(1)} q_0(x) + {c_1^{(1)}}^3 - 3\, {d_1^{(1)}}^2 - 3\, d_2^{(1)} = - \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{m-1} \mu_{j_1}(x)\, \mu_{j_2}(x), \tag{3.114}$$

etc.

$m \equiv 2 \pmod 3$:

$$\frac{1}{6}\, q_{1,x}(x) - q_0(x) + {d_0^{(2)}}^3 - 3 c_1^{(2)} = \sum_{j_1=1}^{m-1} \mu_{j_1}(x),$$

$$\frac{1}{18}\, q_{1,xxxx}(x) + \frac{1}{9}\, q_{0,xxx}(x) + \frac{5}{18}\, q_1(x) q_{1,xx}(x) + \frac{7}{36}\, q_{1,x}(x)^2 + \frac{1}{3}\, q_0(x) q_{1,x}(x) + \frac{1}{9}\, q_1(x) q_{0,x}(x) + \frac{2}{27}\, q_1(x)^3 - q_0(x)^2 + \big(\frac{1}{2}\, c_1^{(2)} - \frac{1}{6}\, {d_0^{(2)}}^3\big)\, q_{1,x}(x) + ({d_0^{(2)}}^3 - 3\, c_1^{(2)}) q_0(x) - 3\, {c_1^{(2)}}^2 - 3\, c_2^{(2)} + 3\, {d_0^{(2)}}^2 d_1^{(2)} = - \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{m-1} \mu_{j_1}(x)\, \mu_{j_2}(x), \tag{3.115}$$

etc.

(ii). For $m = 2$:

$$\frac{1}{3}\, q_{1,x}(x) - 2\, q_0(x) - d_0^{(2)}\, q_1(x) - {d_0^{(2)}}^3 = -\nu_0(x) - \nu_1(x) \tag{3.116}$$

and for $m > 2$:

$m \equiv 1 \pmod 3$:

$$\frac{1}{3}\, q_{1,x}(x) + 3\, d_1^{(1)} = - \sum_{\ell_1=0}^{m-1} \nu_{\ell_1}(x),$$

$$\frac{2}{9}\, q_{0,xxx}(x) + \frac{1}{3}\, q_0(x) q_{1,x}(x) + \frac{1}{18}\, q_{1,x}(x)^2 - \frac{1}{18}\, q_1(x) q_{1,xx}(x) + \frac{5}{9}\, q_{0,x}(x) q_1(x) - \frac{2}{27}\, q_1(x)^3 + d_1^{(1)} q_{1,x}(x) - {c_1^{(1)}}^3 + 3\, {d_1^{(1)}}^2 + 3\, d_2^{(1)} = \sum_{\substack{\ell_1,\ell_2=0 \\ \ell_1<\ell_2}}^{m-1} \nu_{\ell_1}(x)\, \nu_{\ell_2}(x), \tag{3.117}$$

etc.

$m \equiv 2 \pmod 3$:

$$\frac{1}{3}\, q_{1,x}(x) + 3\, c_1^{(2)} - {d_0^{(2)}}^3 = - \sum_{\ell_1=0}^{m-1} \nu_{\ell_1}(x),$$

$$\frac{2}{9}\,q_{0,xxx}(x) + \frac{1}{3}\,q_0(x)q_{1,x}(x) + \frac{1}{18}\,q_{1,x}(x)^2 - \frac{1}{18}\,q_1(x)q_{1,xx}(x) + \frac{5}{9}\,q_{0,x}(x)q_1(x)$$
$$- \frac{2}{27}\,q_1(x)^3 - \big(\frac{1}{3}\,d_0^{(2)^3} - c_1^{(2)}\big)\,q_{1,x}(x) + 3\,c_1^{(2)^2} + 3\,c_2^{(2)} - 3\,d_0^{(2)^2}\,d_1^{(2)} = \sum_{\substack{\ell_1,\ell_2=0\\ \ell_1<\ell_2}}^{m-1} \nu_{\ell_1}(x)\,\nu_{\ell_2}(x), \tag{3.118}$$

etc.

Here $c_1^{(\epsilon)}, c_2^{(\epsilon)}, d_0^{(\epsilon)}, d_1^{(\epsilon)}, d_2^{(\epsilon)}$ can be expressed in terms of zeros of $S_m(z)$ and $T_m(z)$ in analogy to (2.71).

Proof. It suffices to substitute (3.84) and (3.85) into (3.15) and (3.16) (taking into account (3.3)) and comparing powers of z. □

Explicit examples illustrating the formalism of this section are provided in Chapter 4.

3.3. Stationary Algebro-Geometric Solutions of the Boussinesq Hierarchy

In this section we continue our study of the stationary Bsq hierarchy, but now direct our efforts towards obtaining explicit Riemann theta function representations for the fundamental quantities ϕ and ψ, introduced in Section 3.2, and especially, for each of the potentials q_0 and q_1 associated with the differential expression L_3. As a result of our preparatory material in Sections 3.1 and 3.2, we are now able to simultaneously treat the class of algebro-geometric quasi-periodic solutions of the entire Bsq hierarchy, one of our principal aims in this paper.

In the following we freely employ the notation established in Appendices A and B and refer to this material whenever appropriate.

Lemma 3.10. *Let $x \in \mathbb{C}$. Near $P_\infty \in \mathcal{K}_{m-1}$, in terms of the local coordinate $\zeta = z^{-1/3}$, one has*

$$\phi(P,x) \underset{\zeta\to 0}{=} \frac{1}{\zeta}\sum_{j=0}^{\infty} \beta_j(x)\zeta^j \ \text{ as } P \to P_\infty, \tag{3.119}$$

where

$$\beta_0 = 1, \quad \beta_1 = 0, \quad \beta_2 = -\frac{1}{3}q_1, \quad \beta_3 = -\frac{1}{3}q_0 + \frac{1}{6}q_{1,x},$$
$$\beta_j = -\frac{1}{3}\Big(\beta_{j-2,xx} + q_1\beta_{j-2} + \sum_{k=2}^{j-1}(3\beta_{k,x}\beta_{j-k-1} + \beta_k\beta_{j-k}) + \sum_{\ell=1}^{j-1}\sum_{k=0}^{\ell}\beta_k\beta_{\ell-k}\beta_{j-\ell}\Big), \quad j \geq 4. \tag{3.120}$$

Proof. In terms of the local coordinate $\zeta = z^{-1/3}$, (3.89) reads

$$\phi_{xx} + 3\phi\phi_x + \phi^3 + q_1\phi = \zeta^{-3} - q_0 - 2^{-1}\,q_{1,x}. \tag{3.121}$$

A power series ansatz in (3.121) then yields the indicated Laurent series. □

Let $\theta(\underline{z})$ denote the Riemann theta function (cf. (A.59)) associated with $\mathcal{K}_{m-1}$ and an appropriately fixed homology basis. Next, choosing a convenient base point $P_0 \in \mathcal{K}_{m-1} \backslash \{P_\infty\}$, the vector of Riemann constants Ξ_{P_0} is given by (A.66), and the Abel maps $\underline{A}_{P_0}(\,\cdot\,)$ and $\underline{\alpha}_{P_0}(\,\cdot\,)$ are defined by (A.56) and (A.57), respectively. For brevity, define the function $\underline{z} : \mathcal{K}_{m-1} \times \sigma^{m-1}\mathcal{K}_{m-1} \to \mathbb{C}^{m-1}$ by

$$\underline{z}(P, \underline{Q}) = \Xi_{P_0} - \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\underline{Q}), \quad P \in \mathcal{K}_{m-1},\ \underline{Q} = (Q_1, \dots, Q_{m-1}) \in \sigma^{m-1}\mathcal{K}_{m-1}. \tag{3.122}$$

We note that by (A.81) and (A.82), $\underline{z}(\,\cdot\,, \underline{Q})$ is independent of the choice of base point P_0.

The normalized differential $\omega^{(3)}_{P_\infty,\hat{\nu}_0(x)}$ of the third kind (dtk) is the unique differential holomorphic on $\mathcal{K}_{m-1} \backslash \{P_\infty, \nu_0(x)\}$ with simple poles at P_∞ and $\hat{\nu}_0(x)$ with residues ± 1, respectively, that is,

$$\omega^{(3)}_{P_\infty,\hat{\nu}_0(x)}(P) \underset{\zeta\to 0}{=} \big(\zeta^{-1} + O(1)\big) d\zeta \text{ as } P \to P_\infty. \tag{3.123}$$

Then

$$\int_{P_0}^{P} \omega^{(3)}_{P_\infty,\hat{\nu}_0(x)} \underset{\zeta\to 0}{=} \ln(\zeta) + e^{(3)}(P_0) + O(\zeta) \text{ as } P \to P_\infty, \tag{3.124}$$

where $e^{(3)}(P_0)$ is an appropriate constant. Furthermore, let $\omega^{(2)}_{P_\infty,2}$ denote the normalized differential defined by

$$\omega^{(2)}_{P_\infty,2}(P) = -\sum_{j=1}^{m-1} \lambda_j \eta_j(P) - \frac{1}{3y(P)^2 + S_m(z)} \begin{cases} z^{2n} dz, & m = 3n+1, \\ y(P) z^n dz, & m = 3n+2, \end{cases} \tag{3.125}$$

where the constants $\{\lambda_j\}_{j=1,\dots,m-1}$ are determined by the normalization condition

$$\int_{a_j} \omega^{(2)}_{P_\infty,2} = 0, \quad j = 1, \dots, m-1, \tag{3.126}$$

and the differentials $\{\eta_j(P)\}_{j=1,\dots,m-1}$ (defined in (B.7)) form a basis for the space of holomorphic differentials. The b-periods of the differential $\omega^{(2)}_{P_\infty,2}$ are denoted by

$$\underline{U}_2^{(2)} = (U^{(2)}_{2,1}, \dots, U^{(2)}_{2,m-1}), \quad U^{(2)}_{2,j} = \frac{1}{2\pi i} \int_{b_j} \omega^{(2)}_{P_\infty,2}, \quad j = 1, \dots, m-1. \tag{3.127}$$

A straightforward Laurent expansion of (3.125) near P_∞ yields the following result.

Lemma 3.11. *Assume the curve $\mathcal{K}_{m-1}$ is nonsingular. Then the differential $\omega^{(2)}_{P_\infty,2}$ defined in* (3.125) *is a differential of the second kind* (dsk), *holomorphic on $\mathcal{K}_{m-1} \backslash \{P_\infty\}$ with a pole of order 2 at P_∞. In particular, near P_∞ in the local coordinate ζ, the differential $\omega^{(2)}_{P_\infty,2}$ has the Laurent series*

$$\omega^{(2)}_{P_\infty,2}(P) \underset{\zeta\to 0}{=} \big(\zeta^{-2} + u + w\zeta + O(\zeta^2)\big) d\zeta \ \text{ as } P \to P_\infty, \tag{3.128}$$

where

$$u = \begin{cases} \lambda_{m-1} - c_1^{(1)} & \text{for } m \equiv 1 \pmod 3, \\ \lambda_{m-n-1} - (d_0^{(2)})^2 & \text{for } m \equiv 2 \pmod 3, \end{cases} \tag{3.129}$$

and

$$w = \begin{cases} \lambda_{m-n-1} - 2d_1^{(1)} & \text{for } m \equiv 1 \pmod 3, \\ (d_0^{(2)})^3 - c_1^{(2)} - d_0^{(2)}\lambda_{m-n-1} + \lambda_{m-1} & \text{for } m \equiv 2 \pmod 3. \end{cases} \tag{3.130}$$

From Lemma 3.11 one infers

$$\int_{P_0}^{P} \omega_{P_\infty,2}^{(2)} \underset{\zeta\to 0}{=} -\zeta^{-1} + e_2^{(2)}(P_0) + u\zeta + 2^{-1}w\zeta^2 + O(\zeta^3) \text{ as } P \to P_\infty, \tag{3.131}$$

where $e_2^{(2)}(P_0)$ is an appropriate constant.

The theta function representation of $\phi(P,x)$ then reads as follows.

Theorem 3.12. *Let $P = (z,y) \in \mathcal{K}_{m-1}\backslash\{P_\infty\}$, $(z,x) \in \mathbb{C}^2$. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ are nonspecial. Then*

$$\phi(P,x) = \frac{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x)))\,\theta(\underline{z}(P, \underline{\hat{\nu}}(x)))}{\theta(\underline{z}(P_\infty, \underline{\hat{\nu}}(x)))\,\theta(\underline{z}(P, \underline{\hat{\mu}}(x)))} \exp\left(e^{(3)}(P_0) - \int_{P_0}^{P} \omega_{P_\infty,\hat{\nu}_0(x)}^{(3)} \right). \tag{3.132}$$

Proof. Let Φ be defined by the right-hand side of (3.132) with the aim to prove that $\phi = \Phi$. From (3.124) it follows that

$$\exp\left(e^{(3)}(P_0) - \int_{P_0}^{P} \omega_{P_\infty,\hat{\nu}_0(x)}^{(3)} \right) \underset{\zeta\to 0}{=} \zeta^{-1} + O(1). \tag{3.133}$$

Using (3.88) we immediately see that ϕ has simple poles at $\underline{\hat{\mu}}(x)$ and P_∞, and simple zeros at $\hat{\nu}_0(x)$ and $\underline{\hat{\nu}}(x)$. By (3.133) and a special case of Riemann's vanishing theorem (Theorem A.22), we see that Φ has the same properties. Using the Riemann-Roch theorem (Theorem A.12), we conclude that the holomorphic function $\Phi/\phi = c$, a constant with respect to P. Using (3.133) and Lemma 3.10, one computes

$$\frac{\Phi}{\phi} \underset{\zeta\to 0}{=} \frac{(1 + O(\zeta))(\zeta^{-1} + O(1))}{\zeta^{-1} + O(\zeta)} \underset{\zeta\to 0}{=} 1 + O(\zeta) \text{ as } P \to P_\infty, \tag{3.134}$$

from which one concludes $c = 1$. ☐

Similarly, the theta function representation of the Baker-Akhiezer function $\psi(P,x,x_0)$ is summarized in the following theorem.

Theorem 3.13. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular. Let $P = (z,y) \in \mathcal{K}_{m-1}\backslash\{P_\infty\}$ and let x, $x_0 \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ are nonspecial, for $x \in \Omega_\mu$. Then*

$$\psi(P,x,x_0) = \frac{\theta(\underline{z}(P, \underline{\hat{\mu}}(x)))\,\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x_0)))}{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x)))\,\theta(\underline{z}(P, \underline{\hat{\mu}}(x_0)))} \exp\left((x - x_0)\big(e_2^{(2)}(P_0) - \int_{P_0}^{P} \omega_{P_\infty,2}^{(2)}\big) \right). \tag{3.135}$$

Proof. Assume temporarily that

$$\mu_j(x) \neq \mu_{j'}(x) \text{ for } j \neq j' \text{ and } x \in \widetilde{\Omega}_\mu \subseteq \Omega_\mu, \tag{3.136}$$

where $\widetilde{\Omega}_\mu$ is open and connected. For the Baker-Akhiezer function ψ we will use the same strategy as was used in the previous proof. However, the situation is slightly more involved in that ψ has an essential singularity at P_∞. Let Ψ denote the right-hand side of (3.135). In order to prove that $\psi = \Psi$, one first observes that since

$$\psi(P,x,x_0) = \exp\bigg(\int_{x_0}^{x} dx' \phi(P,x')\bigg), \tag{3.137}$$

the zeros and poles of ψ can come only from simple poles in the integrand (with positive and negative residues respectively). Using (3.86) and (3.105), one computes

$$\begin{aligned}
\phi &= \frac{F_m y^2 + A_m y + \frac{2}{3}F_m S_m + \frac{1}{3}\varepsilon(m) D_{m,x}}{\varepsilon(m)D_m} \\
&= \frac{1}{3}\frac{F_m}{\varepsilon(m)D_m}\big(3y^2 + S_m\big) + \frac{1}{3}\frac{3A_m y + F_m S_m}{\varepsilon(m)D_m} + \frac{1}{3}\frac{D_{m,x}}{D_m} \\
&= \frac{2}{3}\frac{F_m}{\varepsilon(m)D_m}\big(3y^2 + S_m\big) - \frac{1}{3}\sum_{k=1}^{m-1}\frac{\mu_{k,x}}{z-\mu_k} + O(1) \\
&= -\frac{\mu_{j,x}}{z-\mu_j} + O(1), \text{ as } P \to \hat{\mu}_j(x).
\end{aligned}$$

More concisely,

$$\phi(P,x') = \frac{\partial}{\partial x'}\ln(z-\mu_j(x')) + O(1) \text{ for } P \text{ near } \hat{\mu}_j(x'). \tag{3.138}$$

Hence

$$\begin{aligned}
&\exp\Big(\int_{x_0}^{x} dx' \big(\frac{\partial}{\partial x'}\ln(z-\mu_j(x')) + O(1)\big)\Big) \\
&= \begin{cases} (z-\mu_j(x))O(1) & \text{for } P \text{ near } \hat{\mu}_j(x) \neq \hat{\mu}_j(x_0), \\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(x) = \hat{\mu}_j(x_0), \\ (z-\mu_j(x_0))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0) \neq \hat{\mu}_j(x), \end{cases}
\end{aligned} \tag{3.139}$$

where $O(1) \neq 0$ in (3.139). Consequently, all zeros of ψ and Ψ on $\mathcal{K}_{m-1}\backslash\{P_\infty\}$ are simple and coincide. It remains to identify the essential singularity of ψ and Ψ at P_∞. From (3.119), we infer

$$\int_{x_0}^{x} dx' \phi(P,x') \underset{\zeta\to 0}{=} (x-x_0)(\zeta^{-1} + O(\zeta)) \text{ as } P \to P_\infty. \tag{3.140}$$

Looking at (3.131) we see that this coincides with the singularity in the exponent of Ψ near P_∞. The uniqueness result in Lemma A.26 for Baker-Akhiezer functions then completes the proof that $\Psi = \psi$ as both functions share the same singularities and zeros. The extension of this result from $x \in \widetilde{\Omega}_\mu$ to $x \in \Omega_\mu$ then simply follows from the continuity of $\underline{\alpha}_{P_0}$ and the hypothesis of $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ being nonspecial for $x \in \Omega_\mu$. □

Next it is necessary to introduce two further polynomials K_m and L_m with respect to the variable $z \in \mathbb{C}$,

$$K_m(z,x) = (\varepsilon(m)N_m(z,x) - J_m(z,x)C_m(z,x))(G_m(z,x) + 2^{-1}F_{m,x}(z,x))^{-1}, \tag{3.141}$$

$$L_m(z,x) = (\varepsilon(m)D_{m-1}(z,x) - (G_m(z,x) - 2^{-1}F_{m,x}(z,x))A_m(z,x))F_m(z,x)^{-1}. \tag{3.142}$$

In analogy to our polynomials A_m–N_m introduced in (3.60)–(3.66), it is possible to derive explicit expressions of K_m and L_m directly in terms of F_m and G_m and their x-derivatives. These expressions then prove, in particular, the polynomial character of K_m and L_m with respect to z, but we here omit the rather lengthy formulas since they can be generated with the help of symbolic calculation programs such as Maple or Mathematica.

Lemma 3.14. *Let $x \in \mathbb{C}$. Then*

$$L_m(\mu_j(x), x) = -\big(G_m(\mu_j(x), x) - 2^{-1}F_{m,x}(\mu_j(x), x)\big)y(\hat{\mu}_j(x)), \tag{3.143}$$

for $j = 1, \dots, m-1$ and

$$K_m(\nu_\ell(x), x) = J_m(\nu_\ell(x), x)y(\hat{\nu}_\ell(x)), \tag{3.144}$$

for $\ell = 0, \dots, m-1$.

The well-known linearization property of the Abel map for completely integrable systems of soliton-type, is next verified in the context of the Bsq hierarchy.

Theorem 3.15. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular and let x, $x_0 \in \mathbb{C}$. Then*

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) + \underline{U}_2^{(2)}(x - x_0), \tag{3.145}$$

$$\underline{A}_{P_0}(\hat{\nu}_0(x)) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x)}) = \underline{A}_{P_0}(\hat{\nu}_0(x_0)) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0)}) + \underline{U}_2^{(2)}(x - x_0). \tag{3.146}$$

Proof. We prove only (3.145) as (3.146) follows *mutatis mutandis* (or from (3.145) and Abel's theorem, Theorem A.14). Assume temporarily that

$$\mu_j(x) \neq \mu_{j'}(x) \text{ for } j \neq j' \text{ and } x \in \widetilde{\Omega}_\mu \subseteq \mathbb{C}, \tag{3.147}$$

where $\widetilde{\Omega}_\mu$ is open and connected. Then using (3.105), (B.7), and (B.9), one computes

$$\begin{aligned}
\frac{d}{dx}\alpha_{P_0,\ell}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) &= \sum_{j=1}^{m-1} \mu_{j,x}(x)\omega_\ell(\hat{\mu}_j(x)) \\
&= -\varepsilon(m) \sum_{k=1}^{m-n-1} e_\ell(k) \sum_{j=1}^{m-1} \mu_j(x)^{k-1} F_m(\mu_j(x), x) \prod_{\substack{p=1 \\ p\neq j}}^{m-1} \big(\mu_j(x) - \mu_p(x)\big)^{-1} \\
&\quad - \varepsilon(m) \sum_{k=1}^{n} e_\ell(k+m-n-1) \sum_{j=1}^{m-1} \mu_j(x)^{k-1} A_m(\mu_j(x), x) \prod_{\substack{p=1 \\ p\neq j}}^{m-1} \big(\mu_j(x) - \mu_p(x)\big)^{-1}.
\end{aligned} \tag{3.148}$$

Next we consider the two cases $m = 3n + 1$ and $m = 3n + 2$ separately and substitute the polynomials $F_m(\mu_j(x), x)$ and $A_m(\mu_j(x), x)$ in the variable $\mu_j(x)$ into (3.148). Using a standard Lagrange interpolation argument then yields

$$\frac{d}{dx}\alpha_{P_0,\ell}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = -\begin{cases} e_\ell(m-1), & m = 3n+1, \\ e_\ell(m-n-1), & m = 3n+2. \end{cases} \tag{3.149}$$

The result now follows for $x \in \widetilde{\Omega}_\mu$, using (3.127), (3.149), (B.11), and (B.16). By continuity of $\underline{\alpha}_{P_0}$, this result extends from $x \in \widetilde{\Omega}_\mu$ to $x \in \mathbb{C}$. $\square$

We conclude this section with the theta function representations for the stationary Bsq solutions q_0, q_1 (the analog of the Its-Matveev formula in the KdV context).

Theorem 3.16. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular and let $x \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ are nonspecial for $x \in \Omega_\mu$. Then*

$$q_0(x) = 3\,\partial_{\underline{U}_3^{(2)}}\partial_x \ln(\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x)))) + (3/2)w, \tag{3.150}$$

$$q_1(x) = 3\,\partial_x^2 \ln(\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x)))) + 3u, \tag{3.151}$$

with u and w defined in (3.129) and (3.130), that is,

$$u = \begin{cases} \lambda_{m-1} - c_1^{(1)} & \text{for } m \equiv 1 \pmod 3, \\ \lambda_{m-n-1} - (d_0^{(2)})^2 & \text{for } m \equiv 2 \pmod 3, \end{cases} \tag{3.152}$$

and

$$w = \begin{cases} \lambda_{m-n-1} - 2d_1^{(1)} & \text{for } m \equiv 1 \pmod 3, \\ (d_0^{(2)})^3 - c_1^{(2)} - d_0^{(2)}\lambda_{m-n-1} + \lambda_{m-1} & \text{for } m \equiv 2 \pmod 3. \end{cases} \tag{3.153}$$

Proof. Using Lemma 3.11 and Theorem 3.13, one can write ψ near P_∞ in the coordinate ζ, as

$$\begin{aligned} \psi(P, x, x_0) \underset{\zeta\to 0}{=} & \big(1 + \alpha_1(x)\zeta + \alpha_2(x)\zeta^2 + O(\zeta^3)\big) \\ & \times \exp\big((x - x_0)(\zeta^{-1} - u\zeta - 2^{-1}w\zeta^2 + O(\zeta^3))\big) \text{ as } P \to P_\infty, \end{aligned} \tag{3.154}$$

where the terms $\alpha_1(x)$ and $\alpha_2(x)$ in (3.154) come from the Taylor expansion about P_∞ of the ratios of the theta functions in (3.135), and the exponential term stems from substituting (3.131) into (3.135). Using (3.154) and its x-derivatives one can show that

$$\psi_{xxx} + 3(u - \alpha_{1,x})\psi_x + 3(2^{-1}w - \alpha_{1,xx} + \alpha_1\alpha_{1,x} - \alpha_{2,x})\psi - \zeta^{-3}\psi = O(\zeta)\psi. \tag{3.155}$$

Since $O(\zeta)\psi$ is another Baker-Akhiezer function with the same essential singularity at P_∞ and the same divisor on $\mathcal{K}_{m-1}\backslash\{P_\infty\}$, the uniqueness theorem for Baker-Akhiezer functions (cf. Lemma A.26) then yields $O(\zeta) = 0$. Hence

$$q_0(x) = 3\big(2^{-1}w - 2^{-1}\alpha_{1,xx}(x) + \alpha_1(x)\alpha_{1,x}(x) - \alpha_{2,x}(x)\big), \tag{3.156}$$

$$q_1(x) = 3(u - \alpha_{1,x}(x)), \tag{3.157}$$

where

$$\alpha_{1,x}(x) = -\partial_x^2 \ln\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x))), \tag{3.158}$$

$$-2^{-1}\alpha_{1,xx}(x) + \alpha_1(x)\alpha_{1,x}(x) - \alpha_{2,x}(x) = \partial_{\underline{U}_3^{(2)}}\partial_x \ln\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x))). \tag{3.159}$$

Here

$$\partial_{\underline{U}_3^{(2)}} = \sum_{j=1}^{m-1} U_{3,j}^{(2)} \frac{\partial}{\partial z_j} \tag{3.160}$$

denotes the directional derivative in the direction of the vector of b-periods $\underline{U}_3^{(2)}$, defined by

$$\underline{U}_3^{(2)} = (U_{3,1}^{(2)}, \dots, U_{3,m-1}^{(2)}), \quad U_{3,j}^{(2)} = \frac{1}{2\pi i}\int_{b_j} \omega_{P_\infty,3}^{(2)}, \quad j = 1, \dots, m-1, \tag{3.161}$$

with $\omega_{P_\infty,3}^{(2)}$ the *dsk* holomorphic on $\mathcal{K}_{m-1}\backslash\{P_\infty\}$ with a pole of order 3 at P_∞,

$$\omega_{P_\infty,3}^{(2)}(P) \underset{\zeta\to 0}{=} \big(\zeta^{-3} + O(1)\big)d\zeta \text{ as } P \to P_\infty. \tag{3.162}$$

Combining (3.156)–(3.159) then proves (3.150) and (3.151). □

For interesting spectral characterizations of third-order (in fact, odd-order) self-adjoint differential operators with quasi-periodic coefficients we refer to [**48**].

3.4. The Time-Dependent Boussinesq Formalism

In this section we return to the recursive approach outlined in Section 3.1 and extend the polynomial approach of Sections 3.2 and 3.3 to the time-dependent Bsq hierarchy.

We start with a stationary algebro-geometric solution $(q_0^{(0)}(x), q_1^{(0)}(x))$ associated with $\mathcal{K}_{m-1}$ satisfying

$$\mathrm{Bsq}_m(q_0^{(0)}, q_1^{(0)}) = \begin{cases} -3\, f_{n+1,x}^{(\varepsilon)} = 0, \\ -3\, g_{n+1,x}^{(\varepsilon)} = 0, \end{cases} \quad x \in \mathbb{C},\ m = 3n + \varepsilon \tag{3.163}$$

for some fixed $\varepsilon \in \{1, 2\}$, $n \in \mathbb{N}_0$, and a given set of integration constants $\{c_\ell^{(\varepsilon)}\}_{\ell=1,\dots,n}$, $\{d_\ell^{(\varepsilon)}\}_{\ell=0,\dots,n}$. Our aim is to construct the rth Bsq flow

$$\mathrm{Bsq}_r(q_0, q_1) = 0, \quad (q_0(x, t_{0,r}), q_1(x, t_{0,r})) = (q_0^{(0)}(x), q_1^{(0)}(x)), \quad x \in \mathbb{C},\ r = 3s + \varepsilon' \tag{3.164}$$

for some fixed $\varepsilon' \in \{1, 2\}$, $s \in \mathbb{N}_0$, and $t_{0,r} \in \mathbb{C}$. In terms of Lax pairs this amounts to solving

$$\frac{d}{d\,t_r} L_3(t_r) - [\widetilde{P}_r(t_r), L_3(t_r)] \quad = \quad 0, \quad t_r \in \mathbb{C}, \tag{3.165}$$

$$[P_m(t_{0,r}), L_3(t_{0,r})] \ = 0. \tag{3.166}$$

As a consequence one obtains

$$[P_m(t_r), L_3(t_r)] = 0, \qquad t_r \in \mathbb{C}, \tag{3.167}$$

$$P_m(t_r)^3 + P_m(t_r)\, S_m(L_3(t_r)) - T_m(L_3(t_r)) = 0, \qquad t_r \in \mathbb{C}, \tag{3.168}$$

since the Bsq flows are isospectral deformations of $L_3(t_{0,r})$.

We emphasize that the integration constants $\{\tilde{c}_\ell^{(\varepsilon')}\}$ and $\{\tilde{d}_\ell^{(\varepsilon')}\}$ in $\widetilde{P}_r$, and $\{c_\ell^{(\varepsilon)}\}$ and $\{d_\ell^{(\varepsilon)}\}$ in P_m, are independent of each other (even for $r = m$). Hence we shall employ the notation $\widetilde{P}_r$, $\widetilde{F}_r$, $\widetilde{G}_r$, $\widetilde{H}_r$, etc., in order to distinguish them from P_m, F_m, G_m, H_m, etc. In addition we follow a more elaborate approach inspired by Hirota's τ-function approach and indicate the individual rth Bsq flow by a separate time variable $t_r \in \mathbb{C}$. (The latter notation suggests considering all Bsq flows simultaneously by introducing $\underline{t} = (t_1, t_2, t_4, t_5, \dots)$.)

Instead of working directly with (3.165) and (3.167) we find it preferable to take the following two equations as our point of departure (never mind their somewhat intimidating size),

$$\begin{aligned} q_{0,t_r} = &-\frac{1}{6}\,\widetilde{F}_{r,xxxxx} - \frac{5}{6}\,q_1\,\widetilde{F}_{r,xxx} - \frac{5}{4}\,q_{1,x}\,\widetilde{F}_{r,xx} - (\frac{3}{4}\,q_{1,xx} + \frac{2}{3}\,q_1^2)\,\widetilde{F}_{r,x} \\ &- (\frac{1}{6}\,q_{1,xxx} + \frac{2}{3}\,q_1 q_{1,x})\,\widetilde{F}_r - 3(z - q_0)\,\widetilde{G}_{r,x} + q_{0,x}\,\widetilde{G}_r, \\ q_{1,t_r} = &\ 2\,\widetilde{G}_{r,xxx} + 2\,q_1\,\widetilde{G}_{r,x} + q_{1,x}\,\widetilde{G}_r - 3\,(z - q_0)\,\widetilde{F}_{r,x} + 2\,q_{0,x}\,\widetilde{F}_r, \qquad (x, t_r) \in \mathbb{C}^2, \end{aligned} \tag{3.169}$$

$$-\frac{1}{6}\,F_{m,xxxx}F_m + \frac{1}{6}\,F_{m,xxx}F_{m,x} - \frac{1}{12}\,F_{m,xx}^2 - \frac{5}{6}\,q_1 F_{m,xx}F_m$$

$$-\frac{5}{12}\,q_{1,x}F_{m,x}F_m+\frac{5}{12}\,q_1F_{m,x}^2-\frac{1}{3}\big(\frac{1}{2}\,q_{1,xx}+q_1^2\big)F_m^2 \\ +2\,G_{m,xx}G_m-G_{m,x}^2+q_1G_m^2-3\,(z-q_0)F_mG_m=S_m(z),\quad (x,t_r)\in\mathbb{C}^2, \tag{3.170}$$

$$\frac{1}{18}\,F_{m,xxxx}F_{m,xx}F_m-\frac{1}{24}\,F_{m,xxxx}F_{m,x}^2+\frac{1}{18}\,q_1F_{m,xxxx}F_m^2+\frac{1}{36}\,F_{m,xxx}F_{m,xx}F_{m,x} \\ -\frac{1}{36}F_mF_{m,xxx}^2-\frac{1}{18}\,q_{1,x}F_{m,xxx}F_m^2-\frac{1}{9}\,q_1F_{m,xxx}F_{m,x}F_m-\frac{1}{108}\,F_{m,xx}^3 \\ +\frac{2}{9}\,q_{1,x}F_{m,xx}F_{m,x}F_m+\frac{1}{18}\,q_{1,xx}F_{m,xx}F_m^2-\frac{7}{72}\,q_1F_{m,xx}F_{m,x}^2+\frac{5}{18}\,q_1^2F_{m,xx}F_m^2 \\ +\frac{7}{36}\,q_1F_{m,xx}^2F_m-\frac{1}{24}\,q_{1,xx}F_{m,x}^2F_m-\frac{7}{48}\,q_{1,x}F_{m,x}^3-\frac{1}{6}\,q_1^2F_{m,x}^2F_m+\frac{1}{12}\,q_{1,x}q_1F_{m,x}F_m^2 \\ +\big(\frac{2}{27}\,q_1^3-\frac{1}{36}\,q_{1,x}^2+\frac{1}{18}\,q_{1,xx}q_1+(z-q_0)^2\big)F_m^3+(z-q_0)G_m^3+\frac{1}{6}\,F_{m,xxxx}G_m^2 \\ -\frac{1}{3}\,F_{m,xxx}G_{m,x}G_m+F_mG_{m,xx}^2+\frac{1}{3}\,F_{m,xx}\big(G_{m,x}^2+G_{m,xx}G_m\big)-F_{m,x}G_{m,xx}G_{m,x} \\ -q_1(z-q_0)F_m^2G_m+\frac{2}{3}\,q_1^2F_mG_m^2+\frac{5}{6}\,q_1F_{m,xx}G_m^2-\frac{4}{3}\,q_1F_{m,x}G_{m,x}G_m+\frac{1}{3}\,q_1F_mG_{m,x}^2 \\ +\frac{7}{12}\,q_{1,x}F_{m,x}G_m^2+\frac{4}{3}\,q_1F_mG_{m,xx}G_m+\frac{1}{6}\,q_{1,xx}F_mG_m^2-\frac{1}{3}\,q_{1,x}F_mG_{m,x}G_m \\ +(z-q_0)F_{m,x}F_mG_{m,x}-\frac{1}{4}\,(z-q_0)F_{m,x}^2G_m-2\,(z-q_0)F_m^2G_{m,xx}=T_m(z), \tag{3.171} \\ (x,t_r)\in\mathbb{C}^2,$$

where (cf. (3.15), (3.16))

$$F_m(z,x,t_r)=\sum_{\ell=0}^{n}f_{n-\ell}^{(\varepsilon)}(x,t_r)z^\ell,\quad F_m(z,x,t_{0,r})=\sum_{\ell=0}^{n}f_{n-\ell}^{(\varepsilon),(0)}(x)z^\ell, \tag{3.172}$$

$$G_m(z,x,t_r)=\sum_{\ell=0}^{n}g_{n-\ell}^{(\varepsilon)}(x,t_r)z^\ell,\quad G_m(z,x,t_{0,r})=\sum_{\ell=0}^{n}g_{n-\ell}^{(\varepsilon),(0)}(x)z^\ell \tag{3.173}$$

for fixed $t_{0,r}\in\mathbb{C}$, $m=3n+\varepsilon$, $r=3s+\varepsilon'$, $n,s\in\mathbb{N}_0$, $\varepsilon,\varepsilon'\in\{1,2\}$. Here $f_\ell^{(\varepsilon)}(x,t_r), g_\ell^{(\varepsilon)}(x,t_r)$ and $f_\ell^{(\varepsilon),(0)}(x), g_\ell^{(\varepsilon),(0)}(x)$ are defined as in (3.3) with $(q_0(x),q_1(x))$ replaced by $(q_0(x,t_r), q_1(x,t_r))$, and $(q_0^{(0)}(x),q_1^{(0)}(x))$, respectively.

In analogy to (3.85) one introduces

$$D_{m-1}(z,x,t_r)=\prod_{j=1}^{m-1}\left(z-\mu_j(x,t_r)\right),\quad N_m(z,x,t_r)=\prod_{\ell=0}^{m-1}\left(z-\nu_\ell(x,t_r)\right), \tag{3.174}$$

where D_{m-1} and N_m are defined as in (3.65) and (3.66). This implies in particular (cf. (3.74)),

$$D_{m-1}(z,x,t_r)N_m(z,x,t_r)=B_m(z,x,t_r)\,E_m(z,x,t_r)-T_m(z)\big(A_m(z,x,t_r) \\ \times\big(G_m(z,x,t_r)+2^{-1}\,F_{m,x}(z,x,t_r)\big)-F_m(z,x,t_r)\,C_m(z,x,t_r)\big), \tag{3.175}$$

and A_m, B_m, C_m, D_{m-1}, E_m, J_m, and N_m are defined as in (3.60)–(3.66). Hence (3.69)–(3.77) also hold in the present context. Moreover, we recall

Lemma 3.17. *Assume* (3.169)–(3.173) *and let* $(z,x,t_r)\in\mathbb{C}^3$. *Then*

$$
\begin{aligned}
(i)\ D_{m-1,t_r}(z,x,t_r) &= D_{m-1,x}(z,x,t_r)\Big(\widetilde{G}_r(z,x,t_r)-\frac{1}{2}\,\widetilde{F}_{r,x}(z,x,t_r)\\
&\quad -\frac{\widetilde{F}_r(z,x,t_r)}{F_m(z,x,t_r)}\big(G_m(z,x,t_r)-\frac{1}{2}\,F_{m,x}(z,x,t_r)\big)\Big)+D_{m-1}(z,x,t_r)\\
&\quad \times 3\,\Big(\widetilde{H}_r(z,x,t_r)-\frac{\widetilde{F}_r(z,x,t_r)}{F_m(z,x,t_r)}\,H_m(z,x,t_r)\Big).
\end{aligned}
\tag{3.176}
$$

$$
\begin{aligned}
(ii)\ N_{m,t_r}(z,x,t_r) &= N_{m,x}(z,x,t_r)\Big(\widetilde{G}_r(z,x,t_r)+\frac{1}{2}\,\widetilde{F}_{r,x}(z,x,t_r)-\frac{\widetilde{J}_r(z,x,t_r)}{J_m(z,x,t_r)}\\
&\quad \times\big(G_m(z,x,t_r)+\frac{1}{2}\,F_{m,x}(z,x,t_r)\big)\Big)-N_m(z,x,t_r)\Big(q_1(x,t_r)\,\widetilde{F}_r(z,x,t_r)\\
&\quad +\widetilde{F}_{r,xx}(z,x,t_r)-\frac{\widetilde{J}_r(z,x,t_r)}{J_m(z,x,t_r)}\big(q_1(x,t_r)\,F_m(z,x,t_r)+F_{m,xx}(z,x,t_r)\big)\Big).
\end{aligned}
\tag{3.177}
$$

Proof. In order to prove (3.176) one combines

$$
\begin{aligned}
\partial_{t_r}\partial_x\big(\ln D_{m-1}(z,x,t_r)\big) &= \partial_x\partial_{t_r}\big(\ln D_{m-1}(z,x,t_r)\big)\\
&= \big(\phi(P,x,t_r)+\phi(P^*,x,t_r)+\phi(P^{**},x,t_r)\big)_{t_r},
\end{aligned}
$$

(3.188), (3.192), and

$$
\begin{aligned}
\phi(P)^2+\phi(P^*)^2+\phi(P^{**})^2 = &-\partial_x\big(\frac{D_{m-1,x}}{D_{m-1}}\big)-\frac{G_m-\frac{1}{2}\,F_{m,x}}{F_m}\frac{D_{m-1,x}}{D_{m-1}}\\
&-\frac{1}{F_m}\,\big(\frac{1}{2}\,F_{m,xx}+2\,q_1F_m-3\,G_{m,x}\big).
\end{aligned}
\tag{3.178}
$$

Similarly, in order to prove (3.177) one combines

$$
\partial_{t_r}\Big(\frac{N_m(z,x,t_r)}{D_{m-1}(z,x,t_r)}\Big)=\big(\phi(P,x,t_r)\phi(P^*,x,t_r)\phi(P^{**},x,t_r)\big)_{t_r},
$$

(3.77), (3.188), (3.178), and

$$
\frac{1}{\phi(P)}+\frac{1}{\phi(P^*)}+\frac{1}{\phi(P^{**})}=\frac{2(G_m+\frac{1}{2}\,F_{m,x})S_m+3\,E_m}{\epsilon(m)N_m}.
\tag{3.179}
$$

□

Similarly, Lemma 3.6 remains valid and one obtains

$$
\phi(P,x,t_r)=\frac{(G_m(z,x,t_r)+\frac{1}{2}\,F_{m,x}(z,x,t_r))y(P)+C_m(z,x,t_r)}{F_m(z,x,t_r)y(P)-A_m(z,x,t_r)}
\tag{3.180}
$$

$$
=\frac{F_m(z,x,t_r)y(P)^2+A_m(z,x,t_r)y(P)+B_m(z,x,t_r)}{\varepsilon(m)D_{m-1}(z,x,t_r)}
\tag{3.181}
$$

$$
=\frac{-\varepsilon(m)N_m(z,x,t_r)}{(G_m(z,x,t_r)+\frac{1}{2}\,F_{m,x}(z,x,t_r))y(P)^2-C_m(z,x,t_r)y(P)-E_m(z,x,t_r)},
\tag{3.182}
$$

$$
P=(z,y)\in\mathcal{K}_{m-1}.
$$

In analogy to (3.86) and (3.87) one then introduces (the analogs of) Dirichlet and Neumann data by

$$\hat{\mu}_j(x,t_r) = \big(\mu_j(x,t_r), \frac{A_m(\mu_j(x,t_r),x,t_r)}{F_m(\mu_j(x,t_r),x,t_r)}\big) \in \mathcal{K}_{m-1}, \quad j=1,\dots,m-1, \quad (x,t_r)\in\mathbb{C}^2, \tag{3.183}$$

$$\hat{\nu}_\ell(x,t_r) = \big(\nu_\ell(x,t_r), -\frac{C_m(\nu_\ell(x,t_r),x,t_r)}{G_m(\nu_\ell(x,t_r),x,t_r)+\frac{1}{2}F_{m,x}(\nu_\ell(x,t_r),x,t_r)}\big) \in \mathcal{K}_{m-1}, \quad \ell=0,\dots,m-1, \quad (x,t_r)\in\mathbb{C}^2 \tag{3.184}$$

and hence infers that the divisor $\big(\phi(P,x,t_r)\big)$ of $\phi(P,x,t_r)$ is given by

$$\big(\phi(P,x,t_r)\big) = \mathcal{D}_{\hat{\nu}_0(x,t_r),\dots,\hat{\nu}_{m-1}(x,t_r)}(P) - \mathcal{D}_{P_\infty,\hat{\mu}_1(x,t_r),\dots,\hat{\mu}_{m-1}(x,t_r)}(P). \tag{3.185}$$

Next we define the time-dependent BA-function $\psi(P,x,x_0,t_r,t_{0,r})$

$$\begin{aligned}\psi(P,x,x_0,t_r,t_{0,r}) &= \exp\bigg(\int_{x_0}^{x} dx'\phi(P,x',t_r) + \int_{t_{0,r}}^{t_r} ds\big(\widetilde{F}_r(z,x_0,s) \\ &\times \big(\phi_x(P,x_0,s)+\phi(P,x_0,s)^2\big) + (\widetilde{G}_r(z,x_0,s) - \frac{1}{2}\widetilde{F}_{r,x}(z,x_0,s))\phi(P,x_0,s) \\ &+ \big(\frac{1}{6}\widetilde{F}_{r,xx}(z,x_0,s) + \frac{2}{3}q_1(x_0,s)\widetilde{F}_r(z,x_0,s) - \widetilde{G}_{r,x}(z,x_0,s)\big)\big)\bigg), \\ &\qquad P\in\mathcal{K}_{m-1}\backslash\{P_\infty\}, \quad (x,t_r)\in\mathbb{C}^2,\end{aligned} \tag{3.186}$$

with fixed $(x_0,t_{0,r})\in\mathbb{C}^2$. The following theorem recalls the basic properties of $\phi(P,x,t_r)$ and $\psi(P,x,x_0,t_r,t_{0,r})$.

Theorem 3.18. *Assume* (3.169)–(3.173), $P=(z,y)\in\mathcal{K}_{m-1}\backslash\{P_\infty\}$ *and let* $(z,x,x_0,t_r,t_{0,r}) \in\mathbb{C}^5$. *Then*

(*i*) $\phi(P,x,t_r)$ *satisfies*

$$\begin{aligned}&\phi_{xx}(P,x,t_r) + 3\,\phi_x(P,x,t_r)\,\phi(P,x,t_r) + \phi(P,x,t_r)^3 + q_1(x,t_r)\,\phi(P,x,t_r) \\ &\qquad = z - q_0(x,t_r) - 2^{-1}\,q_{1,x}(x,t_r),\end{aligned} \tag{3.187}$$

$$\begin{aligned}\phi_{t_r}(P,x,t_r) &= \partial_x\big(\widetilde{F}_r(z,x,t_r)(\phi(P,x,t_r)^2+\phi_x(P,x,t_r)) \\ &\qquad + (\widetilde{G}_r(z,x,t_r) - 2^{-1}\,\widetilde{F}_{r,x}(z,x,t_r))\phi(P,x,t_r) + \widetilde{H}_r(z,x,t_r)\big).\end{aligned} \tag{3.188}$$

(*ii*) $\psi(P,x,x_0,t_r,t_{0,r})$ *satisfies*

$$\begin{aligned}&\psi_{xxx}(P,x,x_0,t_r,t_{0,r}) + q_1(x,t_r)\psi_x(P,x,x_0,t_r,t_{0,r}) \\ &\qquad + (q_0(x,t_r) + 2^{-1}\,q_{1,x}(x,t_r) - z)\psi(P,x,x_0,t_r,t_{0,r}) = 0,\end{aligned} \tag{3.189}$$

$$\begin{aligned}&\psi_{t_r}(P,x,x_0,t_r,t_{0,r}) = \big(\widetilde{F}_r(z,x,t_r)(\phi(P,x,t_r)^2+\phi_x(P,x,t_r)) \\ &\quad + (\widetilde{G}_r(z,x,t_r) - 2^{-1}\widetilde{F}_{r,x}(z,x,t_r))\phi(P,x,t_r) + \widetilde{H}_r(z,x,t_r)\big)\psi(P,x,x_0,t_r,t_{0,r})\end{aligned} \tag{3.190}$$

(*i.e.*, $(L_3-z)\psi=0$, $(P_m-y)\psi=0$, $\psi_{t_r}=\widetilde{P}_r\psi$).

(*iii*) $$\phi(P,x,t_r)\,\phi(P^*,x,t_r)\,\phi(P^{**},x,t_r) = \frac{N_m(z,x,t_r)}{D_{m-1}(z,x,t_r)}. \tag{3.191}$$

(*iv*) $$\phi(P,x,t_r)+\phi(P^*,x,t_r)+\phi(P^{**},x,t_r) = \frac{D_{m-1,x}(z,x,t_r)}{D_{m-1}(z,x,t_r)}. \tag{3.192}$$

(v). $y(P)\,\phi(P,x,t_r)+y(P^*)\,\phi(P^*,x,t_r)$

$$+\,y(P^{**})\,\phi(P^{**},x,t_r)=\frac{3\,T_m(z)\,F_m(z,x,t_r)-2\,S_m(z)\,A_m(z,x,t_r)}{\varepsilon(m)D_{m-1}(z,x,t_r)}. \tag{3.193}$$

(vi) $\psi(P,x,x_0,t_r,t_{0,r})\psi(P^*,x,x_0,t_r,t_{0,r})\psi(P^{**},x,x_0,t_r,t_{0,r})$

$$=\frac{D_{m-1}(z,x,t_r)}{D_{m-1}(z,x_0,t_{0,r})}. \tag{3.194}$$

(vii) $\psi_x(P,x,x_0,t_r,t_{0,r})\psi_x(P^*,x,x_0,t_r,t_{0,r})\psi_x(P^{**},x,x_0,t_r,t_{0,r})$

$$=\frac{N_m(z,x,t_r)}{D_{m-1}(z,x_0,t_{0,r})}. \tag{3.195}$$

(viii)

$$\begin{aligned}\psi(P,x,x_0,t_r,t_{0,r})&=\bigg(\frac{D_{m-1}(z,x,t_r)}{D_{m-1}(z,x_0,t_{0,r})}\bigg)^{1/3}\exp\bigg(\int_{x_0}^{x}d\,x'\varepsilon(m)D_{m-1}(z,x',t_r)^{-1}\\
&\times\big[F_m(z,x',t_r)\,y(P)^2+A_m(z,x',t_r)\,y(P)+\frac{2}{3}\,F_m(z,x',t_r)\,S_m(z)\big]\\
&-\int_{t_{0,r}}^{t_r}d\,s\,\bigg(\varepsilon(m)D_{m-1}(z,x_0,s)^{-1}\big[F_m(z,x_0,s)y(P)^2+A_m(z,x_0,s)\,y(P)\\
&+\frac{2}{3}\,F_m(z,x_0,s)\,S_m(z)\big]\times\big[\widetilde G_r(z,x_0,s)-\frac{1}{2}\,\widetilde F_{r,x}(z,x_0,s)\\
&-\big(G_m(z,x_0,s)-\frac{1}{2}\,F_{m,x}(z,x_0,s)\big)\,\frac{\widetilde F_r(z,x_0,s)}{F_m(z,x_0,s)}\big]+y(P)\,\frac{\widetilde F_r(z,x_0,s)}{F_m(z,x_0,s)}\bigg)\bigg).\end{aligned} \tag{3.196}$$

Proof. (i). (3.187) follows from (3.170), (3.171) and (3.180). In order to prove (3.188) one first derives from (3.169)–(3.171) and (3.180) that

$$\big[\partial_x^2+3\phi\,\partial_x+3(\phi^2+\phi_x)+q_1\big]\Big(\phi_{t_r}-\partial_x\big(\tilde F_r(\phi^2+\phi_x)+(\tilde G_r-\frac{1}{2}\,F_{r,x})\phi+\tilde H_r\big)\Big)=0.$$

Thus

$$\phi_{t_r}-\partial_x\big(\tilde F_r(\phi^2+\phi_x)+(\tilde G_r-\frac{1}{2}\,F_{r,x})\phi+\tilde H_r\big)=C_1f_1+C_2f_2, \tag{3.197}$$

where $f_j,\ j=1,2$ are two linearly independent solutions of

$$\big[\partial_x^2+3\phi\,\partial_x+3(\phi^2+\phi_x)+q_1\big]f=0$$

and $C_j,\ j=1,2$ are independent of x (but may depend on P and t_r). The high-energy behavior of $\phi(P,x,t_r)\underset{|z|\to\infty}{=}O(|z|^{1/3})$ (cf. (3.180)) then proves $C_1=C_2=0$ since the left-hand side of (3.197) is meromorphic on $\mathcal{K}_{m-1}$ (and hence especially near P_∞).
(ii). (3.189) is clear from (3.186) ($\phi=\psi_x/\psi$) and (3.187). (3.190) follows from (3.186) and (3.188). (iii)–(v) follow as in Lemma 3.7 (ii)–(iv). (3.194) follows from (3.186), (3.192), and (3.176). (3.195) follows from (3.191) and (3.194). (3.196) follows from (3.186), (3.181), (3.71), and (3.176). □

The dynamics of the zeros $\mu_j(x,t_r)$ and $\nu_\ell(x,t_r)$ of $D_{m-1}(z,x,t_r)$ and $N_m(z,x,t_r)$, in analogy to Lemma 3.8, are then described in terms of Dubrovin-type equations as follows.

Lemma 3.19. *Suppose* (3.169)–(3.173) *and assume that the curve* $\mathcal{K}_{m-1}$ *is nonsingular.*
(i) Suppose the zeros $\{\mu_j(x,t_r)\}_{j=1,\dots,m-1}$ *of* $D_{m-1}(\,\cdot\,,x,t_r)$ *remain distinct for* $(x,t_r)\in\Omega_\mu$,

where $\Omega_\mu \subseteq \mathbb{C}^2$ *is open and connected. Then* $\{\mu_j(x,t_r)\}_{j=1,\dots,m-1}$ *satisfy the system of differential equations,*

$$\mu_{j,x}(x,t_r) = -\varepsilon(m)\, F_m(\mu_j(x,t_r),x,t_r)\, \frac{\big(3y(\hat\mu_j(x,t_r))^2 + S_m(\mu_j(x,t_r))\big)}{\prod_{\substack{k=1\\k\neq j}}^{m-1}\big(\mu_j(x,t_r)-\mu_k(x,t_r)\big)}, \quad j=1,\dots,m-1, \tag{3.198}$$

$$\begin{aligned}\mu_{j,t_r}(x,t_r) &= -\varepsilon(m)\Big(F_m(\mu_j(x,t_r),x,t_r)\big(\widetilde G_r(\mu_j(x,t_r),x,t_r) - 2^{-1}\widetilde F_{r,x}(\mu_j(x,t_r),x,t_r)\big)\\ &\quad + \widetilde F_r(\mu_j(x,t_r),x,t_r)\big(G_m(\mu_j(x,t_r),x,t_r) - 2^{-1}F_{m,x}(\mu_j(x,t_r),x,t_r)\big)\Big)\\ &\quad \times \frac{\big(3y(\hat\mu_j(x,t_r))^2 + S_m(\mu_j(x,t_r))\big)}{\prod_{\substack{k=1\\k\neq j}}^{m-1}\big(\mu_j(x,t_r)-\mu_k(x,t_r)\big)}, \quad j=1,\dots,m-1,\end{aligned} \tag{3.199}$$

with initial conditions

$$\{\hat\mu_j(x_0,t_{0,r})\}_{j=1,\dots,m-1} \in \mathcal{K}_{m-1}, \tag{3.200}$$

for some fixed $(x_0,t_{0,r}) \in \Omega_\mu$*. The initial value problem* (3.199), (3.200) *has a unique solution satisfying*

$$\hat\mu_j \in C^\infty(\Omega_\mu, \mathcal{K}_{m-1}), \quad j=1,\dots,m-1. \tag{3.201}$$

(ii) Suppose the zeros $\{\nu_\ell(x,t_r)\}_{\ell=0,\dots,m-1}$ *of* $N_m(\,\cdot\,,x,t_r)$ *remain distinct for* $(x,t_r)\in\Omega_\nu$*, where* $\Omega_\nu \subseteq \mathbb{C}^2$ *is open and connected. Then* $\{\nu_\ell(x,t_r)\}_{\ell=0,\dots,m-1}$ *satisfy the system of differential equations,*

$$\nu_{\ell,x}(x,t_r) = -\varepsilon(m)\, J_m(\nu_\ell(x),x,t_r)\, \frac{\big(3y(\hat\nu_\ell(x,t_r))^2 + S_m(\nu_\ell(x,t_r))\big)}{\prod_{\substack{k=0\\k\neq \ell}}^{m-1}\big(\nu_\ell(x,t_r)-\nu_k(x,t_r)\big)}, \quad \ell=0,\dots,m-1, \tag{3.202}$$

$$\begin{aligned}\nu_{\ell,t_r}(x,t_r) &= -\varepsilon(m)\Big(J_m(\nu_\ell(x,t_r),x,t_r)\big(\widetilde G_r(\nu_\ell(x,t_r),x,t_r) + 2^{-1}\widetilde F_{r,x}(\nu_\ell(x,t_r),x,t_r)\big)\\ &\quad - \widetilde J_r(\nu_\ell(x,t_r),x,t_r)\big(G_m(\nu_\ell(x,t_r),x,t_r) + 2^{-1}\,F_{m,x}(\nu_\ell(x,t_r),x,t_r)\big)\Big)\\ &\quad \times \frac{\big(3y(\hat\nu_\ell(x,t_r))^2 + S_m(\nu_\ell(x,t_r))\big)}{\prod_{\substack{k=0\\k\neq \ell}}^{m-1}\big(\nu_\ell(x,t_r)-\nu_k(x,t_r)\big)}, \quad \ell=0,\dots,m-1,\end{aligned} \tag{3.203}$$

with initial conditions

$$\{\hat\nu_\ell(x_0,t_{0,r})\}_{\ell=0,\dots,m-1} \in \mathcal{K}_{m-1}, \tag{3.204}$$

for some fixed $(x_0,t_{0,r}) \in \Omega_\nu$*. The initial value problem* (3.203), (3.204) *has a unique solution satisfying*

$$\hat\nu_\ell \in C^\infty(\Omega_\nu, \mathcal{K}_{m-1}), \quad \ell=0,\dots,m-1. \tag{3.205}$$

(iii) The initial condition

$$(q_0(x, t_{0,r}), q_1(x, t_{0,r})) = (q_0^{(0)}(x), q_1^{(0)}(x)), \quad x \in \mathbb{C} \tag{3.206}$$

effects

$$\hat{\mu}_j(x, t_{0,r}) = \hat{\mu}_j^{(0)}(x), \quad j = 1, \dots, m-1, \quad x \in \mathbb{C}, \tag{3.207}$$

$$\hat{\nu}_\ell(x, t_{0,r}) = \hat{\nu}_\ell^{(0)}(x), \quad \ell = 0, \dots, m-1, \quad x \in \mathbb{C} \tag{3.208}$$

(cf. (3.172)–(3.174)).

Proof. (3.198) and (3.202) are analogous to (3.105) and (3.108). (3.199) follows from (3.176) and (3.203) follows from (3.177). □

The initial condition

$$(q_0(x, t_{0,r}), q_1(x, t_{0,r})) = (q_0^{(0)}(x), q_1^{(0)}(x)), \quad x \in \mathbb{R} \tag{3.209}$$

effects

$$\hat{\mu}_j(x, t_{0,r}) = \hat{\mu}_j^{(0)}(x), \quad 1 \le j \le m-1, \quad x \in \mathbb{R}, \tag{3.210}$$

$$\hat{\nu}_\ell(x, t_{0,r}) = \hat{\nu}_\ell^{(0)}(x), \quad 0 \le \ell \le m-1, \quad x \in \mathbb{R} \tag{3.211}$$

(cf. (3.172)–(3.174)).

Finally, the trace relations in Lemma 3.9 extend in a one-to-one manner to the present time-dependent setting by substituting,

$$\begin{aligned} &(q_0(x), q_1(x)) \to (q_0(x, t_r), q_1(x, t_r)), \\ &\quad \mu_j(x) \to \mu_j(x, t_r), \quad 1 \le j \le m-1, \qquad \nu_\ell(x) \to \nu_\ell(x, t_r), \quad 0 \le \ell \le m-1, \end{aligned} \tag{3.212}$$

keeping $\{c_\ell\}_{1\le\ell\le n}, \{d_\ell\}_{1\le\ell\le n}$ as in Lemma 3.9 since $\mathcal{K}_{m-1}$ is t_r-independent.

3.5. Time-Dependent Algebro-Geometric Solutions of the Boussinesq Hierarchy

In our final and principal section we extend the results of Section 3.3 from the stationary Bsq hierarchy, to the time-dependent case. In particular, we obtain Riemann theta function representations for the time-dependent Baker-Akhiezer function and the time-dependent meromorphic function ϕ. We finish this section with the corresponding theta function representation for general time-dependent algebro-geometric quasi-periodic Bsq solutions q_0, q_1.

We start with the theta function representation of our fundamental object $\phi(P, x, t_r)$.

Theorem 3.20. *Let $P = (z, y) \in \mathcal{K}_{m-1} \backslash \{P_\infty\}$, $(z, x, t_r) \in \mathbb{C}^3$. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ are nonspecial. Then*

$$\phi(P, x, t_r) = \frac{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x, t_r)))}{\theta(\underline{z}(P_\infty, \underline{\hat{\nu}}(x, t_r)))} \frac{\theta(\underline{z}(P, \underline{\hat{\nu}}(x, t_r)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(x, t_r)))} \exp\left(e^{(3)}(P_0) - \int_{P_0}^{P} \omega^{(3)}_{P_\infty, \hat{\nu}_0(x,t_r)}\right). \tag{3.213}$$

Proof. The proof carries over *ad verbatim* from the stationary case, Theorem 3.12. □

Let $\omega^{(2)}_{P_\infty,r}$, $r=3s+\varepsilon'$, $\varepsilon'\in\{1,2\}$, $s\in\mathbb{N}_0$, be the normalized dsk holomorphic on $\mathcal{K}_{m-1}\backslash\{P_\infty\}$, with a pole of order r at P_∞,

$$\omega^{(2)}_{P_\infty,r}(P) \underset{\zeta\to 0}{=} (\zeta^{-r}+O(1))d\zeta \text{ as } P\to P_\infty, \quad r=3s+\varepsilon',\, \varepsilon'\in\{1,2\},\, s\in\mathbb{N}_0. \tag{3.214}$$

Furthermore, define the normalized dsk

$$\widetilde{\Omega}^{(2)}_{P_\infty,r+1} = \sum_{\ell=0}^{s} \tilde{c}^{(\varepsilon')}_{s-\ell}\,(3\ell+2)\,\omega^{(2)}_{P_\infty,3\ell+3} + \sum_{\ell=0}^{s} \tilde{d}^{(\varepsilon')}_{s-\ell}\,(3\ell+1)\,\omega^{(2)}_{P_\infty,3\ell+2}, \\ r=3s+\varepsilon',\, \varepsilon'\in\{1,2\},\, s\in\mathbb{N}_0, \tag{3.215}$$

where (cf. (3.3))

$$(\tilde{c}^{(\varepsilon')}_0, \tilde{d}^{(\varepsilon')}_0) = \begin{cases} (0,1) & \text{for } \varepsilon'=1, \\ (1,\tilde{d}^{(2)}_0) & \text{for } \varepsilon'=2, \end{cases} \qquad \tilde{d}^{(2)}_0\in\mathbb{C}. \tag{3.216}$$

In addition, we define the vector of b-periods of the dsk $\widetilde{\Omega}^{(2)}_{P_\infty,r+1}$

$$\underline{\widetilde{U}}^{(2)}_{r+1} = (\widetilde{U}^{(2)}_{r+1,1},\dots,\widetilde{U}^{(2)}_{r+1,m-1}), \quad \widetilde{U}^{(2)}_{r+1,j} = \frac{1}{2\pi i}\int_{b_j}\widetilde{\Omega}^{(2)}_{P_\infty,r+1}, \quad j=1,\dots,m-1 \\ r=3s+\varepsilon',\, \varepsilon'\in\{1,2\},\, s\in\mathbb{N}_0. \tag{3.217}$$

Motivated by the second integrand in (3.186) one defines the function $I_r(P,x,t_r)$, meromorphic on $\mathcal{K}_{m-1}\times\mathbb{C}^2$ by

$$I_r(P,x,t_r) = \widetilde{F}_r(z,x,t_r)(\phi_x(P,x,t_r)+\phi(P,x,t_r)^2) \\ + (\widetilde{G}_r(z,x,t_r) - 2^{-1}\widetilde{F}_{r,x}(z,x,t_r))\phi(P,x,t_r) + \widetilde{H}_r(z,x,t_r), \tag{3.218}$$

for $r=3s+\varepsilon'$, $\varepsilon'\in\{1,2\}$, $s\in\mathbb{N}_0$. Denote by $\widehat{I}_r(P,x,t_r)$ the associated homogeneous quantity replacing $\widetilde{F}_r,\widetilde{G}_r,\widetilde{H}_r$ by the corresponding homogeneous polynomials $\widehat{\widetilde{F}}_r,\widehat{\widetilde{G}}_r,\widehat{\widetilde{H}}_r$.

Theorem 3.21. *Let $r=3s+\varepsilon'$, $\varepsilon'\in\{1,2\}$, $s\in\mathbb{N}_0$, $(x,t_r)\in\mathbb{C}^2$, and $\zeta=z^{-1/3}$ be the local coordinate near P_∞. Then*

$$\widehat{I}_r(P,x,t_r) \underset{\zeta\to 0}{=} \zeta^{-r}+O(\zeta) \text{ as } P\to P_\infty. \tag{3.219}$$

Proof. One easily verifies (3.219) by direct computation for $r=1$ and $r=2$. Assume (3.219) is true with $r=3s+\varepsilon'$, $\varepsilon'\in\{1,2\}$, $s\in\mathbb{N}_0$. Then one may rewrite (3.219) as

$$\widehat{I}_r(P,x,t_r) \underset{\zeta\to 0}{=} \zeta^{-r}+\sum_{j=1}^{\infty}\delta_j(x,t_r)\,\zeta^j \text{ as } P\to P_\infty, \tag{3.220}$$

for some coefficients $\{\delta_j(x,t_r)\}_{j\in\mathbb{N}}$. Compare coefficients of ζ in (3.119) and (3.220) by means of (3.188) and (3.218) to obtain

$$\delta_{1,x}(x,t_r) = -\frac{1}{3}q_{1,t_r}(x,t_r), \tag{3.221}$$

$$\delta_{2,x}(x,t_r) = \frac{1}{6}q_{1,t_rx}(x,t_r) - \frac{1}{3}q_{0,t_r}(x,t_r), \tag{3.222}$$

$$\delta_{3,x}(x,t_r) = \frac{1}{3}q_{0,t_rx}(x,t_r) - \frac{1}{18}q_{1,t_rxx}(x,t_r). \tag{3.223}$$

From (3.41) one infers

$$\delta_1(x,t_r) = \gamma_1(t_r) - \hat{f}^{(\varepsilon')}_{s+1}(x,t_r), \tag{3.224}$$

$$\delta_2(x,t_r) = \gamma_2(t_r) + 2^{-1}\hat{f}^{(\varepsilon')}_{s+1,x}(x,t_r) - \hat{g}^{(\varepsilon')}_{s+1}(x,t_r), \tag{3.225}$$

$$\delta_3(x,t_r) = \gamma_3(t_r) - 6^{-1}\hat{f}^{(\varepsilon')}_{s+1,xx}(x,t_r) + \hat{g}^{(\varepsilon')}_{s+1,x}(x,t_r), \tag{3.226}$$

where $\gamma_1(t_r)$, $\gamma_2(t_r)$, and $\gamma_3(t_r)$ are integration constants. Next we note that the coefficients of the power series for $\phi(P,x,t_r)$ in the coordinate ζ near P_∞ (cf. Lemma 3.10), and the coefficients of the homogeneous polynomials $\widehat{\widetilde{F}}_r(\zeta,x,t_r)$ and $\widehat{\widetilde{G}}_r(\zeta,x,t_r)$, (and hence those of $\widehat{\widetilde{H}}_r(\zeta,x,t_r)$) are differential polynomials in q_0 and q_1, with no arbitrary integration constants in their construction. From the definition of $\widehat{I}_r$ in (3.218) it follows that it also can have no arbitrary integration constants, and must consist purely of differential polynomials in q_0 and q_1. From these considerations it follows that $\gamma_1(t_r) = \gamma_2(t_r) = \gamma_3(t_r) = 0$. Hence one concludes

$$\begin{aligned} \widehat{I}_r(P,x,t_r) \underset{\zeta\to 0}{=} {} & \zeta^{-r} - \hat{f}^{(\varepsilon')}_{s+1}\zeta + \big(2^{-1}\hat{f}^{(\varepsilon')}_{s+1,x}(x,t_r) - \hat{g}^{(\varepsilon')}_{s+1}(x,t_r)\big)\zeta^2 \\ & + \big(\hat{g}^{(\varepsilon')}_{s+1,x}(x,t_r) - 6^{-1}\hat{f}^{(\varepsilon')}_{s+1,xx}(x,t_r)\big)\zeta^3 + O(\zeta^4) \text{ as } P\to P_\infty, \end{aligned} \tag{3.227}$$

where the functions $f^{(\varepsilon')}_s(x,t_r)$ and $g^{(\varepsilon')}_s(x,t_r)$ are defined as in (3.3) with $(q_0(x),q_1(x))$ replaced by $(q_0(x,t_r),q_1(x,t_r))$. We note that one may write

$$\widehat{\widetilde{F}}_{r+3}(\zeta,x,t_r) = \zeta^{-3}\widehat{\widetilde{F}}_r(\zeta,x,t_r) + \hat{f}^{(\varepsilon')}_{s+1}(x,t_r), \tag{3.228}$$

with analogous expressions for $\widehat{\widetilde{G}}_r$ and $\widehat{\widetilde{H}}_r$. It follows that

$$\begin{aligned} \widehat{I}_{r+3}(P,x,t_r) = {} & \zeta^{-3}\widehat{I}_r(P,x,t_r) + \hat{f}^{(\varepsilon')}_{s+1}(x,t_r)\big(\phi_x(P,x,t_r) + \phi(P,x,t_r)^2\big) \\ & + \big(\hat{g}^{(\varepsilon')}_{s+1}(x,t_r) - \frac{1}{2}\hat{f}^{(\varepsilon')}_{s+1,x}(x,t_r)\big)\phi(P,x,t_r) \\ & + \frac{1}{6}\hat{f}^{(\varepsilon')}_{s+1,xx}(x,t_r) + \frac{2}{3}q_1(x,t_r)\hat{f}^{(\varepsilon')}_{s+1}(x,t_r) - \hat{g}^{(\varepsilon')}_{s+1,x}(x,t_r). \end{aligned} \tag{3.229}$$

Using Lemma 3.10 and (3.227), (3.229) yields

$$\widehat{I}_{r+3}(P,x,t_r) \underset{\zeta\to 0}{=} \zeta^{-r-3} + O(\zeta) \text{ as } P\to P_\infty, \tag{3.230}$$

and the result follows by induction. □

By (3.18) one infers

$$I_r = \sum_{\ell=0}^{s} \tilde{c}^{(\varepsilon')}_{s-\ell}\,\widehat{I}_{3\ell+2} + \sum_{\ell=0}^{s} \tilde{d}^{(\varepsilon')}_{s-\ell}\,\widehat{I}_{3\ell+1}, \quad r = 3s+\varepsilon',\ \varepsilon'\in\{1,2\},\ s\in\mathbb{N}_0. \tag{3.231}$$

Thus,

$$\int_{t_{0,r}}^{t_r} I_r(P,x,\tau)d\tau \underset{\zeta\to 0}{=} (t_r - t_{0,r})\sum_{\ell=0}^{s}\Big(\tilde{c}^{(\varepsilon')}_{s-\ell}\frac{1}{\zeta^{3\ell+2}} + \tilde{d}^{(\varepsilon')}_{s-\ell}\frac{1}{\zeta^{3\ell+1}}\Big) + O(\zeta) \text{ as } P\to P_\infty. \tag{3.232}$$

Furthermore, integrating (3.215) yields

$$\begin{aligned}\int_{P_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,r+1} &\underset{\zeta\to 0}{=} \sum_{\ell=0}^{s} \tilde{c}^{(\varepsilon')}_{s-\ell}\,(3\ell+2)\int_{\zeta_0}^{\zeta}\frac{d\xi}{\xi^{3\ell+3}} + \sum_{\ell=0}^{s} \tilde{d}^{(\varepsilon')}_{s-\ell}\,(3\ell+1)\int_{\zeta_0}^{\zeta}\frac{d\xi}{\xi^{3\ell+2}} \\ &= -\sum_{\ell=0}^{s} \tilde{c}^{(\varepsilon')}_{s-\ell}\,\frac{1}{\zeta^{3\ell+2}} - \sum_{\ell=0}^{s} \tilde{d}^{(\varepsilon')}_{s-\ell}\,\frac{1}{\zeta^{3\ell+1}} + e^{(2)}_{r+1}(P_0) + O(\zeta) \text{ as } P\to P_\infty,\end{aligned} \tag{3.233}$$

where $e^{(2)}_{r+1}(P_0)$ is a constant that arises from evaluating all the integrals at their lowers limits P_0, and summing accordingly. Combining (3.232) and (3.233) yields

$$\int_{t_{0,r}}^{t_r} I_r(P,x,s)ds \underset{\zeta\to 0}{=} (t_r - t_{0,r})\Big(e^{(2)}_{r+1}(P_0) - \int_{P_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,r+1}\Big) + O(\zeta) \text{ as } P\to P_\infty. \tag{3.234}$$

Given these preparations, the theta function representation of $\psi(P,x,x_0,t_r,t_{0,r})$ reads as follows.

Theorem 3.22. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular. Furthermore, let $P=(z,y)\in \mathcal{K}_{m-1}\backslash\{P_\infty\}$, and let (x,t_r), $(x_0,t_{0,r})\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose also that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ are nonspecial for $(x,t_r)\in\Omega_\mu$. Then*

$$\begin{aligned}\psi(P,x,x_0,t_r,t_{0,r}) &= \frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_r)))}\frac{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x_0,t_{0,r})))} \\ &\quad\times \exp\Big((x-x_0)\big(e^{(2)}_2(P_0) - \int_{P_0}^{P}\omega^{(2)}_{P_\infty,2}\big) + (t_r - t_{r,0})\big(e^{(2)}_{r+1}(P_0) - \int_{P_0}^{P}\widetilde{\Omega}^{(2)}_{P_\infty,r+1}\big)\Big).\end{aligned} \tag{3.235}$$

Proof. We present only a proof of the time variation here, and refer the reader to Theorem 3.13 for the argument concerning the space variation. Let $\psi(P,x,x_0,t_r,t_{0,r})$ be defined as in (3.186) and denote the right-hand side of (3.235) by $\Psi(P,x,x_0,t_r,t_{0,r})$. Temporarily assume that

$$\mu_j(x,t_r)\neq\mu'_j(x,t_r) \text{ for } j\neq j' \text{ and } (x,t_r)\in\widetilde{\Omega}_\mu\subseteq\Omega_\mu, \tag{3.236}$$

where $\widetilde{\Omega}_\mu$ is open and connected. In order to prove that $\psi=\Psi$ one uses (3.181), (3.176), the time-dependent analog of (3.71), and

$$F_m(\phi_x+\phi^2)+(G_m-2^{-1}F_{m,x})\phi+H_m=y, \tag{3.237}$$

to compute

$$\begin{aligned}I_r &= \widetilde{F}_r(\phi_x+\phi^2)+(\widetilde{G}_r-\frac{1}{2}\widetilde{F}_{r,x})\phi+\widetilde{H}_r \\ &= \frac{1}{F_m}\Big(y\widetilde{F}_r + (F_m\widetilde{H}_r - \widetilde{F}_rH_m) + \big(F_m(\widetilde{G}_r-\frac{1}{2}\widetilde{F}_{r,x}) - \widetilde{F}_r(G_m-\frac{1}{2}F_{m,x})\big)\phi\Big) \\ &= \frac{1}{3}\frac{D_{m,t_r}}{D_m} + \frac{1}{F_m}\Big(y\widetilde{F}_r + \big(F_m(\widetilde{G}_r-\frac{1}{2}\widetilde{F}_{r,x}) - F_r(G_m-\frac{1}{2}F_{m,x})\big) \\ &\qquad\qquad \times\big(F_my^2+A_my+\frac{2}{3}F_mS_m\big)\varepsilon(m)D_m^{-1}\Big) \\ &= \frac{2}{3}\frac{F_m(\widetilde{G}_r-\frac{1}{2}\widetilde{F}_{r,x})-\widetilde{F}_r(G_m-\frac{1}{2}F_{m,x})}{\varepsilon(m)D_m}\big(3y^2+S_m\big) - \frac{1}{3}\sum_{k=1}^{m-1}\frac{\mu_{j,t_r}}{z-\mu_k} + \frac{y\widetilde{F}_r}{F_m}\end{aligned}$$

$$= -\frac{\mu_{j,t_r}}{z-\mu_j} + \frac{y\widetilde{F}_r}{F_m} + O(1) = -\frac{\mu_{j,t_r}}{z-\mu_j} + O(1) \tag{3.238}$$

as $P \to \hat{\mu}_j(x,t_r)$. More concisely,

$$I_r(P,x_0,s) = \frac{\partial}{\partial s}\ln(z-\mu_j(x_0,s)) \text{ for } P \text{ near } \hat{\mu}_j(x_0,t_r). \tag{3.239}$$

Hence

$$\exp\Big(\int_{t_{0,r}}^{t_r} ds\big(\frac{\partial}{\partial s}\ln(z-\mu_j(x_0,s)) + O(1)\big)\Big)$$
$$= \begin{cases} (z-\mu_j(x_0,t_r))O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0,t_r) \neq \hat{\mu}_j(x_0,t_{0,r}), \\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0,t_r) = \hat{\mu}_j(x_0,t_{0,r}), \\ (z-\mu_j(x_0,t_{0,r}))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0,t_{0,r}) \neq \hat{\mu}_j(x_0,t_r), \end{cases} \tag{3.240}$$

where $O(1) \neq 0$ in (3.240). Consequently, all zeros and poles of ψ and Ψ on $\mathcal{K}_{m-1}\backslash\{P_\infty\}$ are simple and coincide. It remains to identify the essential singularity of ψ and Ψ at P_∞. By (3.234) we see that the singularities in the exponential terms of ψ and Ψ coincide. The uniqueness result in Lemma A.26 for Baker-Akhiezer functions completes the proof that $\psi = \Psi$ on $\widetilde{\Omega}_\mu$. The extension of the result from $(x,t_r) \in \widetilde{\Omega}_\mu$ to $(x,t_r) \in \Omega_\mu$ follows from the continuity of $\underline{\alpha}_{P_0}$ and the hypothesis that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ is nonspecial for $(x,t_r) \in \Omega_\mu$. ☐

The straightening out of the Bsq flows by the Abel map is contained in our next result.

Theorem 3.23. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular, and let (x,t_r), $(x_0,t_{0,r}) \in \mathbb{C}^2$. Then*

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) + \underline{U}_2^{(2)}(x-x_0) + \underline{\widetilde{U}}_{r+1}^{(2)}(t_r - t_{0,r}), \tag{3.241}$$

and

$$\underline{A}_{P_0}(\hat{\nu}_0(x,t_r)) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)})$$
$$= \underline{A}_{P_0}(\hat{\nu}_0(x_0,t_{0,r})) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,r})}) + \underline{U}_2^{(2)}(x-x_0) + \underline{\widetilde{U}}_{r+1}^{(2)}(t_r - t_{0,r}). \tag{3.242}$$

Proof. As in the context of Theorem 3.15, it suffices to prove (3.241). Temporarily assume that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ is nonspecial for $(x,t_r) \in \Omega_\mu \subseteq \mathbb{C}^2$, where Ω_μ is open and connected. Introduce the meromorphic differential

$$\Omega(x,x_0,t_r,t_{0,r}) = \frac{\partial}{\partial z}\ln(\psi(\,\cdot\,,x,x_0,t_r,t_{0,r}))dz. \tag{3.243}$$

From the representation (3.235) one infers

$$\Omega(x,x_0,t_r,t_{0,r}) = -(x-x_0)\omega_{P_\infty,2}^{(2)} - (t_r - t_{0,r})\widetilde{\Omega}_{P_\infty,r+1}^{(2)} - \sum_{j=1}^{m-1}\omega_{\hat{\mu}_j(x_0,t_{0,r}),\hat{\mu}_j(x,t_r)}^{(3)} + \omega, \tag{3.244}$$

where ω denotes a holomorphic differential on $\mathcal{K}_{m-1}$, that is, $\omega = \sum_{j=1}^{m-1} e_j\omega_j$ for some $e_j \in \mathbb{C}$, $j = 1,\dots,m-1$. Since $\psi(\,\cdot\,,x,x_0,t_r,t_{0,r})$ is single-valued on $\mathcal{K}_{m-1}$, all a and b-periods of Ω are integer multiples of $2\pi i$ and hence

$$2\pi i m_k = \int_{a_k}\Omega(x,x_0,t_r,t_{0,r}) = \int_{a_k}\omega = e_k, \quad j = 1,\dots,m-1 \tag{3.245}$$

for some $m_k \in \mathbb{Z}$. Similarly, for some $n_k \in \mathbb{Z}$,

$$\begin{aligned}
2\pi i n_k &= \int_{b_k} \Omega(x, x_0, t_r, t_{0,r}) = -(x - x_0) \int_{b_k} \omega^{(2)}_{P_\infty,2} - (t_r - t_{0,r}) \int_{b_k} \widetilde{\Omega}^{(2)}_{P_\infty,r+1} \\
&\quad - \sum_{j=1}^{m-1} \int_{b_k} \omega^{(3)}_{\hat{\mu}_j(x_0,t_{0,r}),\hat{\mu}_j(x,t_r)} + 2\pi i \sum_{j=1}^{m-1} m_j \int_{b_k} \omega_j \\
&= -(x - x_0) \int_{b_k} \omega^{(2)}_{P_\infty,2} - (t_r - t_{0,r}) \int_{b_k} \widetilde{\Omega}^{(2)}_{P_\infty,r+1} - 2\pi i \sum_{j=1}^{m-1} \int_{\hat{\mu}_j(x,t_r)}^{\hat{\mu}_j(x_0,t_{0,r})} \omega_k \\
&\quad + 2\pi i \sum_{j=1}^{m-1} m_j \int_{b_k} \omega_j = -2\pi i (x - x_0)\, U^{(2)}_{2,k} - 2\pi i (t_r - t_{0,r})\, \widetilde{U}^{(2)}_{r+1,k} \\
&\quad + 2\pi i \alpha_{P_0,k}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) - 2\pi i \alpha_{P_0,k}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) + 2\pi i \sum_{j=1}^{m-1} m_j \tau_{j,k},
\end{aligned} \tag{3.246}$$

where we used (A.36). By symmetry of τ (see Theorem A.4) this is equivalent to

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) + \underline{U}^{(2)}_2 (x - x_0) + \underline{\widetilde{U}}^{(2)}_{r+1} (t_r - t_{0,r}), \tag{3.247}$$

for $(x, t_r) \in \Omega_\mu$. This result extends from $(x, t_r) \in \Omega_\mu$ to $(x, t_r) \in \mathbb{C}^2$ using the continuity of $\underline{\alpha}_{P_0}$ and the fact that positive nonspecial divisors are dense in the space of positive divisors (cf. [**30**], p. 95). ☐

Our principal result, the theta function representation of the class of time-dependent algebro-geometric quasi-periodic Bsq solutions now quickly follows from the material prepared thus far.

Theorem 3.24. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular and let $(x, t_r) \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}^2$ is open and connected. Suppose also that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ are nonspecial. Then*

$$q_0(x, t_r) = 3\, \partial_{\underline{U}^{(2)}_3} \partial_x \ln(\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x, t_r)))) + (3/2) w, \tag{3.248}$$

$$q_1(x, t_r) = 3\, \partial_x^2 \ln(\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x, t_r)))) + 3u, \tag{3.249}$$

where u and w are defined by (3.152) *and* (3.153)*, respectively, and $\partial_{\underline{U}^{(2)}_3}$ denotes the directional derivative introduced in* (3.160).

Proof. The proof carries over *ad verbatim* from the stationary case, Theorem 3.16. ☐

Chapter 4

Halphen potentials

4.1. Halphen potentials associated with the Bsq hierarchy

In this section we study in detail Halphen potentials

$$q_1(x) = h_g - g(g+2)\wp(x), \quad h_g \in \mathbb{C}, \quad g \in \mathbb{N}, \quad g \not\equiv 2 \pmod 3. \tag{4.1}$$

and the associated linear third-order differential equation

$$\psi'''(z,x) + (h_g - g(g+2)\,\wp(x))\,\psi'(z,x) - \Big(\frac{1}{2}\,g(g+2)\wp'(x) + z\Big)\psi(z,x) = 0, \tag{4.2}$$
$$z \in \mathbb{C}, \quad h_g \in \mathbb{C}, \quad g \not\equiv 2 \pmod 3.$$

Here $\wp(x) = \wp(x, \omega_1, \omega_3)$ denotes the elliptic Weierstrass function with fundamental periods $2\,\omega_1, 2\,\omega_3$, and invariants g_2, g_3, (see, e.g., [**1**]). The potentials (4.1) were introduced by Halphen [**52**, Ch. IV, p. 179] in the case $h_g = 0, g_2 = 0$, $(g = n - 1)$.

From the work of Segal and Wilson [**86**] one may obtain that solutions of $L_3\psi = z\psi$ are necessarily meromorphic if the coefficients of L_3 are algebro-geometric potentials. That this condition is also sufficient for elliptic algebro-geometric solutions of the KdV hierarchy was recently proven by Gesztesy and Weikard in [**45**] (see also [**43**], [**46**]).

Theorem 4.1. *Let q be an elliptic function. Then q is an elliptic algebro-geometric KdV potential if and only if the equation $y''(x)+q(x)y(x) = \tilde{z}y(x)$ has a meromorphic fundamental system of solutions with respect to x for all values of the spectral parameter $\tilde{z} \in \mathbb{C}$.*

Recently Weikard [**92**] (cf. [**91**]) proved an analogous theorem for the entire Gelfand-Dickii hierarchy for rational and simply periodic algebro-geometric potentials. It is assumed that this is also true for elliptic algebro-geometric potentials.

Since we expect that (4.2) will lead to algebro-geometric Bsq potentials only when the fundamental system is meromorphic, we investigate when (4.2) possesses a meromorphic fundamental system around $x = 0$. We distinguish two cases.

(i) $g_2 = 0, h_g = 0$. If $g_2 = 0$ the Laurent series ([**1**], p. 656) for $\wp(x)$ reduces to

$$\wp(x) = \frac{1}{x^2}\Big(1 + \sum_{m=1}^{\infty} c_{3m} x^{6m}\Big). \tag{4.3}$$

According to the theory of Fuchs, $x = 0$ is a regular singular point of (4.2). By the method of Frobenius (see, e.g., [**56**])

$$\psi(z,x) = x^{\rho} \sum_{\ell=0}^{\infty} r_\ell x^\ell, \quad r_0 \in \mathbb{C} \setminus \{0\} \tag{4.4}$$

then yields from the indicial equation $\rho = -g, 1, (g+2)$. This directly leads to the following three linear independent meromorphic solutions

$$\psi_1(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell+g+2}, \quad r_{3\ell+1} = r_{3\ell+2} = 0, \tag{4.5}$$

$$r_{3\ell+3} = \frac{z r_{3\ell} + g(g+2) \sum_{m=1}^{[(\ell+1)/2]} (g + 3\ell + 4 - 3m) c_{3m} r_{3\ell+3-6m}}{(3\ell+3)(3\ell+g+4)(3\ell+2g+5)}, \quad \ell \in \mathbb{N}_0,$$

$$\psi_2(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell+1}, \quad r_{3\ell+1} = r_{3\ell+2} = 0, \tag{4.6}$$

$$r_{3\ell+3} = \frac{z r_{3\ell} + g(g+2) \sum_{m=1}^{[(\ell+1)/2]} 3\,(\ell + 1 - m) c_{3m} r_{3\ell+3-6m}}{(3\ell+3)(3\ell+g+4)(3\ell-g+2)}, \quad \ell \in \mathbb{N}_0,$$

$$\psi_3(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell-g}, \quad r_{3\ell+1} = r_{3\ell+2} = 0, \tag{4.7}$$

$$r_{3\ell+3} = \frac{z r_{3\ell} + g(g+2) \sum_{m=1}^{[(\ell+1)/2]} (3\ell + 2 - g - 3m) c_{3m} r_{3\ell+3-6m}}{(3\ell+3)(3\ell-g+2)(3\ell-2g+1)}, \quad \ell \in \mathbb{N}_0$$

(where $[s]$ denotes the integer part of $s \in \mathbb{R}$.) Note that the denominators in the coefficients $r_{3\ell+3}$ in (4.6) and (4.7) can not become zero since $g \not\equiv 2 \pmod 3$. Thus we have proven that (4.2) possesses a meromorphic fundamental systems, whenever $g_2 = 0, h_g = 0$.

Remark 4.2. *Halphen studied invariants of $X^m + Y^n = Z^p$, $m,n,p \in \mathbb{N}$ and applied this to differential equations to prove the meromorphy of their fundamental systems. In the case of equation (4.2) this polynomial reads $h^3 = A\ell^2 + B$ with $h^3 = \frac{(-g(g+2))^3}{4z^2} \wp(x)^3, \ell = \frac{-g(g+2))}{2z} \wp'(x), A = \frac{-g(g+2)}{4}, B = \frac{(-g(g+2))^3}{16z^2} g_3$.*

(ii) If $g_2 \neq 0$, direct computations show that meromorphic fundamental systems exist for the following six cases (cf. Example 1–4)

$$\begin{aligned} &g = 1,\ h_1 = 0, \\ &g = 3,\ h_3 = \pm 2\sqrt{3g_2}, \\ &g = 4,\ h_4 = 0, \\ &g = 6,\ h_6 = \pm\frac{30}{7}\sqrt{3g_2}. \end{aligned} \tag{4.8}$$

In general however, if $g \geq 7$ and $g_2 \neq 0$, a constant $h_g \in \mathbb{C}$ does not exist such that the fundamental system is meromorphic for arbitrary spectral parameters $z \in \mathbb{C}$.

Setting $\psi_2(z,x) = \sum_{\ell=0}^{\infty} r_{\ell,2} x^{\ell+1}$ yields, for $r_{g+1,2}$ being finite, the condition

$$0 = z\, r_{g-2,2} - g\, h_g\, r_{g-1,2} + g(g+2) \sum_{m=2}^{[(g+1)/2]} (g+1-m)\, c_m r_{g+1-2m,2}. \tag{4.9}$$

The solution of this equation for h_g will in general always contain a term dependent of z if $g \geq 7$.

Remark 4.3. *Let $\psi_1(z,x)$ be a solution of*

$$\psi'''(z,x) + q_1(x)\,\psi'(z,x) + \big(\frac{1}{2}\, q_{1,x}(x) - z\big)\psi(z,x) = 0, \quad z \in \mathbb{C}, \tag{4.10}$$

and define $\hat{\psi}_1(z,x)$ by $\hat{\psi}_1(z,x) = \psi_1(-z,x)$. Then $\psi_2(z,x)$ given by

$$\psi_2(z,x) = \psi_1(z,x) \int^x \frac{\hat{\psi}_1(z,x')}{\psi_1^2(z,x')} dx' \tag{4.11}$$

yields a second linearly independent solution of (4.10). It is well known (cf. Ince [**56**]*, p. 122, or* [**38**]*) that the third linearly independent solution can be represented as*

$$\psi_3(z,x) = -\psi_1(z,x) \int^x \frac{\psi_2(z,x')}{W(\psi_1,\psi_1;x')^2} dx' + \psi_2(z,x) \int^x \frac{\psi_1(z,x')}{W(\psi_1,\psi_1;x')^2} dx' \tag{4.12}$$

where $W(f,g;x) = f g_x - f_x g$ denotes the Wronskian of f and g.

Note that if $z = 0$ (4.10) reduces to the well known third order differential equation which is fulfilled by the product of two solutions of a second order differential equation of the type $y''(x) + q(x)\,y(x) = \tilde{z}\,y(x)$, (see, e.g., [**31**]*, Part III, Chapt. V, Section 71, Ex. 1, or* [**54**]*, p. 511, equation (3.15).)*

According to a theorem of Picard ([**78**], [**79**], [**80**], see also [**55**], [**2**], p. 182–187, [**56**], p. 375–376) a differential equation with doubly-periodic coefficients and a meromorphic fundamental system possesses solutions which are in general elliptic of the second kind. Since there exists at least one solution which is elliptic of the second kind and every elliptic function can be expressed in terms of σ functions, we write

$$\psi_a(z,x) = e^{\lambda_a(z)x} \prod_{j=1}^{g} \frac{\sigma(x - a_j(z))}{\sigma(x)\sigma(a_j(z))}, \quad a(z) = (a_1(z), \dots, a_g(z)), \tag{4.13}$$

which yields

$$\begin{aligned}
&\frac{1}{\psi_a}\left(\psi_a''' + (h_g - g(g+2)\wp(x))\ \psi_a' - \frac{g(g+2)}{2}\wp'(x)\psi_a\right) \\
&= (2g^2+g)\wp(x)\Big(\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big) \\
&\quad + g\zeta(x)\left((2-2g)\sum_{j=1}^{g}\wp(a_j) - h_g - 3\Big(\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big)^2\right) \\
&\quad + \sum_{j=1}^{g}\zeta(x-a_j)\left(3\Big(\sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big)^2 + \wp(a_j)(g-g^2)\right.
\end{aligned}$$

$$+ h_g + 3\sum_{\substack{\ell=1\\ \ell\neq j}}^{g} \wp(a_\ell - a_j)\Bigg) + c_1 = z, \qquad c_1 \in \mathbb{C} \tag{4.14}$$

if and only if

$$z = (g^2 - \frac{5}{2}g + 1)\sum_{j=1}^{g} \wp'(a_j), \tag{4.15}$$

$$\lambda_a = \sum_{j=1}^{g} \zeta(a_j), \tag{4.16}$$

$$h_g = (2-2g)\sum_{j=1}^{g} \wp(a_j), \tag{4.17}$$

$$\begin{aligned} 0 &= 3\Big(\sum_{\substack{\ell=1\\ \ell\neq j}}^{g} \zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big)^2 + (2-2g)\sum_{\ell=1}^{g} \wp(a_\ell) \\ &\quad + (g-g^2)\wp(a_j) + 3\sum_{\substack{\ell=1\\ \ell\neq j}}^{g} \wp(a_\ell - a_j) \\ &= -(2g+1)\sum_{\ell=1}^{g} \wp(a_\ell) - (g^2+2g-6)\wp(a_j) \\ &\quad + \frac{3}{4}\sum_{\substack{\ell=1\\ \ell\neq j}}^{g} \left(\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\right)^2 + \frac{3}{4}\left(\sum_{\substack{\ell=1\\ \ell\neq j}}^{g} \frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\right)^2, \quad 1 \le j \le g. \end{aligned} \tag{4.18}$$

In order to derive (4.14) we used

$$\frac{\psi_a'''}{\psi_a} = \left(\frac{\psi_a'}{\psi_a}\right)'' + \left(\frac{\psi_a'}{\psi_a}\right)^3 + 3\left(\frac{\psi_a'}{\psi_a}\right)\left(\frac{\psi_a'}{\psi_a}\right)', \tag{4.19}$$

$$\frac{\psi_a'}{\psi_a} = \lambda_a + \sum_{j=1}^{g} \zeta(x - a_j) - g\zeta(x), \tag{4.20}$$

$$\left(\frac{\psi_a'}{\psi_a}\right)' = g\wp(x) - \sum_{j=1}^{g} \wp(x - a_j), \tag{4.21}$$

$$\begin{aligned} \frac{\psi_a'}{\psi_a}\left(\frac{\psi_a'}{\psi_a}\right)' &= -\sum_{j=1}^{g} \wp(a_j)\Bigg(\Big[\sum_{\substack{\ell=1\\ \ell\neq j}}^{g} \zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big] + \Big[\lambda_a - \sum_{j=1}^{g} \zeta(a_j)\Big]\Bigg) \\ &\quad + \frac{1-g}{2}\sum_{j=1}^{g} \wp'(a_j) + \frac{g^2}{2}\wp'(x) + g\wp(x)\Big[\lambda_a - \sum_{j=1}^{g} \zeta(a_j)\Big] \\ &\quad + \frac{1}{2}\sum_{j=1}^{g} \wp'(x - a_j) + \sum_{j=1}^{g} \wp(x - a_j)\Big[\sum_{\substack{\ell=1\\ \ell\neq j}}^{g} \zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big], \end{aligned} \tag{4.22}$$

$$\wp(x)\left(\frac{\psi_a'}{\psi_a}\right) = \frac{1}{2}\sum_{j=1}^{g}\wp'(a_j) + \sum_{j=1}^{g}\wp(a_j)\zeta(a_j) + \frac{g}{2}\wp'(x) \\ + \wp(x)\Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big] - \zeta(x)\sum_{j=1}^{g}\wp(a_j) + \sum_{j=1}^{g}\zeta(x-a_j)\wp(a_j), \tag{4.23}$$

$$\left(\frac{\psi_a'}{\psi_a}\right)^3 = \frac{g^3}{2}\wp'(x) + 3g^2\wp(x)\Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big] \tag{4.24}$$

$$- 3g\zeta(x)\left(\Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big]^2 + g\sum_{j=1}^{g}\wp(a_j)\right)$$

$$- \frac{1}{2}\sum_{j=1}^{g}\wp'(x-a_j) - 3\sum_{j=1}^{g}\wp(x-a_j)\Big[\sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big]$$

$$+ 3\sum_{j=1}^{g}\zeta(x-a_j)\left(\Big[\sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big]^2 + g\wp(a_j) + \sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\wp(a_\ell - a_j)\right)$$

$$+ \frac{3g^2-1}{2}\sum_{j=1}^{g}\wp'(a_j) + \Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big]^3 + 6g\sum_{j=1}^{g}\wp(a_j)\Big[\lambda_a - \sum_{j=1}^{g}\zeta(a_j)\Big]$$

$$+ 3\sum_{j=1}^{g}\zeta(a_j)\left(\Big[\sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big]^2 + g\wp(a_j) + \sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\wp(a_\ell - a_j)\right)$$

$$+ 3\sum_{j=1}^{g}\wp(a_j)\Big[\sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\zeta(a_\ell - a_j) + g\zeta(a_j) - \lambda_a\Big].$$

Remark 4.4. *The transformation $a \to -a$ (i.e., $a_j \to -a_j$, $1 \le j \le g$) in (4.13) yields a solution of $L_3\psi_{-a} = -z\,\psi_{-a}$. By Remark 4.3 this yields two further solutions of (4.2).*

4.1.1. The equiharmonic case $g_2 = 0, h_g = 0$. In the equiharmonic case where $g_2 = 0, h_g = 0$, the two other solutions of (4.2) can be obtained in the following way.

We start with

Remark 4.5. *Given $\wp'(v) = z, z \neq 0$ there exist 3 different points $v_j, j = 1,2,3$ with $\wp'(v_j) = z$ and $v_1 + v_2 + v_3 = 0$. Assume $\wp'(v_j) = \wp'(v_k), v_j \neq v_k,\ j,k = 1,2,3$. Then*

$$\wp(v_2) = \wp(-v_3 - v_1) = -\wp(v_3) - \wp(v_1) + \frac{1}{4}\left(\frac{\wp'(v_3) - \wp'(v_1)}{\wp(v_3) - \wp(v_1)}\right)^2. \tag{4.25}$$

This implies

$$\wp(v_1) + \wp(v_2) + \wp(v_3) = 0 \text{ and } \zeta(v_1) + \zeta(v_2) + \zeta(v_3) = 0. \tag{4.26}$$

Now

$$\wp'^2(v_j) = 4\wp^3(v_j) - g_2\wp(v_j) - g_3, \quad j = 1,2,3 \tag{4.27}$$

yields

$$\frac{g_2}{4} = \wp^2(v_j) + \wp(v_j)\wp(v_k) + \wp^2(v_k), \qquad j,k = 1,2,3 \;\; j \neq k. \tag{4.28}$$

From that we conclude that for $g_2 = 0$

$$\wp(v_2) = \wp(v_1)\alpha_3, \quad \wp(v_3) = \wp(v_1)\alpha_3^2, \qquad \alpha_3 = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \tag{4.29}$$

It follows that $v_2 = \alpha_3\, v_1$ *and* $v_3 = \alpha_3^2\, v_1$.

We define

$$\begin{aligned} \psi_{a,1}(z,x) &= \psi_a(a_{1,1},\dots,a_{g,1},z,x),\\ \psi_{a,2}(z,x) &= \psi_a(a_{1,2},\dots,a_{g,2},z,x),\\ \psi_{a,3}(z,x) &= \psi_a(a_{1,3},\dots,a_{g,3},z,x), \end{aligned} \tag{4.30}$$

where $\wp'(a_{j,1}) = \wp'(a_{j,2}) = \wp'(a_{j,3}),\ a_{j,\ell} = \alpha_3^{\ell-1} a_j,\ \ell = 1,2,3,\ 1 \le j \le g.$

One immediately recognizes that the conditions (4.15)-(4.17) are fulfilled if $g_2 = 0, h_g = 0$ and hence $\psi_{a,k}(z,x), k = 1,2,3$ are solutions of (4.2).

The product $D_g(z,x) = \psi_{a,1}(z,x)\,\psi_{a,2}(z,x)\,\psi_{a,3}(z,x)$ of all three solutions then reads

$$\begin{aligned} D_g(z,x) &= \prod_{j=1}^{g} \frac{\sigma(x-a_{j,1}(z))}{\sigma(x)\sigma(a_{j,1}(z))} \prod_{j=1}^{g} \frac{\sigma(x-a_{j,2}(z))}{\sigma(x)\sigma(a_{j,2}(z))} \prod_{j=1}^{g} \frac{\sigma(x-a_{j,3}(z))}{\sigma(x)\sigma(a_{j,3}(z))}\\ &= \prod_{j=1}^{g} \frac{\sigma(x-a_{j,1}(z))}{\sigma(x)\sigma(a_{j,1}(z))} \frac{\sigma(x-a_{j,2}(z))}{\sigma(x)\sigma(a_{j,2}(z))} \frac{\sigma(x-a_{j,3}(z))}{\sigma(x)\sigma(a_{j,3}(z))}\\ &= \prod_{j=1}^{g} \frac{1}{2}\left(\wp'(x) - \wp'(a_{j,1})\right). \end{aligned} \tag{4.31}$$

The Wronskian $W(\psi_{a,1},\psi_{a,2},\psi_{a,3})$ is given by

$$\begin{aligned} W(\psi_{a,1},\psi_{a,2},\psi_{a,3}) = D_g(z,x)\Big(&\frac{\psi'_{a,2}}{\psi_{a,2}}\frac{\psi''_{a,3}}{\psi_{a,3}} + \frac{\psi'_{a,3}}{\psi_{a,3}}\frac{\psi''_{a,1}}{\psi_{a,1}} + \frac{\psi'_{a,1}}{\psi_{a,1}}\frac{\psi''_{a,2}}{\psi_{a,2}}\\ &- \frac{\psi'_{a,2}}{\psi_{a,2}}\frac{\psi''_{a,1}}{\psi_{a,1}} - \frac{\psi'_{a,1}}{\psi_{a,1}}\frac{\psi''_{a,3}}{\psi_{a,3}} - \frac{\psi'_{a,3}}{\psi_{a,3}}\frac{\psi''_{a,2}}{\psi_{a,2}}\Big). \end{aligned} \tag{4.32}$$

With

$$\frac{\psi'_{a,j}}{\psi_{a,j}} = \frac{1}{2}\sum_{\ell=1}^{g} \frac{\wp'(x) + \wp'(a_{\ell,j})}{\wp(x) - \wp(a_{\ell,j})}, \quad j = 1,2,3 \tag{4.33}$$

and

$$\frac{\psi''_{a,k}}{\psi_{a,k}} = 2\,g\,\wp(x) + \frac{1}{2}\sum_{\substack{\ell,s=1\\ \ell<s}}^{g} \frac{\wp'(x) + \wp'(a_{\ell,k})}{\wp(x) - \wp(a_{\ell,k})}\frac{\wp'(x) + \wp'(a_{s,k})}{\wp(x) - \wp(a_{s,k})}, \quad k = 1,2,3, \tag{4.34}$$

we may evaluate $W(\psi_{a,1},\psi_{a,2},\psi_{a,3})$ at $x = a_{j,1}$ since it is independent of x. This yields

$$W(\psi_{a,1},\psi_{a,2},\psi_{a,3}) = \frac{3}{2^g}\wp(a_{j,1})^2 \prod_{\substack{\ell=1\\ \ell\neq j}}^{g} (\wp'(a_{j,1}) - \wp'(a_{\ell,1}))$$

$$\Big[\sum_{\substack{\ell,s=1\\ \ell<s}}^{g}\Big(\frac{\wp'(a_{j,1})+\wp'(a_{\ell,2})}{\wp(a_{j,1})-\wp(a_{\ell,2})}\frac{\wp'(a_{j,1})+\wp'(a_{s,2})}{\wp(a_{j,1})-\wp(a_{s,2})} - \frac{\wp'(a_{j,1})+\wp'(a_{\ell,3})}{\wp(a_{j,1})-\wp(a_{\ell,3})}\frac{\wp'(a_{j,1})+\wp'(a_{s,3})}{\wp(a_{j,1})-\wp(a_{s,3})}\Big) + \Big(\sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\frac{\wp'(a_{j,1})+\wp'(a_{\ell,1})}{\wp(a_{j,1})-\wp(a_{\ell,1})}\Big)\Big(\sum_{\ell=1}^{g}\frac{\wp'(a_{j,1})+\wp'(a_{\ell,3})}{\wp(a_{j,1})-\wp(a_{\ell,3})} - \frac{\wp'(a_{j,1})+\wp'(a_{\ell,2})}{\wp(a_{j,1})-\wp(a_{\ell,2})}\Big)\Big]. \tag{4.35}$$

Note that

$$\frac{\wp'(x)^2-\wp'(v)^2}{\wp(x)-\wp(v)} = 4(\wp(x)^2+\wp(x)\wp(v)+\wp(v)^2), \qquad (g_2=0) \tag{4.36}$$

and hence all remaining fractions in(4.35) will cancel out. Thus $\psi_{a,1}(z,x)$, $\psi_{a,2}(z,x)$, $\psi_{a,3}(z,x)$ will not form a fundamental system when one of the values $\wp(a_{j,1})=0,\ 1\le j\le g$. In this case we can apply either Remark 4.3 or the results from Subsection 4.1.2 to obtain a fundamental system.

Remark 4.6. *Halphen used the following ansatz*

$$\psi(z,x) = e^{vx}\sum_{j=0}^{g-1}\alpha_j(z,\tilde z,v)\frac{d^j\phi(\tilde z,x)}{dx^j} \tag{4.37}$$

to solve (4.2), where $\phi(\tilde z,x)$, which he called "élément simple", is a solution of

$$\phi'' - (2\,\wp(x)+\tilde z)\phi = 0. \tag{4.38}$$

An extended version of this ansatz was used by Eilbeck and Enol'skii [**26**] to compute a solution in the case $g=3,(g_2=0,h_3=0)$ and by Enol'skii and Kostov [**28**] in the case $g=4,(g_2=0,h_4=0)$, (cf. [**35**]).

4.1.2. Reduction of the order of the differential equation. Here we shortly discuss the well known process of the reduction of the order of a differential equation when one solution is known and apply it to Halphen's equation.

Having determined the solution $\psi_a(z,x)=\psi_1(z,x)$, we now consider the reduced equation (d' Alembert's method) setting

$$\psi_2(z,x) = \psi_1(z,x)\int^x u(z,x')dx'. \tag{4.39}$$

This yields

$$u'' + 3\,\frac{\psi_a'}{\psi_a}\,u' + (3\,\frac{\psi_a''}{\psi_a}+q_1)u = 0. \tag{4.40}$$

Picard's Theorem applies again and hence we write

$$u_b(z,x) = e^{\lambda_b(z)x}\prod_{j=1}^{g}\frac{\sigma(x-b_j(z))\sigma(x)\sigma(a_j(z))^2}{\sigma(x-a_j(z))^2\sigma(-b_j(z))}, \quad b(z)=(b_1(z),\dots,b_g(z)). \tag{4.41}$$

A similar analysis as before yields

$$\frac{1}{u_b}\left(u_b'' + 3\,\frac{\psi_a'}{\psi_a}u_b' + (3\,\frac{\psi_a''}{\psi_a}+q_1)u_b\right)$$

$$\begin{aligned}
&= c + g\zeta(x)\Big(-\lambda_b + \sum_{j=1}^{g}\zeta(b_j) - 2\sum_{j=1}^{g}\zeta(a_j)\Big)\\
&\quad + \sum_{j=1}^{g}\zeta(x-a_j)\left(2\sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\zeta(a_j-a_\ell) - g\zeta(a_j) - \lambda_b - \sum_{\ell=1}^{g}\zeta(a_j-b_\ell)\right)\\
&\quad + \sum_{j=1}^{g}\zeta(x-b_j)\left(2\sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\zeta(b_j-b_\ell) - g\zeta(b_j) + 2\lambda_b + 3\lambda_a\right.\\
&\quad \left. - \sum_{\ell=1}^{g}\zeta(b_j-a_\ell)\right) = 0, \qquad c\in\mathbb{C}. \qquad (4.42)
\end{aligned}$$

Equation (4.42) is fulfilled if and only if the following conditions hold

$$\lambda_b = \sum_{j=1}^{g}\zeta(b_j) - 2\sum_{j=1}^{g}\zeta(a_j), \tag{4.43}$$

$$0 = 2\sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\zeta(a_j-a_\ell) - g\zeta(a_j) - \lambda_b - \sum_{\ell=1}^{g}\zeta(a_j-b_\ell), \quad 1\le j\le g, \tag{4.44}$$

$$0 = 2\sum_{\substack{\ell=1\\ \ell\neq j}}^{g}\zeta(b_j-b_\ell) - g\zeta(b_j) + 2\lambda_b + 3\lambda_a - \sum_{\ell=1}^{g}\zeta(b_j-a_\ell), \quad 1\le j\le g, \tag{4.45}$$

$$0 = (1-g)(\sum_{\ell=1}^{g}\wp(a_\ell) - \sum_{\ell=1}^{g}\wp(b_\ell)). \tag{4.46}$$

The second solution u_2 of (4.40) can be obtained either by the transformation $a\to -a, b\to -b$, (i.e., $a_j\to -a_j, b_j\to -b_j,\ 1\le j\le g$), or by

$$u_2(z,x) = u_b(z,x)\int^x \frac{1}{u_b^2(z,x')\psi_a^3(z,x')}\,dx'. \tag{4.47}$$

4.2. Examples

4.2.1. Example 1. $g=1$.

Differential expressions

$$L_3 = \frac{d^3}{d\,x^3} - 3\,\wp(x)\,\frac{d}{d\,x} - \frac{3}{2}\,\wp'(x),$$
$$P_2 = \frac{d^2}{d\,x^2} - 2\wp(x).$$

Curve

$$\mathcal{F}_1(z,y) = y^3 - \frac{g_2}{4}\,y - z^2 - \frac{g_3}{4} = 0. \tag{4.48}$$

Elliptic solutions of the second kind

$$\psi_{a,j}(z,x) = \frac{\sigma(x-a_{1,j}(z))}{\sigma(x)\sigma(a_{1,j}(z))}\,e^{x\zeta(a_{1,j})},$$

$$z = -\frac{1}{2}\wp'(a_{1,j}) \quad z \neq 0, \quad j = 1, 2, 3.$$

Product of solutions

$$F_2(z,x) = 1, \qquad G_2(z,x) = 0, \tag{4.49}$$

$$D_1(z,x) = z + \frac{1}{2}\,\wp'(x), \qquad N_2(z,x) = \big(z - \frac{1}{2}\,\wp'(x)\big)^2, \tag{4.50}$$

$$\phi_j(z,x) = \frac{z - \frac{1}{2}\,\wp'(x)}{y_j - \wp(x)} \tag{4.51}$$

$$= \frac{y_j^2 + y_j\,\wp(x) + \wp(x)^2 - \frac{g_2}{4}}{z + \frac{1}{2}\,\wp'(x)} \tag{4.52}$$

$$= \frac{(z - \frac{1}{2}\,\wp'(x))^2}{(z - \frac{1}{2}\,\wp'(x))y_j - \wp(x)(z - \frac{1}{2}\,\wp'(x))}, \qquad 1 \leq j \leq 3 \tag{4.53}$$

where $y_j,\ 1 \leq j \leq 3$ denote the roots of (4.48).

4.2.2. Example 2. $g = 3$.

Differential expressions

$$L_3 = \frac{d^3}{d\,x^3} + \big(2\,\sqrt{3g_2} - 15\,\wp(x)\big)\,\frac{d}{d\,x} - \frac{15}{2}\,\wp'(x),$$
$$P_4 = \frac{d^4}{d\,x^4} + \big(\frac{\sqrt{3g_2}}{3} - 20\,\wp(x)\big)\,\frac{d^2}{d\,x^2} - 20\,\wp'(x)\frac{d}{d\,x} + \big(10\,\sqrt{3g_2}\,\wp(x) - \frac{5}{2}\,g_2\big).$$

Curve

$$\mathcal{F}_3(z,y) = y^3 + y\left(-\frac{375}{16}\,{g_2}^2 - \frac{225}{4}\,\sqrt{3g_2}\,g_3 + 7\,\sqrt{3g_2}\,z^2\right) + \frac{1375}{32}\,{g_2}^3 + \frac{2625}{16}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,g_3 + \frac{3375}{16}\,g_3^2 + \frac{1505}{36}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,z^2 + \frac{55}{2}\,g_3\,z^2 - z^4 = 0. \tag{4.54}$$

Product of solutions

$$F_4(z,x) = \big(-\frac{5}{3}\,\sqrt{3g_2} - 5\,\wp(x)\big), \qquad G_4(z,x) = z, \tag{4.55}$$

$$\begin{aligned}
D_3(z,x) &= z^3 + \frac{5}{2}\,\wp'(x)\,z^2 + z\left(\frac{1025}{36}\,\sqrt{3}\,g_2^{\frac{3}{2}} + \frac{25}{4}\,g_3 + 100g_2\,\wp(x)\right.\\
&\quad \left. - 50\,\sqrt{3g_2}\,\wp(x)^2 - 200\,\wp(x)^3\right) + \frac{125}{8}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,\wp'(x) + \frac{125}{8}\,g_3\,\wp'(x)\\
&\quad + 250g_2\,\wp(x)\wp'(x) + 375\,\sqrt{3g_2}\,\wp(x)^2\wp'(x) + 500\,\wp(x)^3\wp'(x),\\
N_4(z,x) &= z^4 - 5\,\wp'(x)z^3 + z^2\left(\frac{1145}{36}\,\sqrt{3}\,g_2^{\frac{3}{2}} - 40g_3 + \frac{135}{4}\,g_2\,\wp(x)\right.\\
&\quad \left. - 90\,\sqrt{3g_2}\,\wp(x)^2 + 225\,\wp(x)^3\right) + z\left(\frac{75}{4}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,\wp'(x) + \frac{675}{4}\,g_3\,\wp'(x)\right.\\
&\quad \left. - 900g_2\wp(x)\wp'(x) - 450\,\sqrt{3g_2}\,\wp(x)^2\wp'(x)\right) + \frac{375}{16}\,\sqrt{3}\,g_2^{\frac{3}{2}}g_3 + \frac{3375}{16}\,g_3^2\\
&\quad + \frac{375}{16}\,\sqrt{3}\,g_2^{\frac{5}{2}}\,\wp(x) + \frac{3375}{16}\,g_2g_3\wp(x) - \frac{3375}{2}\,\sqrt{3g_2}\,g_3\,\wp(x)^2 - 6750\,g_2\,\wp(x)^4\\
&\quad - (\frac{7125}{4}\,\sqrt{3}\,g_2^{\frac{3}{2}} + \frac{30375}{4}\,g_3)\,\wp(x)^3 + 6750\,\sqrt{3g_2}\,\wp(x)^5 + 27000\,\wp(x)^6,
\end{aligned} \tag{4.56}$$

$$\begin{aligned}
\phi_j(z,x) = &\Big(y_j\,\big(z - \frac{5}{2}\,\wp'(x)\big) + \frac{175}{12}\,g_2\,z + \frac{20}{3}\,\sqrt{3g_2}z\,\wp(x) - 20z\wp(x)^2 - \frac{25}{8}\,g_2\,\wp'(x) \\
&- 50\,\sqrt{3g_2}\,\wp(x)\wp'(x) - 150\,\wp(x)^2\wp'(x)\Big)\Big(y_j\,\big(-\frac{5}{3}\,\sqrt{3g_2} - 5\wp(x)\big) \\
&+ z^2 + \frac{25}{6}\,\sqrt{3}g_2^{\frac{3}{2}} + \frac{25}{4}\,g_3 - \frac{25}{4}\,g_2\,\wp(x) - 50\sqrt{3g_2}\,\wp(x)^2 - 100\wp(x)^3\Big)^{-1} \qquad (4.57)\\
= &\frac{1}{D_3(z,x)}\Big(y_j^2\,\big(\frac{5}{3}\,\sqrt{3g_2} + 5\,\wp(x)\big) + y_j\,\Big(\frac{25}{6}\,\sqrt{3}\,g_2^{\frac{3}{2}} + \frac{25}{4}\,g_3 + z^2 - \frac{25}{4}\,g_2\,\wp(x) \\
&- 50\,\sqrt{3g_2}\,\wp(x)^2 - 100\,\wp(x)^3\Big) + \frac{275}{12}\,g_2\,z^2 - \frac{875}{2}\,\sqrt{3g_2}\,g_3\,\wp(x) + 5\,z^2\,\wp(x)^2 \\
&- \frac{1625}{8}\,g_2^2\,\wp(x) + \frac{70}{3}\,\sqrt{3g_2}\,z^2\,\wp(x) - \frac{1125}{4}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,\wp(x)^2 - \frac{1875}{4}\,g_3\,\wp(x)^2 \\
&+ 250\,g_2\,\wp(x)^3 + 1750\,\sqrt{3g_2}\,\wp(x)^4 + \frac{100}{3}\,g_2\,z\,\wp'(x) - \frac{100}{3}\,\sqrt{3g_2}\,z\,\wp(x)\,\wp'(x) \\
&- 200\,z\,\wp(x)^2\,\wp'(x) - \frac{1375}{48}\,\sqrt{3}\,g_2^{\frac{5}{2}} - \frac{4375}{16}\,g_2\,g_3 + 3000\,\wp(x)^5\Big) \qquad (4.58)\\
= &N_4(z,x)\Big(y_j^2\,\big(z - \frac{5}{2}\,\wp'(x)\big) + y_j\,\Big(-\frac{175}{12}\,g_2\,z - \frac{20}{3}\,\sqrt{3g_2}\,z\,\wp(x) + 20\,z\,\wp(x)^2 \\
&+ \frac{25}{8}\,g_2\,\wp'(x) + 50\,\sqrt{3g_2}\,\wp(x)\,\wp'(x) + 150\,\wp(x)^2\,\wp'(x)\Big) + \big(\frac{16}{3}\,\sqrt{3g_2} - 5\,\wp(x)\big)z^3 \\
&+ \frac{725}{24}\,g_2^2\,z - \frac{125}{4}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,z\,\wp(x) + \frac{675}{4}\,g_3\,z\,\wp(x) - 200\,g_2\,z\,\wp(x)^2 - 600\,z\,\wp(x)^4 \\
&+ \frac{125}{16}\,g_2^2\,\wp'(x) - \frac{40}{3}\,\sqrt{3g_2}\,z^2\,\wp'(x) - \frac{1375}{8}\,\sqrt{3}\,g_2^{\frac{3}{2}}\,\wp(x)\,\wp'(x) + \frac{25}{2}\,z^2\,\wp(x)\,\wp'(x) \\
&- \frac{3375}{8}\,g_3\,\wp(x)\,\wp'(x) + 1875\,\sqrt{3g_2}\,\wp(x)^3\,\wp'(x) - 150\,\sqrt{3g_2}\,z\,\wp(x)^3 \\
&+ 4500\,\wp(x)^4\,\wp'(x)\Big)^{-1}, \qquad 1 \le j \le 3 \qquad (4.59)
\end{aligned}$$

where $y_j,\ 1 \le j \le 3$ denote the roots of (4.54).

Elliptic solution of the second kind

$$\begin{aligned}
\psi_a(z,x) &= e^{\lambda_a(z)x}\prod_{j=1}^{3}\frac{\sigma(x-a_j(z))}{\sigma(x)\sigma(a_j(z))}, \quad a(z) = (a_1(z),\dots,a_3(z)),\\
z &= \frac{5}{2}(\wp'(a_1) + \wp'(a_2) + \wp'(a_3)),\\
\lambda_a &= \zeta(a_1) + \zeta(a_2) + \zeta(a_3),\\
2\sqrt{3g_2} &= -4(\wp(a_1) + \wp(a_2) + \wp(a_3)),\\
0 &= -7\sum_{\ell=1}^{3}\wp(a_\ell) - 9\wp(a_j)\\
&\quad + \frac{3}{4}\sum_{\substack{\ell=1\\ \ell\neq j}}^{3}\left(\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\right)^2 + \frac{3}{4}\left(\sum_{\substack{\ell=1\\ \ell\neq j}}^{3}\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\right)^2, \quad 1 \le j \le 3.
\end{aligned}$$

Series solutions

$$\begin{aligned}
\psi_1 &= x^5 - \frac{\sqrt{3\,g_2}}{12} x^7 + O(x^8),\\
\psi_2 &= x + \frac{\sqrt{3\,g_2}}{12}\,x^3 - \frac{z}{21}x^4 + r_{4,2}\,x^5 + \frac{13}{1260}\,\sqrt{3\,g_2}\,z\,x^6 + O(x^7),\ \ r_{4,2}\in\mathbb{C},\\
\psi_3 &= \frac{1}{x^3} + \frac{\sqrt{3\,g_2}}{4}\,\frac{1}{x} + \frac{z}{15} + r_{4,3}\,x - \frac{\sqrt{3\,g_2}}{60}\,z\,x^2\\
&\quad + \left(\frac{\sqrt{3\,g_2}}{12}\,r_{4,3} + \frac{5\,g_3}{224} - \frac{z^2}{360}\right)\,x^3 - \left(\frac{r_{4,3}}{21} + \frac{g_2}{84}\right)\,z\,x^4 + O(x^5),\ \ r_{4,3}\in\mathbb{C}.
\end{aligned}$$

To obtain the results for the case $h_3 = -2\sqrt{3g_2}$ simply replace $\sqrt{g_2}$ by $-\sqrt{g_2}$ in all expressions above.

4.2.3. Example 3. $g = 4$.

Differential expressions

$$\begin{aligned}
L_3 &= \frac{d^3}{d\,x^3} - 24\,\wp(x)\,\frac{d}{d\,x} - 12\wp'(x),\\
P_5 &= \frac{d^5}{d\,x^5} - 40\,\wp(x)\,\frac{d^3}{d\,x^3} - 60\,\wp'(x)\frac{d^2}{d\,x^2} + \left(38g_2 + 40\wp(x)^2\right)\frac{d}{d\,x} + 160\wp(x)\wp'(x).
\end{aligned}$$

Curve

$$\begin{aligned}
\mathcal{F}_4(z,y) = y^3 - z^5 + 208\,g_3\,z^3 + y\,\left(3136\,g_2\,g_3 - 44\,g_2\,z^2\right)&\\
- 3136\,\left({g_2}^3 + 4\,{g_3}^2\right)\,z = 0.& \qquad (4.60)
\end{aligned}$$

Elliptic solution of the second kind

$$\begin{aligned}
\psi_a(z,x) &= e^{\lambda_a(z)x}\prod_{j=1}^{4}\frac{\sigma(x-a_j(z))}{\sigma(x)\sigma(a_j(z))},\quad a(z) = (a_1(z),\ldots,a_4(z)),\\
z &= 7(\wp'(a_1)+\wp'(a_2)+\wp'(a_3)+\wp'(a_4)),\\
\lambda_a &= \zeta(a_1)+\zeta(a_2)+\zeta(a_3)+\zeta(a_4),\\
0 &= (\wp(a_1)+\wp(a_2)+\wp(a_3)+\wp(a_4)),\\
18\wp(a_j) &= \frac{3}{4}\sum_{\substack{\ell=1\\ \ell\neq j}}^{4}\left(\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\right)^2 + \frac{3}{4}\left(\sum_{\substack{\ell=1\\ \ell\neq j}}^{4}\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\right)^2,\quad 1\le j\le 4.
\end{aligned}$$

Series solutions

$$\begin{aligned}
\psi_1 &= x^6 + \frac{z}{312}x^9 + O(x^{10}),\\
\psi_2 &= x - \frac{z}{48}\,x^4 - \frac{g_2}{15}\,x^5 + r_{5,2}\,x^6 + \left(\frac{3\,g_3}{77} - \frac{z^2}{3168}\right)\,x^7 + O(x^8),\ \ r_{5,2}\in\mathbb{C},\\
\psi_3 &= \frac{1}{x^4} + \frac{z}{42}\,\frac{1}{x} - \frac{3\,g_2}{20} + r_{5,3}\,x + O(x^2),\ \ r_{5,3}\in\mathbb{C}.
\end{aligned}$$

4.2.4. Example 4. $g = 6$.

Differential expressions

$$L_3 = \frac{d^3}{d\,x^3} + \big(\frac{30}{7}\sqrt{3g_2} - 48\,\wp(x)\big)\frac{d}{d\,x} - 24\wp'(x),$$

$$\begin{aligned}P_7 = {}& \frac{d^7}{d\,x^7} + \big(\frac{4}{3}\sqrt{3g_2} - 112\,\wp(x)\big)\frac{d^5}{d\,x^5} - 280\,\wp'(x)\,\frac{d^4}{d\,x^4} \\ &+ \Big(\frac{316\,g_2}{3} + \frac{160}{3}\sqrt{3\,g_2}\,\wp(x) + 1120\,\wp(x)^2\Big)\frac{d^3}{d\,x^3} + \Big(80\sqrt{3\,g_2}\,\wp'(x) \\ &+ 6720\,\wp(x)\,\wp'(x)\Big)\frac{d^2}{d\,x^2} - \frac{8}{49}\Big(3333\sqrt{3}\,g_2{}^{\frac{3}{2}} + 23030\,g_3 + 5614\,g_2\,\wp(x) \\ &+ 30380\sqrt{3\,g_2}\,\wp(x)^2 - 150920\,\wp(x)^3\Big)\frac{d}{d\,x} \\ &+ \frac{512}{7}\Big(19\,g_2 - 70\sqrt{3\,g_2}\,\wp(x)\Big)\;\wp'(x).\end{aligned}$$

Curve

$$\begin{aligned}\mathcal{F}_6(z,y) = {}& y^3 - z^7 + \Big(\frac{1172432}{441}\sqrt{3}\,g_2{}^{\frac{3}{2}} + 2992\,g_3\Big)\,z^5 \\ &- \Big(\frac{389275254016}{453789}\,g_2{}^3 + \frac{8716731904}{3087}\sqrt{3}\,g_2{}^{\frac{3}{2}}\,g_3 + 2972416\,g_3{}^2\Big)\,z^3 \\ &+ \Big(26\sqrt{3\,g_2}\,z^4 - \frac{20521280}{1029}\,g_2{}^2\,z^2 + \frac{308472947200}{823543}\sqrt{3}\,g_2{}^{\frac{7}{2}} \\ &- \frac{225472}{7}\sqrt{3\,g_2}\,g_3\,z^2 + \frac{14301619200}{2401}\,g_2{}^2\,g_3 + \frac{41817600}{7}\sqrt{3\,g_2}\,g_3{}^2\Big)\,y \\ &+ \Big(\frac{791904252620800}{17294403}\sqrt{3}\,g_2{}^{\frac{9}{2}} + \frac{77133027840000}{117649}\,g_2{}^3\,g_3 \\ &+ \frac{346472755200}{343}\sqrt{3}\,g_2{}^{\frac{3}{2}}\,g_3{}^2 + 1003622400\,g_3{}^3\Big)\,z = 0. \end{aligned} \tag{4.61}$$

Elliptic solution of the second kind

$$\psi_a(z,x) = e^{\lambda_a(z)x}\prod_{j=1}^{6}\frac{\sigma(x-a_j(z))}{\sigma(x)\sigma(a_j(z))}, \quad a(z) = (a_1(z),\dots,a_6(z)),$$

$$z = 22(\wp'(a_1) + \wp'(a_2) + \wp'(a_3) + \wp'(a_4) + \wp'(a_5) + \wp'(a_6)),$$

$$\lambda_a = \zeta(a_1) + \zeta(a_2) + \zeta(a_3) + \zeta(a_4) + \zeta(a_5) + \zeta(a_6),$$

$$\frac{30}{7}\sqrt{3g_2} = -10(\wp(a_1) + \wp(a_2) + \wp(a_3) + \wp(a_4) + \wp(a_5) + \wp(a_6)),$$

$$\begin{aligned}0 = {}& -13\sum_{j=1}^{6}\wp(a_j) - 42\wp(a_j) \\ &+ \frac{3}{4}\sum_{\substack{\ell=1\\ \ell\neq j}}^{6}\left(\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\right)^2 + \frac{3}{4}\left(\sum_{\substack{\ell=1\\ \ell\neq j}}^{6}\frac{\wp'(a_\ell)+\wp'(a_j)}{\wp(a_\ell)-\wp(a_j)}\right)^2, \quad 1\le j\le 6.\end{aligned}$$

Series solutions

$$\psi_1 = x^8 + O(x^{10}),$$

$$\psi_2 = x + \frac{\sqrt{3\,g_2}}{21}x^3 - \frac{z}{120}x^4 - \frac{11\,g_2}{490}x^5 - \frac{z\sqrt{3\,g_2}}{630}x^6$$
$$+ \left(\frac{z^2}{9360} - \frac{1609}{133770}\sqrt{3}\,{g_2}^{\frac{3}{2}} - \frac{6\,g_3}{91}\right)x^7 + r_{7,2}x^8 + O(x^9),\ r_{7,2} \in \mathbb{C},$$
$$\psi_3 = \frac{1}{x^6} + \frac{3}{14}\sqrt{3\,g_2}\,\frac{1}{x^4} + \frac{z}{132}\frac{1}{x^3} - \frac{2\,g_2}{245}\frac{1}{x^2} + \frac{4}{1155}\sqrt{3\,g_2}\,z\,\frac{1}{x} + \frac{z^2}{6336}$$
$$- \frac{461}{13720}\sqrt{3}\,{g_2}^{\frac{3}{2}} - \frac{g_3}{7} + r_{7,3}\,x + O(x^2),\ r_{7,3} \in \mathbb{C}.$$

To obtain the results for the case $h_6 = -\frac{30}{7}\sqrt{3g_2}$ simply replace $\sqrt{g_2}$ by $-\sqrt{g_2}$ in all expressions above.

Remark 4.7. *If $g_2 = 0$, all curves above degenerate into cyclic coverings of the line (see, e.g., [**70**]), i.e.,*

$$\mathcal{F}_g(z,y) = y^3 - T_{g+1}(z) = 0. \tag{4.62}$$

4.2.5. Example 5. $g = 7$.

$$L_3 = \frac{d^3}{d\,x^3} + \big(h_7 - 63\,\wp(x)\big)\frac{d}{d\,x} - \frac{63}{2}\wp'(x).$$

Series solutions

$$\psi_1 = x^9 + O(x^{11}).$$

The ansatz

$$\psi_2 = \sum_{j=0}^{\infty} r_{j,2}x^{j+1}$$

leads to the condition

$$0 = 54054000\,{g_2}^2 + \big(55296\,z^2 - 49420800\,g_3\big)\,h_7 - 1801800\,g_2\,{h_7}^2 + 3575\,{h_7}^4$$

for $r_{8,2}$ being finite. This equation does not have a solution h_7 which is independent of z if $g_2 \neq 0$.

Remark 4.8. *This result does not imply that there exist no commuting pairs of differential expressions (L_3, P_7) with elliptic coefficients where $g_2 \neq 0$. For example, choose in (3.1) for the coefficients $(q_1(x), q_0(x))$ one of the pairs $\{(-6\,\wp(x), 3\,\wp'(x)),\ (-18\,\wp(x), \pm 15\,\wp'(x)),\ (-12\,\wp(x), \pm 6\,\wp'(x))\}$. Then there exist corresponding differential expressions P_7 such that the commutator $[L_3, P_7] = 0$.*

4.3. The rational limit $\omega_1 \to \infty, \omega_3 \to \infty$

In the limiting case, where the half-periods $\omega_1 \to \infty, \omega_3 \to \infty$, equation (4.2) degenerates into

$$\psi'''(z,x) - \frac{g(g+2)}{x^2}\psi'(z,x) + \big(\frac{g(g+2)}{x^3} - z\big)\psi(z,x) = 0, \tag{4.63}$$
$$z \in \mathbb{C},\quad g \in \mathbb{N},\quad g \not\equiv 2 \pmod 3).$$

According to the theory of Fuchs, $x = 0$ is a regular singular point of (4.63). By the method of Frobenius

$$\psi(z,x) = x^{\rho} \sum_{\ell=0}^{\infty} r_\ell x^\ell, \quad r_0 \neq 0 \tag{4.64}$$

then yields from the indicial equation $\rho = -g, 1, (g+2)$. This directly leads to the following three linear independent meromorphic solutions

$$\psi_1(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell+g+2}, \quad r_{3\ell+1} = r_{3\ell+2} = 0, \tag{4.65}$$
$$r_{3\ell+3} = \frac{z r_{3\ell}}{(3\ell+3)(3\ell+g+4)(3\ell+2g+5)}, \quad \ell \in \mathbb{N}_0,$$

$$\psi_2(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell+1}, \quad r_{3\ell+1} = r_{3\ell+2} = 0, \tag{4.66}$$
$$r_{3\ell+3} = \frac{z r_{3\ell}}{(3\ell+3)(3\ell+g+4)(3\ell-g+2)}, \quad \ell \in \mathbb{N}_0,$$

$$\psi_3(z,x) = \sum_{\ell=0}^{\infty} r_\ell x^{\ell-g}, \quad r_{3\ell+1} = r_{3\ell+2} = 0, \tag{4.67}$$
$$r_{3\ell+3} = \frac{z r_{3\ell}}{(3\ell+3)(3\ell-g+2)(3\ell-2g+1)}, \quad \ell \in \mathbb{N}_0.$$

Note that the denominators in the coefficients $r_{3\ell+3}$ in (4.66) and (4.67) can not become zero since $g \not\equiv 2 \pmod 3$. Thus (4.63) possesses a meromorphic fundamental system. By another theorem of Halphen [**51**], [**56**, p. 272–275] the general solution of (4.63) must therefore have the following form

$$\psi(z,x) = \sum_{j=1}^{3} c_j \frac{p_{g,j}(x)}{x^g} e^{\beta_j x} \tag{4.68}$$

where $c_j, \beta_j \in \mathbb{C}, j = 1,2,3$ and $p_{g,j}(x), j = 1,2,3$ are polynomials of degree g. Equation (4.63) is invariant under the transformation $x \to \alpha_3^j k x, j = 1,2,3,\ k = z^{1/3},\ \alpha_3 = e^{2\pi i/3}$, which finally yields the general solution of (4.63)

$$\psi(z,x) = \sum_{j=1}^{3} \frac{c_j p_g(\alpha_3^j k x)}{x^g} e^{\alpha_3^j k x}, \qquad p_g(\tilde{x}) = \sum_{\ell=0}^{g} \tilde{r}_\ell \tilde{x}^\ell, \tag{4.69}$$
$$\tilde{r}_{\ell+3} = \frac{(6g\ell + 11g - 3\ell^2 - 9\ell - 6 - 2g^2)\tilde{r}_{\ell+2} + 3(g-\ell-1)\tilde{r}_{\ell+1}}{(\ell+3)(\ell-g+2)(\ell-2g+1)},$$
$$\ell = 0, \dots, g-3, \quad \tilde{r}_1 = -\tilde{r}_0, \quad \tilde{r}_2 = \tilde{r}_0/2.$$

Remark 4.9. *Halphen solved equation (4.63) by using a Darboux-type transformation expressing a solution ψ corresponding to $g+3$ in terms of a solution ψ for g, i.e.,*

$$\psi_{g+3} = z\,\psi_g - \frac{2g+3}{x}\,\psi_g'' + \frac{(2g+3)(g+1)}{x^2}\,\psi_g' - \frac{(2g+3)(g+1)}{x^3}\,\psi_g. \tag{4.70}$$

4.3.1. Examples. Rational Bsq potentials. We abbreviate $y_j = \omega_3^j y,\ 1 \le j \le 3,\ \omega_3 = \exp(2\pi i/3)$.

(i). $r = 2$ (genus $g = 1$):

$$L_3 = \frac{d^3}{d\,x^3} - \frac{3}{x^2}\frac{d}{d\,x} + \frac{3}{x^3}, \qquad P_2 = \frac{d^2}{d\,x^2} - \frac{2}{x^2}, \tag{4.71}$$

$$\mathcal{F}_1(z,y) = y^3 - z^2 = 0, \tag{4.72}$$

$$F_2(z,x) = 1, \qquad G_2(z,x) = 0, \tag{4.73}$$

$$D_1(z,x) = z - \frac{1}{x^3}, \qquad N_2(z,x) = z^2 + \frac{2}{x^3}\,z + \frac{1}{x^6}, \tag{4.74}$$

$$\phi_j(z,x) = \frac{(z + \frac{1}{x^3})}{y_j - \frac{1}{x^2}} \tag{4.75}$$

$$= \frac{y_j^2 + y_j\frac{1}{x^2} + \frac{1}{x^4}}{z - \frac{1}{x^3}} \tag{4.76}$$

$$= \frac{(z + \frac{1}{x^3})^2}{(z + \frac{1}{x^3})y_j - \frac{1}{x^2}(z + \frac{1}{x^3})}, \quad 1 \le j \le 3. \tag{4.77}$$

(ii). $r = 4$ (genus $g = 3$):

$$L_3 = \frac{d^3}{d\,x^3} - \frac{15}{x^2}\frac{d}{d\,x} + \frac{15}{x^3}, \quad P_4 = \frac{d^4}{d\,x^4} - \frac{20}{x^2}\frac{d^2}{d\,x^2} + \frac{40}{x^3}\frac{d}{d\,x}, \tag{4.78}$$

$$\mathcal{F}_3(z,y) = y^3 - z^4 = 0, \tag{4.79}$$

$$F_4(z,x) = -\frac{5}{x^2}, \qquad G_4(z,x) = z, \tag{4.80}$$

$$D_3(z,x) = z^3 - \frac{5}{x^3}\,z^2 - \frac{200}{x^6}\,z - \frac{1000}{x^9},$$

$$N_4(z,x) = z^4 + \frac{10}{x^3}\,z^3 + \frac{225}{x^6}\,z^2 + \frac{27000}{x^{12}}, \tag{4.81}$$

$$\phi_j(z,x) = \frac{(z + \frac{5}{x^3})\,y_j + (\frac{300}{x^7} - \frac{20}{x^4}\,z)}{-\frac{5}{x^2}\,y_j - (\frac{100}{x^6} - z^2)} \tag{4.82}$$

$$= \frac{\frac{5}{x^2}\,y_j^2 + (z^2 - \frac{100}{x^6})\,y_j + (\frac{5}{x^4}\,z^2 + \frac{400}{x^7}\,z + \frac{3000}{x^{10}})}{z^3 - \frac{5}{x^3}\,z^2 - \frac{200}{x^6}\,z - \frac{1000}{x^9}} \tag{4.83}$$

$$= \frac{z^4 + \frac{10}{x^3}\,z^3 + \frac{225}{x^6}\,z^2 + \frac{27000}{x^{12}}}{(z + \frac{5}{x^3})\,y_j^2 + (\frac{20}{x^4}\,z - \frac{300}{x^7})\,y_j - (\frac{5}{x^2}\,z^3 + \frac{25}{x^5}\,z^2 + \frac{600}{x^8}\,z + \frac{9000}{x^{11}})},$$
$$1 \le j \le 3. \tag{4.84}$$

(iii). $r = 5$ (genus $g = 4$):

$$L_3 = \frac{d^3}{d\,x^3} - \frac{24}{x^2}\frac{d}{d\,x} + \frac{24}{x^3},$$

$$P_5 = \frac{d^5}{dx^5} - \frac{40}{x^2}\frac{d^3}{dx^3} + \frac{120}{x^3}\frac{d^2}{dx^2} + \frac{40}{x^4}\frac{d}{dx} - \frac{320}{x^5}, \tag{4.85}$$

$$\mathcal{F}_4(z,y) = y^3 - z^5 = 0, \tag{4.86}$$

$$F_5(z,x) = z, \qquad G_5(z,x) = -\frac{56}{x^4}, \tag{4.87}$$

$$D_4(z,x) = z^4 - \frac{8}{x^3}\,z^3 - \frac{224}{x^6}\,z^2 + \frac{12544}{x^9}\,z + \frac{175616}{x^{12}},$$

$$N_5(z,x) = z^5 + \frac{16}{x^3}\,z^4 + \frac{960}{x^6}\,z^3 - \frac{17920}{x^9}\,z^2 - \frac{200704}{x^{12}}\,z - \frac{11239424}{x^{15}}, \tag{4.88}$$

$$\phi_j(z,x) = \frac{-\frac{56}{x^4}\,y_j + (z^3 + \frac{8}{x^3}\,z^2 + \frac{224}{x^6}\,z - \frac{12544}{x^9})}{z\,y_j - (\frac{8}{x^2}\,z^2 - \frac{3136}{x^8})} \tag{4.89}$$

$$= \frac{z\,y_j^2 + \big(\frac{8}{x^2}z^2 - \frac{3136}{x^8}\big)\,y_j + \frac{8}{x^4}\,z^3 + \frac{448}{x^7}\,z^2 - \frac{37632}{x^{10}}\,z - \frac{702464}{x^{13}}}{z^4 - \frac{8}{x^3}\,z^3 - \frac{224}{x^6}\,z^2 + \frac{12544}{x^9}\,z + \frac{175616}{x^{12}}} \tag{4.90}$$

$$= \frac{z^5 + \frac{16}{x^3}\,z^4 + \frac{960}{x^6}\,z^3 - \frac{17920}{x^9}\,z^2 - \frac{200704}{x^{12}}\,z - \frac{11239424}{x^{15}}}{\frac{56}{x^4}\,y_j^2 + (z^3 + \frac{8}{x^3}\,z^2 + \frac{224}{x^6}\,z - \frac{12544}{x^9})\,y_j + (\frac{-8}{x^2}\,z^4 - \frac{64}{x^5}\,z^3 + \frac{896}{x^8}\,z^2 + \frac{100352}{x^{11}}\,z + \frac{2809856}{x^{14}})},$$

$$1 \leq j \leq 3. \tag{4.91}$$

Chapter 5

On The Asymptotics Of A Diagonal Green's Function

For almost any $z \in \mathbb{C}$ let $\{\psi_\ell(z,x)\}_{\ell=1}^3$ be a fundamental system for the differential equation

$$L_3\psi(z,x) = z\psi(z,x). \tag{5.1}$$

Define the on diagonal Green's function $G(z,x) = G(z,x,x')|_{x=x'}$ to be

$$G(z,x) = -\frac{W(\psi_1,\psi_2)(z,x)\,\psi_3(z,x)}{W(\psi_1,\psi_2,\psi_3)(z)}. \tag{5.2}$$

Lemma 5.1. *The diagonal Green's function* (5.2) *satisfies a linear differential equation of order eight.*

Proof. We first recall that the Wronskian in the denominator of (5.2) is a constant, and can henceforth be ignored in the pursuit of a linear differential equation. Define $W_1 = \psi_2\psi_{3,x} - \psi_{2,x}\psi_3$, $W_2 = \psi_1\psi_{3,xx} - \psi_{1,xx}\psi_3$, and $W_3 = \psi_{2,x}\psi_{1,xx} - \psi_{2,xx}\psi_{1,x}$. Observe that repeated use of (3.28) and (5.1) enables one to express the derivatives of order three and higher of $\psi_\ell W_\ell$ as linear combinations of terms of the form $\psi_1^{(j)}W_1^{(k)}$, $0 \le j,\, k \le 2$. This linear system can be solved to obtain a linear eighth order differential equation for $G(z,x)$. $\square$

Remark 5.2. *We do not provide the reader with this differential equation due to the complexity of its coefficients.*

The linear differential equation for $G(z,x)$ in the special case $q_0(x) = 0$ reads

$$\begin{aligned}
&q_{1,x}(x)\,G_{xxxxxxxx}(z,x) - q_{1,xx}(x)\,G_{xxxxxxx}(z,x) + 6\,q_1(x)\,q_{1,x}(x)\,G_{xxxxxx}(z,x)\\
&+ \big(24\,q_{1,x}(x)^2 - 6\,q_1(x)\,q_{1,xx}(x)\big)\,G_{xxxxx}(z,x) + \big(9\,q_1(x)^2\,q_{1,x}(x)\\
&+ \frac{49}{2}\,q_{1,x}(x)\,q_{1,xx}(x)\big)G_{xxxx}(z,x) + \big(48\,q_1(x)\,q_{1,x}(x)^2 - 9\,q_1(x)^2\,q_{1,xx}(x)\\
&- \frac{49}{2}\,q_{1,xx}(x)^2 42\,q_{1,x}(x)\,q_{1,xxx}(x)\big)G_{xxx}(z,x) + \big(27\,z^2\,q_{1,x}(x) + 4\,q_1(x)^3\,q_{1,x}(x)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{189}{4}\,q_{1,x}(x)^3+\frac{41}{2}\,q_1(x)\,q_{1,x}(x)\,q_{1,xx}(x)-\frac{35}{2}\,q_{1,xx}(x)\,q_{1,xxx}(x)\\
&+24\,q_{1,x}(x)\,q_{1,xxxx}(x)\big)G_{xx}(z,x)+\big(18\,q_1(x)^2\,q_{1,x}(x)^2-27\,z^2\,q_{1,xx}(x)\\
&-4\,q_1(x)^3\,q_{1,xx}(x)+\frac{203}{4}\,q_{1,x}(x)^2\,q_{1,xx}(x)-\frac{41}{2}\,q_1(x)\,q_{1,xx}(x)^2\\
&+\frac{51}{2}\,q_1(x)\,q_{1,x}(x)\,q_{1,xxx}(x)-\frac{13}{2}\,q_{1,xx}(x)\,q_{1,xxxx}(x)+\frac{15}{2}\,q_{1,x}(x)\,q_{1,xxxxx}(x)\big)G_x(z,x)\\
&+\big(10\,q_1(x)\,q_{1,x}(x)^3+\frac{35}{2}\,q_{1,x}(x)^2\,q_{1,xxx}(x)-5\,q_1(x)\,q_{1,xx}(x)\,q_{1,xxx}(x)\\
&+5\,q_1(x)\,q_{1,x}(x)\,q_{1,xxxx}(x)-q_{1,xx}(x)\,q_{1,xxxxx}(x)+q_{1,x}(x)\,q_{1,xxxxxx}(x)\big)G(x)=0,\\
&\qquad q_0(x)=0. \qquad (5.3)
\end{aligned}
$$

Lemma 5.3. *The diagonal Green's function* (5.2) *is of the form*

$$
G(z,x)=\frac{-F_r(z,x)\,y(z)+A_r(z,x)}{3y(z)^2+S_r(z)}. \qquad (5.4)
$$

Proof. Straight forward but lengthy. (Hint: write the Wronskian in terms of ϕ). □

Corollary 5.4. *$F_r(z,x)$ and $A_r(z,x)$ are both solutions of the linear differential equation of Lemma 5.1.*

Proof. This follows by equating coefficients on y in the differential equation. Direct computation via symbolic mathematics software also verifies this result immediately. □

Theorem 5.5. *The diagonal Green's function* (5.2) *satisfies the fourth order non-linear differential equation:*

$$
\begin{aligned}
108\,&\big(4\,G+4\,q_1\,G^2-3\,G_x^2+4\,G\,G_{xx}\big)^3\,(z-q_0)^2\\
&+\big(-16+48\,q_1^2\,G^2+32\,q_1^3\,G^3-60\,q_1\,G_x^2-72\,q_1^2\,G\,G_x^2+60\,q_{1,x}\,G\,G_x\\
&+36\,q_1\,q_{1,x}\,G^2\,G_x-63\,q_{1,x}\,G_x^3-12\,q_{1,x}^2\,G^3+120\,q_1\,G\,G_{xx}\\
&+120\,q_1^2\,G^2\,G_{xx}-42\,q_1\,G_x^2\,G_{xx}+96\,q_{1,x}\,G\,G_x\,G_{xx}+12\,G_{xx}^2\\
&+84\,q_1\,G\,G_{xx}^2-4\,G_{xx}^3+24\,q_{1,xx}\,G^2+24\,q_1\,q_{1,xx}\,G^3-18\,q_{1,xx}\,G\,G_x^2\\
&+24\,q_{1,xx}\,G^2\,G_{xx}-24\,G_x\,G_{xxx}-48\,q_1\,G\,G_x\,G_{xxx}-24\,q_{1,x}\,G^2\,G_{xxx}\\
&+12\,G_x\,G_{xx}\,G_{xxx}-12\,G\,G_{xxx}^2+24\,G\,G_{xxxx}+24\,q_1\,G^2\,G_{xxxx}\\
&-18\,G_x^2\,G_{xxxx}+24\,G\,G_{xx}\,G_{xxxx}\big)^2=0. \qquad (5.5)
\end{aligned}
$$

Proof. The Green's function on the diagonal reads

$$
G(z,x)=-\frac{W(\psi_1,\psi_2)\psi_3}{W(\psi_1,\psi_2,\psi_3)}=\psi_3\psi_3^* \qquad (5.6)
$$

where

$$
\psi_3^*=-\frac{W(\psi_1,\psi_2)}{W(\psi_1,\psi_2,\psi_3)}. \qquad (5.7)
$$

Define

$$a_0 = \psi_3, \qquad a_1 = a_0' = \psi_3', \qquad a_2 = a_1' = a_0'' = \psi_3'', \tag{5.8}$$

and

$$b_0 = \psi_3^*, \qquad b_1 = b_0' = \psi_3'^*, \qquad b_2 = b_1' = b_0'' = \psi_3^{*''}. \tag{5.9}$$

Then

$$a_2' = \psi_3''' = -q_1\psi_3' - (\frac{1}{2}q_{1,x} + q_0 - z)\psi_3, \quad b_2' = \psi_3'''^* = -q_1\psi_3'^* - (\frac{1}{2}q_{1,x} - q_0 + z)\psi_3^*. \tag{5.10}$$

This yields the following system of equations for $G(z,x)$

$$\begin{aligned}
G &= a_0b_0, \\
G_x &= a_1b_0 + a_0b_1, \\
G_{xx} &= a_2b_0 + 2\,a_1b_1 + a_0b_2, \\
G_{xxx} &= 3\,a_2b_1 + 3\,a_1b_2 - q_1G_x - q_{1,x}G, \\
G_{xxxx} &= -q_{1,xx}G - \frac{7}{2}\,q_1G_x - q_1G_{xx} - 6\,q_1a_1b_1 + 6\,a_2b_2 + 3\,(q_0 - z)(a_1b_0 - a_0b_1), \\
-1 &= G_{xx} - 3\,a_1b_1 + q_1G,
\end{aligned} \tag{5.11}$$

where the last equation is derived from the Wronskian $W(\psi_1, \psi_2, \psi_3)$.

Eliminating $a_0, a_1, a_2, b_0, b_1, b_2$ yields the fourth order differential equation for $G(z,x)$.

□

Finally, we would like to mention the following

Conjecture 5.6. *The diagonal Green's function* (5.2) *has the asymptotic expansion*

$$G(z,x) \underset{z\to\infty}{=} -\frac{1}{3}\left(z^{-1/3}\sum_{j=0}^{\infty}\hat{f}_j^{(1)}z^{-j} + z^{-2/3}\sum_{j=0}^{\infty}\hat{f}_j^{(2)}z^{-j}\right) \tag{5.12}$$

where $\hat{f}_j^{(k)}$, $k = 1,2$ *are the homgeneous* $\hat{f}_j^{(k)}$ *satisfing the Boussinesq recursion.* $k = 1$ *corresponds to* $r \equiv 1 \pmod 3$ *and* $k = 2$ *corresponds to* $r \equiv 2 \pmod 3$*.*

Appendix A

Algebraic Curves and their Theta Functions in a Nutshell

This appendix treats some of the basic aspects of complex algebraic curves and their theta functions as used at numerous places in this paper. The material below is standard (see, e.g., [**11**], [**30**], [**50**], [**60**], and [**70**], (actually Appendix A in [**90**] contains all we need)), and we include it for two major reasons: On the one hand it allows us to introduce a large part of the notation used in Sections 2.3 and 3.2 (which otherwise would take up considerable space and disrupt the flow of arguments in these sections) and on the other hand, it permits a fairly self-contained presentation of the Bsq hierarchy and its algebro-geometric solutions in this paper.

Definition A.1. *An affine plane (complex) algebraic curve $\mathcal{K}$ is the locus of zeros in $\mathbb{C}^2$ of a (nonconstant) polynomial $\mathcal{F}(z,y)$ in two variables. The polynomial $\mathcal{F}$ is called nonsingular at a root (z_0,y_0) if*

$$\nabla\mathcal{F}(z_0,y_0)=(\mathcal{F}_z(z_0,y_0),\mathcal{F}_y(z_0,y_0))\neq 0. \tag{A.1}$$

The affine plane curve $\mathcal{K}$ of roots of $\mathcal{F}$ is called nonsingular at $P_0=(z_0,y_0)$ if $\mathcal{F}$ is nonsingular at P_0. The curve $\mathcal{K}$ is called nonsingular, or smooth, if it is nonsingular at each of its points.

The Implicit Function Theorem allows one to conclude that a smooth affine curve $\mathcal{K}$ is locally a graph and to introduce complex charts on $\mathcal{K}$ as follows. If $\mathcal{F}(P_0)=0$ with $\mathcal{F}_y(P_0)\neq 0$, there is a holomorphic function $g_{P_0}(z)$ such that in a neighborhood U_{P_0} of P_0, the curve $\mathcal{K}$ is characterized by the graph $y=g_{P_0}(z)$. Hence the projection

$$\tilde\pi_z\colon U_{P_0}\to\tilde\pi_z(U_{P_0})\subset\mathbb{C},\quad (z,y)\mapsto z, \tag{A.2}$$

yields a complex chart on $\mathcal{K}$. If, on the other hand, $\mathcal{F}(P_0)=0$ with $\mathcal{F}_z(P_0)\neq 0$, then the projection

$$\tilde\pi_y\colon U_{P_0}\to\tilde\pi_y(U_{P_0})\subset\mathbb{C},\quad (z,y)\mapsto y, \tag{A.3}$$

defines a chart on $\mathcal{K}$. In this way, as long as $\mathcal{K}$ is nonsingular, one arrives at a complex atlas on $\mathcal{K}$. The space $\mathcal{K} \subset \mathbb{C}^2$ is second countable and Hausdorff. In order to obtain a Riemann surface one needs connectedness of $\mathcal{K}$ which is implied by adding the assumption of irreducibility of the polynomial $\mathcal{F}$. Thus, $\mathcal{K}$ equipped with charts (A.2) and (A.3) is a Riemann surface if $\mathcal{F}$ is nonsingular and irreducible. Affine plane curves $\mathcal{K}$ are unbounded as subsets of $\mathbb{C}^2$, and hence noncompact. The compactification of $\mathcal{K}$ is conveniently described in terms of the projective plane $\mathbb{CP}^2$, the set of all one-dimensional (complex) subspaces of $\mathbb{C}^3$.

In order to simplify notations, we temporarily abbreviate $x_0 = x$, $x_1 = y$, and $x_2 = z$. Moreover, we denote the linear span of $(x_2, x_1, x_0) \in \mathbb{C}^3\backslash\{0\}$ by $[x_2 : x_1 : x_0]$. In particular, $[x_2 : x_1 : x_0] \in \mathbb{CP}^2$ with $L_\infty = \{[x_2 : x_1 : x_0] \in \mathbb{CP}^2 \mid x_0 = 0\}$ representing the line at infinity. Since the homogeneous coordinates $[x_2 : x_1 : x_0]$ satisfy

$$[x_2 : x_1 : x_0] = [cx_2 : cx_1 : cx_0], \quad c \in \mathbb{C}\backslash\{0\}, \tag{A.4}$$

the space $\mathbb{CP}^2$ can be viewed as the quotient space of $\mathbb{C}^3\backslash\{0\}$ by the multiplicative action of $\mathbb{C}\backslash\{0\}$, that is, $\mathbb{CP}^2 = (\mathbb{C}^3\backslash\{0\})/(\mathbb{C}\backslash\{0\})$, and hence $\mathbb{CP}^2$ inherits a Hausdorff topology which is the quotient topology induced by the natural map

$$\iota\colon \mathbb{C}^3\backslash\{0\} \to \mathbb{CP}^2, \quad (x_2, x_1, x_0) \mapsto [x_2 : x_1 : x_0]. \tag{A.5}$$

Next, define the open sets

$$U^m = \{[x_2 : x_1 : x_0] \in \mathbb{CP}^2 \mid x_m \neq 0\}, \quad m = 0, 1, 2. \tag{A.6}$$

Then

$$f^0\colon U^0 \to \mathbb{C}^2, \quad [x_2 : x_1 : x_0] \mapsto \left(\frac{x_2}{x_0}, \frac{x_1}{x_0}\right) \tag{A.7}$$

with inverse

$$(f^0)^{-1}\colon \mathbb{C}^2 \to U^0, \quad (x_2, x_1) \mapsto [x_2 : x_1 : 1], \tag{A.8}$$

and analogously for functions f^1 and f^2 (relative to sets U^1 and U^2, respectively), are homeomorphisms. In particular, U^0, U^1, and U^2 together cover $\mathbb{CP}^2$. Moreover, $\mathbb{CP}^2$ is compact since it is covered by the closed unit (poly)disks in U^0, U^1, and U^2.

Let $\mathcal{P}$ be a (nonconstant) homogeneous polynomial of degree d in (x_2, x_1, x_0), that is,

$$\mathcal{P}(cx_2, cx_1, cx_0) = c^d \mathcal{P}(x_2, x_1, x_0), \tag{A.9}$$

and introduce

$$\overline{\mathcal{K}} = \{[x_2 : x_1 : x_0] \in \mathbb{CP}^2 \mid \mathcal{P}(x_2, x_1, x_0) = 0\}. \tag{A.10}$$

The set $\overline{\mathcal{K}}$ is well-defined (even though $\mathcal{P}(u, v, w)$ is not for $[u : v : w] \in \mathbb{CP}^2$) and closed in $\mathbb{CP}^2$. The intersections,

$$\mathcal{K}^m = \overline{\mathcal{K}} \cap U^m, \quad m = 0, 1, 2 \tag{A.11}$$

are affine plane curves when transported to $\mathbb{C}^2$, that is,

$$\mathcal{K}^0 \cong \{(x_2, x_1) \in \mathbb{C}^2 \mid \mathcal{P}(x_2, x_1, 1) = 0\} \tag{A.12}$$

represents the affine curve $\mathcal{F}(z, y) = 0$, where $\mathcal{F}(x_2, x_1) = \mathcal{P}(x_2, x_1, 1)$, and analogously for $\mathcal{K}^1$ and $\mathcal{K}^2$. We recall that $\mathcal{F}(x_2, x_1)$ is irreducible if and only if $\mathcal{P}(x_2, x_1, x_0)$ is irreducible.

Given the affine curve defined by $\mathcal{F}(x_2, x_1) = 0$, the associated homogeneous polynomial $\mathcal{P}(x_2, x_1, x_0)$ can be obtained from

$$\mathcal{P}(x_2, x_1, x_0) = x_0^d \mathcal{F}\left(\frac{x_2}{x_0}, \frac{x_1}{x_0}\right), \tag{A.13}$$

where d denotes the degree of $\mathcal{F}$ (and $\mathcal{P}$).

The element $[x_2 : x_1 : 0] \in \mathbb{CP}^2$ represents the point at infinity along the direction $x_2 : x_1$ in $\mathbb{C}^2$ (identifying $[x_2 : x_1 : 0] \in \mathbb{CP}^2$ and $[x_2 : x_1] \in \mathbb{CP}^1$). The set of all such elements then represents the line at infinity, L_∞, and yields the compactification $\mathbb{CP}^2$ of $\mathbb{C}^2$. In other words, $\mathbb{CP}^2 \cong \mathbb{C}^2 \cup L_\infty$, $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$, and $L_\infty \cong \mathbb{CP}^1$. The projective plane curve $\overline{\mathcal{K}}$ then intersects L_∞ in a finite number of points (the points at infinity).

Definition A.2. *A projective plane (complex) algebraic curve $\overline{\mathcal{K}}$ is the locus of zeros in $\mathbb{CP}^2$ of a homogeneous polynomial $\mathcal{P}$ in three variables.*

A homogeneous (nonconstant) polynomial $\mathcal{P}(x_2, x_1, x_0)$ is called nonsingular if there are no common solutions $(x_{2,0}, x_{1,0}, x_{0,0}) \in \mathbb{C}^3 \backslash \{0\}$ of

$$\mathcal{P}(x_{2,0}, x_{1,0}, x_{0,0}) = 0, \tag{A.14}$$

$$\nabla\mathcal{P}(x_{2,0}, x_{1,0}, x_{0,0}) = (\mathcal{P}_{x_2}, \mathcal{P}_{x_1}, \mathcal{P}_{x_0})(x_{2,0}, x_{1,0}, x_{0,0}) = 0. \tag{A.15}$$

The set $\overline{\mathcal{K}}$ is called a smooth projective plane curve (of degree $d \in \mathbb{N}$) if $\mathcal{P}$ is nonsingular (and of degree $d \in \mathbb{N}$).

One verifies that the homogeneous polynomial $\mathcal{P}(x_2, x_1, x_0)$ is nonsingular if and only if each $\mathcal{K}^m$ is a smooth affine plane curve in $\mathbb{C}^2$. Moreover, any nonsingular homogeneous polynomial $\mathcal{P}(x_2, x_1, x_0)$ is irreducible and consequently each $\mathcal{K}^m$ is a Riemann surface for $m = 0$, 1, and 2. The coordinate charts on each $\mathcal{K}^m$ are simply the projections, that is, x_2/x_0 and x_1/x_0 for $\mathcal{K}^0$, x_2/x_1 and x_0/x_1 for $\mathcal{K}^1$, and finally, x_1/x_2 and x_0/x_2 for $\mathcal{K}^2$. These separate complex structures on $\mathcal{K}^m$ are compatible on $\overline{\mathcal{K}}$ and hence induce a complex structure on $\overline{\mathcal{K}}$.

The zero locus in $\mathbb{CP}^2$ of a nonsingular homogeneous polynomial $\mathcal{P}(x_2, x_1, x_0)$ defines a smooth projective plane curve $\overline{\mathcal{K}}$ which is a compact Riemann surface. Topologically, this Riemann surface is a sphere with g handles where

$$g = (d-1)(d-2)/2, \tag{A.16}$$

with d the degree of $\mathcal{P}(x_2, x_1, x_0)$. In particular, $\overline{\mathcal{K}}$ has topological genus g and we indicate this by writing $\overline{\mathcal{K}}_g$ in our main text, or simply $\mathcal{K}_g$ if no confusion can arise. In general, the projective curve $\mathcal{K}_g$ can be singular even though the associated affine curve $\mathcal{K}_g^0$ is nonsingular. In this case one has to account for the singularities at infinity and properly amend the genus formula (A.16) according to results of Clebsch, Noether, and Plücker.

If $\mathcal{K}_g$ is a nonsingular projective curve, associated with the homogeneous polynomial $\mathcal{P}(z, y, x)$ of degree d, the set of finite branch points of $\mathcal{K}_g$ is given by

$$\{[z : y : 1] \in \mathbb{CP}^2 \mid \mathcal{P}(z, y, 1) = \mathcal{P}_y(z, y, 1) = 0\}. \tag{A.17}$$

Similarly, branch points at infinity are defined by

$$\{[z : y : 0] \in \mathbb{CP}^2 \mid \mathcal{P}(z, y, 0) = \mathcal{P}_y(z, y, 0) = 0\}. \tag{A.18}$$

The set of branch points $\mathcal{B}$ of $\mathcal{K}_g$ then being the union of points in (A.17) and (A.18). Given $\mathcal{B} = \{P_1, \ldots, P_r\}$ one can cut the complex plane along smooth nonintersecting curves $\mathcal{C}_q$ (e.g., straight lines if $P_1, \dots, P_r$ are arranged suitably) connecting P_q and P_{q+1} for $q = 1, \dots, r-1$, and defines holomorphic functions $f_1, \dots, f_d$ on the cut plane $\Pi = \mathbb{C} \setminus \cup_{q=1}^{r-1} \mathcal{C}_q$ such that

$$\mathcal{P}(z, y, 1) = 0 \text{ for } y \in \Pi \text{ if and only if } y = f_j(z) \text{ for some } j \in \{1, \dots, d\}. \tag{A.19}$$

This yields a topological construction of $\mathcal{K}_g$ by appropriately gluing together d copies of the cut plane Π, the result being a sphere with g handles (g depending on the order of the branch points in $\mathcal{B}$). If $\mathcal{K}_g$ is singular, this procedure requires appropriate modifications.

Next, choose a homology basis $\{a_j, b_j\}_{j=1}^g$ on $\mathcal{K}_g$ for some $g \in \mathbb{N}$ in such a way that the intersection matrix of the cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad j, k = 1, \dots, g \tag{A.20}$$

(with a_j and b_k intersecting to form a right-handed coordinate system).

Turning briefly to meromorphic differentials (1-forms) on $\mathcal{K}_g$, we state the following result.

Theorem A.3 (Riemann's period relations)**.** *Let $g \in \mathbb{N}$ and suppose ω and ν to be closed differentials (1-forms) on $\mathcal{K}_g$. Then*
(i)

$$\iint_{\mathcal{K}_g} \omega \wedge \nu = \sum_{j=1}^g \bigg(\Big(\int_{a_j} \omega\Big)\Big(\int_{b_j} \nu\Big) - \Big(\int_{b_j} \omega\Big)\Big(\int_{a_j} \nu\Big) \bigg). \tag{A.21}$$

If, in addition ω and ν are holomorphic 1-forms on $\mathcal{K}_g$, then

$$\sum_{j=1}^g \bigg(\Big(\int_{a_j} \omega\Big)\Big(\int_{b_j} \nu\Big) - \Big(\int_{b_j} \omega\Big)\Big(\int_{a_j} \nu\Big) \bigg) = 0. \tag{A.22}$$

(ii) If ω is a nonzero holomorphic 1-form on $\mathcal{K}_g$, then

$$\operatorname{Im} \bigg(\sum_{j=1}^g \Big(\int_{a_j} \omega\Big)\Big(\int_{b_j} \omega\Big) \bigg) > 0. \tag{A.23}$$

The proof of Theorem A.3 is usually based on Stokes' theorem and a canonical dissection of $\mathcal{K}_g$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_g$ of the fundamental polygon $\partial \widehat{\mathcal{K}}_g$ given by

$$\partial \widehat{\mathcal{K}}_g = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g^{-1} b_g^{-1}. \tag{A.24}$$

Given the cycles $\{a_j, b_j\}_{j=1}^g$, we denote by $\{\omega_j\}_{j=1}^g$ a normalized basis of the space of holomorphic differentials (also called Abelian differentials of the first kind, denoted *dfk*) on $\mathcal{K}_g$, that is,

$$\int_{a_j} \omega_k = \delta_{j,k}, \quad j, k = 1, \dots, g. \tag{A.25}$$

The b-periods of ω_k are then defined by

$$\tau_{j,k} = \int_{b_j} \omega_k, \quad j, k = 1, \dots, g. \tag{A.26}$$

Theorem A.3 then implies the following result.

Theorem A.4. *The matrix τ is symmetric, that is,*

$$\tau_{j,k} = \tau_{k,j}, \quad j,k = 1,\dots,g, \tag{A.27}$$

with a positive definite imaginary part,

$$\mathrm{Im}(\tau) = (\tau - \tau^*)/(2i) > 0. \tag{A.28}$$

Abelian differentials of the second kind (abbreviated *dsk*), say $\omega^{(2)}$, are characterized by the property that all their residues vanish. They are normalized by the vanishing of all their a-periods (achieved by adding a suitable linear combination of *dfk*'s)

$$\int_{a_j} \omega^{(2)} = 0, \quad j = 1,\dots,g, \tag{A.29}$$

which determines them uniquely. (We will always assume that the poles of *dsk*'s on $\mathcal{K}_g$ lie in $\widehat{\mathcal{K}}_g$, that is, do not lie on $\partial\widehat{\mathcal{K}}_g$. This can always be achieved by an appropriate choice of the cycles a_j and b_j.) We may add in this context that the sum of the residues of any meromorphic differential ν on $\mathcal{K}_g$ vanishes, the residue at a pole $Q_0 \in \mathcal{K}_g$ of ν being defined by

$$\mathrm{res}_{Q_0}(\nu) = \frac{1}{2\pi i} \int_{\gamma_{Q_0}} \nu, \tag{A.30}$$

where γ_{Q_0} is a smooth, simple, closed contour, oriented counter-clockwise, encircling Q_0, but no other pole of ν.

Theorem A.5. *Let $g \in \mathbb{N}$. Assume $\omega^{(2)}_{Q_1,n}$ to be a dsk on $\mathcal{K}_g$, whose only pole is $Q_1 \in \widehat{\mathcal{K}}_g$ with principal part $\zeta_{Q_1}^{-n} d\zeta_{Q_1}$ for some $n \in \mathbb{N}_0$ and $\omega^{(1)}$ a dfk on $\mathcal{K}_g$ of the type $\omega^{(1)} = \sum_{m=0}^{\infty} c_m(Q_1)\zeta_{Q_1}^m d\zeta_{Q_1}$ near Q_1. Then*

$$\sum_{j=1}^{g} \bigg(\Big(\int_{a_j} \omega^{(1)}\Big)\Big(\int_{b_j} \omega^{(2)}_{Q_1,n}\Big) - \Big(\int_{b_j} \omega^{(1)}\Big)\Big(\int_{a_j} \omega^{(2)}_{Q_1,n}\Big) \bigg) = \frac{2\pi i}{(n-1)} c_{n-2}(Q_1), \quad n \geq 2. \tag{A.31}$$

In particular, if $\omega^{(2)}_{Q_1,n}$ is normalized and $\omega^{(1)} = \omega_j = \sum_{m=0}^{\infty} c_{j,m}(Q_1)\zeta_{Q_1}^m \, d\zeta_{Q_1}$, then

$$\int_{b_j} \omega^{(2)}_{Q_1,n} = \frac{2\pi i}{(n-1)} c_{j,n-2}(Q_1), \quad n \geq 2, \quad j = 1,\dots,g. \tag{A.32}$$

Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_g$ not of the first or second kind is said to be of the third kind, written *dtk*. It is common to normalize a *dtk* $\omega^{(3)}$, by the vanishing of its a-periods, that is, by

$$\int_{a_j} \omega^{(3)} = 0, \quad j = 1,\dots,g. \tag{A.33}$$

A normal *dtk*, denoted $\omega^{(3)}_{Q_1,Q_2}$, associated with two distinct points $Q_1, Q_2 \in \widehat{\mathcal{K}}_g$ by definition has simple poles at Q_ℓ with residues $(-1)^{\ell+1}$ for $\ell = 1$ and 2, vanishing a-periods, and is holomorphic anywhere else.

Theorem A.6. *Let $g \in \mathbb{N}$. Suppose $\omega^{(3)}$ to be a dtk on $\mathcal{K}_g$ whose only singularities are simple poles at $Q_n \in \widehat{\mathcal{K}}_g$ with residues c_n for $n = 1, \dots, N$. Denote by $\omega^{(1)}$ a dfk on $\mathcal{K}_g$. Then*

$$\sum_{j=1}^{g} \bigg(\Big(\int_{a_j} \omega^{(1)} \Big) \Big(\int_{b_j} \omega^{(3)} \Big) - \Big(\int_{b_j} \omega^{(1)} \Big) \Big(\int_{a_j} \omega^{(3)} \Big) \bigg) = 2\pi i \sum_{n=1}^{N} c_n \int_{Q_0}^{Q_n} \omega^{(1)}, \tag{A.34}$$

where $Q_0 \in \widehat{\mathcal{K}}_g$ is any fixed base point. In particular, if $\omega^{(3)}$ is normalized and $\omega^{(1)} = \omega_j$, then

$$\int_{b_j} \omega^{(3)} = 2\pi i \sum_{n=1}^{N} c_n \int_{Q_0}^{Q_n} \omega_j, \quad j = 1, \dots, g. \tag{A.35}$$

Moreover, if $\omega^{(3)}_{Q_1,Q_2}$ is a normal dtk on $\mathcal{K}_g$ holomorphic on $\mathcal{K}_g \backslash \{Q_1, Q_2\}$, then

$$\int_{b_j} \omega^{(3)}_{Q_1,Q_2} = 2\pi i \int_{Q_2}^{Q_1} \omega_j, \quad j = 1, \dots, g. \tag{A.36}$$

We shall always assume (without loss of generality) that all poles of *dsk*'s and *dtk*'s on $\mathcal{K}_g$ lie on $\widehat{\mathcal{K}}_g$ (i.e., not on $\partial\widehat{\mathcal{K}}_g$) and that integration paths on the right hand side of (A.34)–(A.36) do not touch any cycles a_j or b_k.

Next, we turn to divisors on $\mathcal{K}_g$ and the Jacobi variety $J(\mathcal{K}_g)$ of $\mathcal{K}_g$. Let $\mathcal{H}(\mathcal{K}_g)$ ($\mathcal{M}(\mathcal{K}_g)$) and $\mathcal{H}^1(\mathcal{K}_g)$ ($\mathcal{M}^1(\mathcal{K}_g)$) denote the set of holomorphic (meromorphic) functions (i.e., 0-forms) and holomorphic (meromorphic) 1-forms on $\mathcal{K}_g$ for some $g \in \mathbb{N}_0$.

Definition A.7. *Let $g \in \mathbb{N}_0$. Suppose $f \in \mathcal{M}(\mathcal{K}_g)$, $\omega = h(\zeta_{Q_0}) d\zeta_{Q_0} \in \mathcal{M}^1(\mathcal{K}_g)$, and (U_{Q_0}, ζ_{Q_0}) a chart near $Q_0 \in \mathcal{K}_g$.*
(i) If $(f \circ \zeta_{Q_0}^{-1})(\zeta) = \sum_{n=m_0}^{\infty} c_n(Q_0)\zeta^n$ for some $m_0 \in \mathbb{Z}$ (which turns out to be independent of the chosen chart), the order $\nu_f(Q_0)$ of f at Q_0 is defined by

$$\nu_f(Q_0) = m_0. \tag{A.37}$$

One defines $\nu_f(P) = \infty$ for all $P \in \mathcal{K}_g$ if f is identically zero on $\mathcal{K}_g$.
(ii) If $h_{Q_0}(\zeta_{Q_0}) = \sum_{n=m_0}^{\infty} d_n(Q_0)\zeta_{Q_0}^n$ for some $m_0 \in \mathbb{Z}$ (which again is independent of the chart chosen), the order $\nu_\omega(Q_0)$ of ω at Q_0 is defined by

$$\nu_\omega(Q_0) = m_0. \tag{A.38}$$

Definition A.8. *Let $g \in \mathbb{N}_0$.*
(i) A divisor $\mathcal{D}$ on $\mathcal{K}_g$ is a map $\mathcal{D}\colon \mathcal{K}_g \to \mathbb{Z}$, where $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_g$. On the set of all divisors $\operatorname{Div}(\mathcal{K}_g)$ on $\mathcal{K}_g$ one introduces the partial ordering

$$\mathcal{D} \geq \mathcal{E} \text{ if } \mathcal{D}(P) \geq \mathcal{E}(P), \quad P \in \mathcal{K}_g. \tag{A.39}$$

(ii) The degree $\deg(\mathcal{D})$ of $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$ is defined by

$$\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_g} \mathcal{D}(P). \tag{A.40}$$

(iii) $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$ is called nonnegative (or effective) if

$$\mathcal{D} \geq 0, \tag{A.41}$$

where 0 *denotes the zero divisor* $0(P) = 0$ *for all* $P \in \mathcal{K}_g$.
(iv) Let $\mathcal{D}, \mathcal{E} \in \operatorname{Div}(\mathcal{K}_g)$. *Then* $\mathcal{D}$ *is called a multiple of* $\mathcal{E}$ *if*

$$\mathcal{D} \geq \mathcal{E}. \tag{A.42}$$

$\mathcal{D}$ *and* $\mathcal{E}$ *are called relatively prime if*

$$\mathcal{D}(P)\mathcal{E}(P) = 0, \quad P \in \mathcal{K}_g. \tag{A.43}$$

(v) If $f \in \mathcal{M}(\mathcal{K}_g)\backslash\{0\}$ *and* $\omega \in \mathcal{M}^1(\mathcal{K}_g)\backslash\{0\}$, *then the divisor* (f) *of* f *is defined by*

$$(f)\colon \mathcal{K}_g \to \mathbb{Z}, \quad P \mapsto \nu_f(P) \tag{A.44}$$

(thus f *is holomorphic,* $f \in \mathcal{H}(\mathcal{K}_g)$, *if and only if* $(f) \geq 0$*), and the divisor of* ω *is defined by*

$$(\omega)\colon \mathcal{K}_g \to \mathbb{Z}, \quad P \mapsto \nu_\omega(P) \tag{A.45}$$

(thus ω *is a dfk,* $\omega \in \mathcal{H}^1(\mathcal{K}_g)$, *if and only if* $(\omega) \geq 0$*). The divisor* (f) *is called a principal divisor, and* (ω) *a canonical divisor.*
(vi) The divisors $\mathcal{D}, \mathcal{E} \in \operatorname{Div}(\mathcal{K}_g)$ *are called equivalent, written* $\mathcal{D} \sim \mathcal{E}$, *if*

$$\mathcal{D} - \mathcal{E} = (f) \tag{A.46}$$

for some $f \in \mathcal{M}(\mathcal{K}_g)\backslash\{0\}$. *The divisor class* $[\mathcal{D}]$ *of* $\mathcal{D}$ *is defined by*

$$[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_g) \mid \mathcal{E} \sim \mathcal{D}\}. \tag{A.47}$$

Clearly, $\operatorname{Div}(\mathcal{K}_g)$ forms an Abelian group with respect to addition of divisors. The principal divisors form a subgroup $\operatorname{Div}_{\mathrm{P}}(\mathcal{K}_g)$ of $\operatorname{Div}(\mathcal{K}_g)$. The quotient group $\operatorname{Div}(\mathcal{K}_g)/\operatorname{Div}_{\mathrm{P}}(\mathcal{K}_g)$ consists of the cosets of divisors, the divisor classes defined in (A.47). Also the set of divisors of degree zero, $\operatorname{Div}_0(\mathcal{K}_g)$, forms a subgroup of $\operatorname{Div}(\mathcal{K}_g)$. Since $\operatorname{Div}_{\mathrm{P}}(\mathcal{K}_g) \subset \operatorname{Div}_0(\mathcal{K}_g)$, one can introduce the quotient group $\operatorname{Pic}(\mathcal{K}_g) = \operatorname{Div}_0(\mathcal{K}_g)/\operatorname{Div}_{\mathrm{P}}(\mathcal{K}_g)$ called the Picard group of $\mathcal{K}_g$.

Theorem A.9. *Let* $g \in \mathbb{N}_0$. *Suppose* $f \in \mathcal{M}(\mathcal{K}_g)$ *and* $\omega \in \mathcal{M}^1(\mathcal{K}_g)$. *Then*

$$\deg((f)) = 0 \text{ and } \deg((w)) = 2(g-1). \tag{A.48}$$

Definition A.10. *Let* $g \in \mathbb{N}_0$, *and define*

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_g) \mid (f) \geq \mathcal{D}\}, \quad \mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_g) \mid (\omega) \geq \mathcal{D}\}. \tag{A.49}$$

Both $\mathcal{L}(\mathcal{D})$ and $\mathcal{L}^1(\mathcal{D})$ are linear spaces over $\mathbb{C}$. We denote their (complex) dimensions by

$$r(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}), \quad i(\mathcal{D}) = \dim \mathcal{L}^1(\mathcal{D}). \tag{A.50}$$

$i(\mathcal{D})$ is also called the index of specialty of $\mathcal{D}$.

Lemma A.11. *Let* $g \in \mathbb{N}_0$ *and* $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$. *Then* $\deg(\mathcal{D})$, $r(\mathcal{D})$, *and* $i(\mathcal{D})$ *only depend on the divisor class* $[\mathcal{D}]$ *of* $\mathcal{D}$ *(and not on the particular representative* $\mathcal{D}$*). Moreover, for* $\omega \in \mathcal{M}^1(\mathcal{K}_g)\backslash\{0\}$ *one infers*

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad \mathcal{D} \in \operatorname{Div}(\mathcal{K}_g). \tag{A.51}$$

Theorem A.12 (Riemann-Roch)**.** *Let* $g \in \mathbb{N}_0$ *and* $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$. *Then* $r(-\mathcal{D})$ *and* $i(\mathcal{D})$ *are finite and*

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - g + 1. \tag{A.52}$$

In particular, Riemann's inequality

$$r(-\mathcal{D}) \geq \deg(\mathcal{D}) - g + 1 \tag{A.53}$$

holds.

Next we turn to the Jacobi variety and the Abel map.

Definition A.13. *Let $g \in \mathbb{N}$ and define the period lattice L_g in $\mathbb{C}^g$ by*

$$L_g = \{\underline{z} \in \mathbb{C}^g \mid \underline{z} = \underline{N} + \tau\underline{M},\ \underline{N}, \underline{M} \in \mathbb{Z}^g\}. \tag{A.54}$$

Then the Jacobi variety $J(\mathcal{K}_g)$ of $\mathcal{K}_g$ is defined by

$$J(\mathcal{K}_g) = \mathbb{C}^g / L_g, \tag{A.55}$$

and the Abel maps are defined by

$$\begin{aligned}\underline{A}_{P_0}\colon \mathcal{K}_g \to J(\mathcal{K}_g), \quad P \mapsto \underline{A}_{P_0}(P) &= (A_{P_0,1}(P), \dots, A_{P_0,g}(P)) \\ &= \Big(\int_{P_0}^P \omega_1, \dots, \int_{P_0}^P \omega_g\Big) \,(mod\ L_g),\end{aligned} \tag{A.56}$$

and

$$\underline{\alpha}_{P_0}\colon \operatorname{Div}(\mathcal{K}_g) \to J(\mathcal{K}_g), \quad \mathcal{D} \mapsto \underline{\alpha}_{P_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_g} \mathcal{D}(P)\underline{A}_{P_0}(P), \tag{A.57}$$

where $P_0 \in \mathcal{K}_g$ is a fixed base point and (for convenience only) the same path is chosen from P_0 to P for all $j = 1, \dots, g$ in (A.56) *and* (A.57)[1].

Clearly, $\underline{A}_{P_0}$ is well-defined since changing the path from P_0 to P amounts to adding a closed cycle whose contribution in the integral (A.56) consists in adding a vector in L_g. Moreover, $\underline{\alpha}_{P_0}$ is a group homomorphism and $J(\mathcal{K}_g)$ is a complex torus of (complex) dimension g that depends on the choice of the homology basis $\{a_j, b_j\}_{j=1}^g$. However, different homology bases yield isomorphic Jacobians, see [**30**], p. 137, and [**50**], Section 8(b).

Theorem A.14 (Abel's theorem)**.** *Let $g \in \mathbb{N}$. Then $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$ is principal if and only if*

$$\deg(\mathcal{D}) = 0 \ \text{and}\ \underline{\alpha}_{P_0}(\mathcal{D}) = \underline{0}. \tag{A.58}$$

Next, we turn to Riemann theta functions and a constructive approach to the Jacobi inversion problem. We assume $g \in \mathbb{N}$ for the remainder of this appendix.

Given the curve $\mathcal{K}_g$, the homology basis $\{a_j, b_j\}_{j=1}^g$, and the matrix τ of b-periods of the *dfk*'s $\{\omega_j\}_{j=1}^g$, the Riemann theta function associated with $\mathcal{K}_g$ and the homology basis is defined as

$$\theta(\underline{z}) = \sum_{\underline{n} \in \mathbb{Z}^g} \exp\big(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \tau\underline{n})\big), \quad \underline{z} \in \mathbb{C}^g, \tag{A.59}$$

where $(\underline{u}, \underline{v}) = \sum_{j=1}^g \overline{u}_j v_j$ denotes the scalar product in $\mathbb{C}^g$. Because of (A.28), θ is well-defined and represents an entire function on $\mathbb{C}^g$. Elementary properties of θ are, for instance,

$$\theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = \theta(\underline{z}), \quad \underline{z} = (z_1, \dots, z_g) \in \mathbb{C}^g, \tag{A.60}$$

$$\theta(\underline{z} + \underline{m} + \tau\underline{n}) = \theta(\underline{z}) \exp\big(-2\pi i(\underline{n}, \underline{z}) - \pi i(\underline{n}, \tau\underline{n})\big), \quad \underline{m}, \underline{n} \in \mathbb{Z}^n,\ \underline{z} \in \mathbb{C}^g. \tag{A.61}$$

[1] This convention allows one to avoid the multiplicative version of the Riemann-Roch Theorem at various places in this paper.

Lemma A.15. *Let* $\underline{\xi} \in \mathbb{C}^g$ *and define*

$$F\colon \widehat{\mathcal{K}}_g \to \mathbb{C}, \quad P \mapsto \theta(\widehat{\underline{A}}_{P_0}(P) - \underline{\xi}), \tag{A.62}$$

where

$$\begin{aligned} \widehat{\underline{A}}_{P_0}\colon \widehat{\mathcal{K}}_g \to \mathbb{C}^g, \quad P \mapsto \widehat{\underline{A}}_{P_0}(P) &= \big(\widehat{A}_{P_0,1}(P), \dots, \widehat{A}_{P_0,g}(P)\big) \\ &= \bigg(\int_{P_0}^P \omega_1, \dots, \int_{P_0}^P \omega_g\bigg). \end{aligned} \tag{A.63}$$

Suppose F *is not identically zero on* $\widehat{\mathcal{K}}_g$*, that is,* $F \not\equiv 0$*. Then* F *has precisely* g *zeros on* $\widehat{\mathcal{K}}_g$ *counting multiplicities.*

Lemma A.15 is traditionally proven by integrating $d\ln(F)$ along $\partial\widehat{\mathcal{K}}_g$.

Theorem A.16. *Let* $\underline{\xi} \in \mathbb{C}^g$ *and define* F *as in* (A.62)*. Assume that* F *is not identically zero on* $\widehat{\mathcal{K}}_g$*, and let* $Q_1, \dots, Q_g \in \mathcal{K}_g$ *be the zeros of* F *(multiplicities included) given by Lemma A.15. Define the corresponding positive divisor* $\mathcal{D}_{\underline{Q}}$ *of degree* g *on* $\mathcal{K}_g$ *by*

$$\begin{aligned} &\mathcal{D}_{\underline{Q}}\colon \mathcal{K}_g \to \mathbb{N}_0, \\ &P \mapsto \mathcal{D}_{\underline{Q}}(P) = \begin{cases} m & \text{if } P \text{ occurs } m \text{ times in } \{Q_1, \dots, Q_g\}, \\ 0 & \text{if } P \notin \{Q_1, \dots, Q_g\}, \end{cases} \\ &\underline{Q} = (Q_1, \dots, Q_g), \end{aligned} \tag{A.64}$$

and recall the Abel map $\underline{\alpha}_{P_0}$ *in* (A.57)*. Then there exists a vector* $\underline{\Xi}_{P_0} \in \mathbb{C}^g$*, the vector of Riemann constants, such that*

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{Q}}) = (\underline{\xi} - \underline{\Xi}_{P_0}) (mod\ L_g). \tag{A.65}$$

The vector $\underline{\Xi}_{P_0} = (\Xi_{P_{0,1}}, \dots, \Xi_{P_{0,g}})$ *is given by*

$$\Xi_{P_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{g} \int_{a_\ell} \omega_\ell(P) \int_{P_0}^{P} \omega_j, \quad j = 1, \dots, g. \tag{A.66}$$

For the proof of Theorem A.16 one integrates $\widehat{A}_{P_{0,j}}(P)d\ln(F(P))$ along $\partial\widehat{\mathcal{K}}_g$. Clearly, $\underline{\Xi}_{P_0}$ depends on the base point P_0 and on the choice of the homology basis $\{a_j, b_j\}_{j=1}^g$.

Remark A.17. *Theorem A.16 yields a partial solution of Jacobi's inversion problem which can be stated as follows: Given* $\underline{\xi} \in \mathbb{C}^g$*, find a divisor* $\mathcal{D}_{\underline{Q}} \in \mathrm{Div}(\mathcal{K}_g)$ *such that*

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{Q}}) = \underline{\xi} (mod\ L_g). \tag{A.67}$$

Indeed, if $\widetilde{F}(P) = \theta(\underline{\Xi}_{P_0} - \widehat{\underline{A}}_{P_0}(P) + \underline{\xi}) \not\equiv 0$ *on* $\widehat{\mathcal{K}}_g$*, the zeros* $Q_1, \dots, Q_g \in \widehat{\mathcal{K}}_g$ *of* $\widetilde{F}$ *(guaranteed by Lemma A.15) satisfy Jacobi's inversion problem by* (A.65)*. Thus it remains to specify conditions such that* $\widetilde{F} \not\equiv 0$ *on* $\widehat{\mathcal{K}}_g$*.*

Remark A.18. *While* $\theta(\underline{z})$ *is well-defined (in fact, entire) for* $\underline{z} \in \mathbb{C}^g$*, it is not well-defined on* $J(\mathcal{K}_g) = \mathbb{C}^g/L_g$ *because of* (A.61)*. Nevertheless,* θ *is a "multiplicative function" on* $J(\mathcal{K}_g)$ *since the multipliers in* (A.61) *cannot vanish. In particular, if* $\underline{z}_1 = \underline{z}_2 (mod\ L_g)$*, then* $\theta(\underline{z}_1) = 0$ *if and only if* $\theta(\underline{z}_2) = 0$*. Hence it is meaningful to state that* θ *vanishes at points of* $J(\mathcal{K}_g)$*. Since the Abel map* $\underline{A}_{P_0}$ *maps* $\mathcal{K}_g$ *into* $J(\mathcal{K}_g)$*, the function* $\theta(\underline{A}_{P_0}(P) - \underline{\xi})$ *for* $\underline{\xi} \in \mathbb{C}^g$*,* *becomes a multiplicative function on* $\mathcal{K}_g$. *Again it makes sense to say that* $\theta(\underline{A}_{P_0}(\,\cdot\,) - \underline{\xi})$ *vanishes at points of* $\mathcal{K}_g$.

In the following we use the obvious notation

$$\begin{aligned} X + Y &= \{(\underline{x} + \underline{y}) \in J(\mathcal{K}_g) \mid \underline{x} \in X, \underline{y} \in Y\}, \\ -X &= \{-\underline{x} \in J(\mathcal{K}_g) \mid \underline{x} \in X\}, \\ X + \underline{z} &= \{(\underline{x} + \underline{z}) \in J(\mathcal{K}_g) \mid \underline{x} \in X\}, \end{aligned} \tag{A.68}$$

for $X, Y \subset J(\mathcal{K}_g)$ and $\underline{z} \in J(\mathcal{K}_g)$. Furthermore, we may identify the nth symmetric power of $\mathcal{K}_g$, denoted $\sigma^n\mathcal{K}_g$, with the set of nonnegative divisors of degree $n \in \mathbb{N}$ on $\mathcal{K}_g$. Moreover, we introduce the convenient notation ($N \in \mathbb{N}$)

$$\mathcal{D}_{P_0\underline{Q}} = \mathcal{D}_{P_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_N}, \quad \underline{Q} = (Q_1, \dots, Q_N) \in \sigma^N\mathcal{K}_g, \tag{A.69}$$

where for any $Q \in \mathcal{K}_g$,

$$\mathcal{D}_Q \colon \mathcal{K}_g \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_g \backslash \{Q\}. \end{cases} \tag{A.70}$$

Definition A.19. *(i) Define*

$$\underline{W}_0 = \{\underline{0}\} \subset J(\mathcal{K}_g), \quad \underline{W}_n = \underline{\alpha}_{P_0}(\sigma^n\mathcal{K}_g), \quad n \in \mathbb{N}. \tag{A.71}$$

(ii) A positive divisor $\mathcal{D} \in \mathrm{Div}(\mathcal{K}_g)$ *is called special if* $i(\mathcal{D}) \geq 1$, *otherwise* $\mathcal{D}$ *is called nonspecial.*
(iii) $Q \in \mathcal{K}_g$ *is called a Weierstrass point of* $\mathcal{K}_g$ *if* $i(g\mathcal{D}_Q) \geq 1$, *where* $g\mathcal{D}_Q = \sum_{j=1}^g \mathcal{D}_Q$.

Remark A.20. *(i) Since* $i(\mathcal{D}_P) = 0$ *for all* $P \in \mathcal{K}_1$, *the curve* $\mathcal{K}_1$ *has no Weierstrass points. For* $g \geq 2$, *and* $\mathcal{K}_g$ *hyperelliptic, the Weierstrass points of* $\mathcal{K}_g$ *are given precisely by the* $2g+2$ *branch points of* $\mathcal{K}_g$.
(ii) The special divisors of the type $\mathcal{D}_{\underline{Q}}$ *with* $\underline{Q} = (Q_1, \dots, Q_N) \in \sigma^N\mathcal{K}_g$ *and* $\deg(\underline{Q}) = N \geq g$ *are precisely the critical points of the Abel map* $\underline{\alpha}_{P_0} \colon \sigma^N\mathcal{K}_g \to J(\mathcal{K}_g)$, *that is, the set of points* $\mathcal{D}$ *at which the rank of the differential* $d\underline{\alpha}_{P_0}$ *is less than* g.
(iii) While $\sigma^m\mathcal{K}_g \not\subset \sigma^n\mathcal{K}_g$ *for* $m < n$, *one has* $\underline{W}_m \subseteq \underline{W}_n$ *for* $m < n$. *Thus* $\underline{W}_n = J(\mathcal{K}_g)$ *for* $n \geq g$ *by Theorem A.23 below.*

Theorem A.21. *The set* $\underline{W}_{g-1} + \underline{\Xi}_{P_0} \subset J(\mathcal{K}_g)$ *is the complete set of zeros of* θ *on* $J(\mathcal{K}_g)$, *that is,*

$$\theta(X) = 0 \textit{ if and only if } X \in \underline{W}_{g-1} + \underline{\Xi}_{P_0} \tag{A.72}$$

(i.e., if and only if $X = \big(\underline{\alpha}_{P_0}(\mathcal{D}) + \underline{\Xi}_{P_0}\big) (\textit{mod } L_g)$ *for some* $\mathcal{D} \in \sigma^{g-1}\mathcal{K}_g$*). The set* $\underline{W}_{g-1} + \underline{\Xi}_{P_0}$ *has complex dimension* $g-1$.

Theorem A.22 (Riemann's vanishing theorem)**.** *Let* $\underline{\xi} \in \mathbb{C}^g$.
(i) If $\theta(\underline{\xi}) \neq 0$, *then there exists a unique* $\mathcal{D} \in \sigma^g\mathcal{K}_g$ *such that*

$$\underline{\xi} = \big(\underline{\alpha}_{P_0}(\mathcal{D}) + \underline{\Xi}_{P_0}\big) \,(\textit{mod } L_g) \tag{A.73}$$

and

$$i(\mathcal{D}) = 0. \tag{A.74}$$

(ii) If $\theta(\underline{\xi}) = 0$ and $g = 1$, then

$$\underline{\xi} = \underline{\Xi}_{P_0} (mod\ L_1) = 2^{-1}(1+\tau)(mod\ L_1), \quad L_1 = \mathbb{Z} + \tau\mathbb{Z}, \quad -i\tau > 0. \tag{A.75}$$

(iii) Assume $\theta(\underline{\xi}) = 0$ and $g \geq 2$. Let $s \in \mathbb{N}$ with $s \leq g-1$ be the smallest integer such that $\theta(\underline{W}_s - \underline{W}_s - \underline{\xi}) \neq 0$ (i.e., there exist $\mathcal{E}, \mathcal{F} \in \sigma^s \mathcal{K}_g$ with $\mathcal{E} \neq \mathcal{F}$ such that $\theta(\underline{\alpha}_{P_0}(\mathcal{E}) - \underline{\alpha}_{P_0}(\mathcal{F}) - \underline{\xi}) \neq 0$). Then there exists a $\mathcal{D} \in \sigma^{g-1}\mathcal{K}_g$ such that

$$\underline{\xi} = \big(\underline{\alpha}_{P_0}(\mathcal{D}) + \underline{\Xi}_{P_0}\big)\ (mod\ L_g) \tag{A.76}$$

and

$$i(\mathcal{D}) = s. \tag{A.77}$$

All partial derivatives of θ with respect to $A_{P_0,j}$ for $j = 1, \dots, g$ of order strictly less than s vanish at $\underline{\xi}$, whereas at least one partial derivative of θ of order s is nonzero at $\underline{\xi}$. Moreover, $s \leq (g+1)/2$ and the integer s is the same for $\underline{\xi}$ and $-\underline{\xi}$.

Note that there is no explicit reference to the base point P_0 in the formulation of Theorem A.22 since the set $\underline{W}_s - \underline{W}_s \subset J(\mathcal{K}_g)$ is independent of the base point while $\underline{W}_s$ alone is not.

Theorem A.23 (Jacobi's inversion theorem)**.** *The map $\underline{\alpha}_{P_0}$ is surjective. More precisely, given $\tilde{\underline{\xi}} = (\underline{\xi} + \underline{\Xi}_{P_0}) \in \mathbb{C}^g$, the divisors $\mathcal{D}$ in* (A.73) *and* (A.76) *(resp. $\mathcal{D} = \mathcal{D}_{P_0}$ if $g = 1$) solve the Jacobi inversion problem for $\underline{\xi} \in \mathbb{C}^g$.*

We summarize some of this analysis in the following remark.

Remark A.24. *Consider the function*

$$G(P) = \theta\big(\underline{\Xi}_{P_0} - \widehat{\underline{A}}_{P_0}(P) + \sum_{j=1}^{g} \widehat{\underline{A}}_{P_0}(Q_j)\big), \quad P, Q_j \in \mathcal{K}_g, \quad j = 1, \dots, g \tag{A.78}$$

on $\mathcal{K}_g$. Then

$$G(Q_k) = \theta(\underline{\Xi}_{P_0} + \sum_{\substack{j=1 \\ j\neq k}}^{g} \widehat{\underline{A}}_{P_0}(Q_j)) = \theta\big(\underline{\Xi}_{P_0} + \underline{\alpha}_{P_0}(\mathcal{D}_{(Q_1,\dots,Q_{k-1},Q_{k+1},\dots,Q_g)})\big) = 0, \tag{A.79}$$

$$k = 1, \dots, g$$

by Theorem A.21. Moreover, by Lemma A.15 and Theorem A.22, the points $Q_1, \dots, Q_g$ are the only zeros of G on $\mathcal{K}_g$ if and only if $\mathcal{D}_{\underline{Q}}$ is nonspecial, that is, if and only if

$$i(\mathcal{D}_{\underline{Q}}) = 0, \quad \underline{Q} = (Q_1, \dots, Q_g) \in \sigma^g \mathcal{K}_g. \tag{A.80}$$

Conversely, $G \equiv 0$ on $\mathcal{K}_g$ if and only if $\mathcal{D}_{\underline{Q}}$ is special, that is, if and only if $i(\mathcal{D}_{\underline{Q}}) \geq 1$.

We also mention the elementary change in the Abel map and in Riemann's vector if one changes the base point,

$$\underline{A}_{P_1} = \big(\underline{A}_{P_0} - \underline{A}_{P_0}(P_1)\big)\ (\text{mod } L_g), \tag{A.81}$$

$$\underline{\Xi}_{P_1} = \big(\underline{\Xi}_{P_0} + (g-1)\underline{A}_{P_0}(P_1)\big)\ (\text{mod } L_g), \quad P_0, P_1 \in \mathcal{K}_g. \tag{A.82}$$

Remark A.25. *Let $\underline{\xi} \in J(\mathcal{K}_g)$ be given, assume that $\theta(\underline{\Xi}_{P_0} - \underline{A}_{P_0}(\,\cdot\,) + \underline{\xi}) \not\equiv 0$ on $\mathcal{K}_g$ and suppose that $\underline{A}_{P_0}^{-1}(\underline{\xi}) = (Q_1, \dots, Q_g) \in \sigma^g \mathcal{K}_g$ is the unique solution of Jacobi's inversion problem. Let $f \in \mathcal{M}(\mathcal{K}_g)\backslash\{0\}$ and suppose $f(Q_j) \neq \infty$ for $j = 1, \dots, g$. Then $\underline{\xi}$ uniquely determines the values $f(Q_1), \dots, f(Q_g)$. Moreover, any symmetric function of these values is a single-valued meromorphic function of $\underline{\xi} \in J(\mathcal{K}_g)$, that is, an Abelian function on $J(\mathcal{K}_g)$. Any such meromorphic function on $J(\mathcal{K}_g)$ can be expressed in terms of the Riemann theta function on $\mathcal{K}_g$. For instance, for the elementary symmetric functions of the second kind (Newton polynomials) one obtains from the residue theorem in analogy to the proof of Lemma A.15 that*

$$\sum_{j=1}^{g} f(Q_j)^n = \sum_{j=1}^{g} \int_{a_j} f(P)^n \omega_j(P) - \sum_{\substack{P_r \in \mathcal{K}_g \\ f(P_r)=\infty}} \operatorname{res}_{P=P_r} \big(f(P)^n d \ln(\theta(\underline{\Xi}_{P_0} - \underline{A}_{P_0} + \underline{\xi})) \big), \tag{A.83}$$

where an appropriate homology basis $\{a_j, b_j\}_{j=1}^g$ with $\partial \widehat{\mathcal{K}}_g = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g^{-1} b_g^{-1}$ avoiding $\{Q_1, \dots, Q_g\}$ and the poles $\{P_r\}$ of f has been chosen. (We also note that Lemma A.15 just corresponds to the case $n = 0$ in (A.83).*)*

Finally, we formulate the following auxiliary result (cf., e.g., Lemma 3.4 in [**39**]).

Lemma A.26. *Let $\psi(\,\cdot\,, x)$, $x \in \mathcal{U}$, $\mathcal{U} \subseteq \mathbb{R}$ open, be meromorphic on $\mathcal{K}_g \backslash \{P_\infty\}$ with an essential singularity at P_∞ (and $\widetilde{\Omega}^{(2)}_{P_\infty, r+1}$ defined as in* (3.215)*) such that $\widetilde{\psi}(\,\cdot\,, x)$ defined by*

$$\widetilde{\psi}(\,\cdot\,, x) = \psi(\,\cdot\,, x) \exp\bigg(- i(x - x_0) \int_{P_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty, r+1} \bigg) \tag{A.84}$$

is multi-valued meromorphic on $\mathcal{K}_n$ and its divisor satisfies

$$(\widetilde{\psi}(\,\cdot\,, x)) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x)}. \tag{A.85}$$

Define a divisor $\mathcal{D}_0(x)$ by

$$(\widetilde{\psi}(\,\cdot\,, x)) = \mathcal{D}_0(x) - \mathcal{D}_{\underline{\hat{\mu}}(x)}. \tag{A.86}$$

Then

$$\mathcal{D}_0(x) \in \sigma^g \mathcal{K}_g, \ \mathcal{D}_0(x) \geq 0, \ \deg(\mathcal{D}_0(x)) = g. \tag{A.87}$$

Moreover, if $\mathcal{D}_0(x)$ is nonspecial for all $x \in \mathcal{U}$, that is, if $i(\mathcal{D}_0(x)) = 0$, then $\psi(\,\cdot\,, x)$ is unique up to a constant multiple (which may depend on $x \in \mathcal{U}$).

Trigonal Curves of Boussinesq-Type

We give a brief summary of some of the fundamental properties and notations needed from the theory of trigonal curves of Boussinesq-type (i.e., those with a triple point at infinity).

First we investigate what happens at the point (or possibly points) at infinity on our Bsq-type curves. Fix $g \in \mathbb{N}$. The Bsq-type curve $\mathcal{K}_g$ of arithmetic genus $g = m - 1$ is defined by

$$\begin{gathered} \mathcal{F}_{m-1}(z,y) = y^3 + y\,S_m(z) - T_m(z) = 0, \\ S_m(z) = \sum_{p=0}^{2n-1+\varepsilon} s_{m,p} z^p, \quad T_m(z) = z^m + \sum_{q=0}^{m-1} t_{m,q} z^q, \\ m = 3n+\varepsilon,\ \varepsilon \in \{1,2\},\ n \in \mathbb{N}_0. \end{gathered} \tag{B.1}$$

Following the treatment in [**75**] one substitutes the variable $u = z^{-1}$ into (B.1) to obtain

$$u^{3n+\varepsilon} y^3 + \big(s_{m,0} u^{2n-1+\varepsilon} + \cdots + s_{m,2n-1+\varepsilon}\big) u^{n+1} y - \big(t_{m,0} u^{3n+\varepsilon} + \cdots + t_{m,m-1} u + 1\big) = 0. \tag{B.2}$$

Let $v = u^{n+1} y$ in (B.2) to obtain

$$v^3 + (s_{m,0} u^{2n-1+\varepsilon} + \cdots + s_{m,2n-1+\varepsilon}) u^{3-\varepsilon} v - (t_{m,0} u^{3n+\varepsilon} + \cdots + t_{m,3n-1+\varepsilon} u + 1) u^{3-\varepsilon} = 0. \tag{B.3}$$

Let $u \to 0$ (corresponding to $z \to \infty$) in (B.3) to obtain $v^3 = 0$. This corresponds to one point of multiplicity three at infinity (in both cases $\varepsilon = 1$ and $\varepsilon = 2$), given by $(u,v) = (0,0)$. We therefore use the coordinate $\zeta = z^{-1/3}$ at the branch point at infinity, denoted by P_∞.

The curve (B.1) is compactified by adding the point P_∞ at infinity. In homogeneous coordinates, the point at infinity we add is $[1:0:0] \in \mathbb{CP}^2$ if $g = 0$ or $g = 1$, otherwise the point at infinity we add is $[0:1:0] \in \mathbb{CP}^2$. The point P_∞ is singular in all cases except when $g = 1$, or when $g = 2$ and $s_{m,0} = -1/3$.

Although not directly associated with the Bsq hierarchy, we note that the case $\varepsilon = 0$ in (B.1) is analogous to AKNS, Toda, and Thirring-type hyperelliptic curves, which are not branched at infinity. In fact, a similar argument to that above, with the coordinate $v = u^n y$ in (B.2), yields the equation $v^3 = 1$ as $u \to 0$. This corresponds to three distinct points $P_{\infty,j}$, $j = 1, 2, 3$ at infinity (each with multiplicity one), given by the three points $(u, v) = (0, \omega_j)$ for $j = 1, 2, 3$, where the ω_1, ω_2, and ω_3 are the third roots of unity. As each point at infinity has multiplicity one, none are branch points, and consequently each admits the local coordinate $u = 1/z$ for $|z|$ sufficiently large.

In [**14**], p. 561, Burchnall and Chaundy define the g-number of an algebraic curve as the maximum number of double points possible in the finite plane. For Bsq-type curves the g-number is $g = m-1$. For a curve that is smooth in the finite plane, the g-number coincides with the arithmetic genus of the curve, but in the presence of double points, the g-number remains the same, while the genus is diminished (according to results of Clebsch, Noether, and Plücker, see, e.g., [**11**] and [**70**]). We now prove that the g-number of $\mathcal{K}_g$, and hence the arithmetic genus of $\mathcal{K}_g$ if $\mathcal{K}_g$ is smooth in the finite plane, is $m-1$ using a special case of the Riemann-Hurwitz theorem.

Theorem B.1. *Let $\tilde{\pi}_z\colon \mathcal{K}_g \to \mathbb{CP}^1$ be the projection map with respect to the z coordinate. Then*

$$\sum_{P\in\mathcal{K}_g} \big[\nu_P(\tilde{\pi}_z) - 1\big] = 2g + 4, \tag{B.4}$$

where $\nu_P(\tilde{\pi}_z)$ denotes the multiplicity of $\tilde{\pi}_z$ at $P \in \mathcal{K}_g$, and g is the arithmetic genus of the curve $\mathcal{K}_g$.

If equation (B.1) has only double points, this implies that the discriminant $\Delta(z)$ of the curve (B.1), defined by

$$\Delta(z) = 27T_m(z)^2 + 4\,S_m(z)^3 \tag{B.5}$$

(modulo constants), is non-zero. $\Delta(z)$ is easily seen to be a polynomial of degree $2m$. Hence in the finite complex plane, the Riemann surface defined by the compactification of (B.1) can have at most $2m$ double points, corresponding to the possible $2m$ zeros of $\Delta(z)$. If all finite branch points are distinct double points (taking into account the triple point at infinity) one obtains $\sum_{P\in\mathcal{K}_g} \big[\nu_P(\tilde{\pi}_z) - 1\big] = 2m + 2$, and so by (B.4), one infers $g = m - 1$.

Let $\mathcal{B}$ denote the set of branch points and let $|\mathcal{B}|$ denote the number of branch points counted according to multiplicity. In the case of Bsq-type curves, $\deg(\tilde{\pi}_z) = 3$, and $\nu_P(\tilde{\pi}_z) = 1$ for all $P \in \mathcal{K}_g \backslash \mathcal{B}$. Moreover, $\nu_P(\tilde{\pi}_z) \in \{2, 3\}$ for all $P \in \mathcal{B}$. Hence $|\mathcal{B}| \leq \sum_{P\in\mathcal{K}_g} \big[\nu_P(\tilde{\pi}_z) - 1\big] \leq 2|\mathcal{B}|$, and (B.4) reduces to

$$g + 2 \leq |\mathcal{B}| \leq 2g + 4. \tag{B.6}$$

Thus one arrives at an upper and lower bound on the number of branch points on $\mathcal{K}_g$.

When $m = 1$, corresponding to $g = 0$, there are no non-zero holomorphic differentials on $\mathcal{K}_g$. When $m = 2$, corresponding to $g = 1$, the only holomorphic differential on $\mathcal{K}_g$ is $dz/(3y(P)^2 + S_m(z))$. Recall also that $m \not\equiv 0 \pmod 3$, so we need not consider holomorphic differentials for the case $m = 3$. One verifies that $dz/(3y(P)^2 + S_m(z))$ and $y(P)dz/(3y(P)^2 + S_m(z))$ are holomorphic differentials $\mathcal{K}_g$ with zeros at P_∞ of order $2(m-2)$ and $(m-4)$, respectively, for $m \geq 4$. It follows that the differentials ($m = 3n + \varepsilon$, $\varepsilon \in \{1,2\}$)

$$\eta_\ell(P) = \frac{1}{3y(P)^2 + S_m(z)} \begin{cases} z^{\ell-1}dz & \text{for } 1 \leq \ell \leq g-n, \\ y(P)z^{\ell+n-g-1}dz & \text{for } g-n+1 \leq \ell \leq g, \end{cases} \tag{B.7}$$

form a basis in the space of holomorphic differentials $\mathcal{H}^1(\mathcal{K}_g)$. Introducing the invertible matrix $\Upsilon \in GL(g,\mathbb{C})$,

$$\begin{aligned} \Upsilon &= (\Upsilon_{j,k})_{j,k=1,\dots,g}, \quad \Upsilon_{j,k} = \int_{a_k} \eta_j, \\ \underline{e}(k) &= (e_1(k),\dots,e_g(k)), \quad e_j(k) = \left(\Upsilon^{-1}\right)_{j,k}, \end{aligned} \tag{B.8}$$

the normalized differentials ω_j for $j = 1,\dots,g$,

$$\omega_j = \sum_{\ell=1}^{g} e_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k = 1,\dots,g \tag{B.9}$$

form a canonical basis for $\mathcal{H}^1(\mathcal{K}_g)$. Near P_∞ one infers

$$\underline{\omega} = (\omega_1,\dots,\omega_g) \underset{\zeta\to 0}{=} \big(\underline{\alpha}_0^{(\varepsilon)} + \underline{\alpha}_1^{(\varepsilon)}\zeta + \underline{\alpha}_3^{(\varepsilon)}\zeta^3 + O(\zeta^4)\big)d\zeta, \tag{B.10}$$

where

$$\underline{\alpha}_0^{(\varepsilon)} = -\begin{cases} \underline{e}(g), & \varepsilon = 1, \\ \underline{e}(g-n), & \varepsilon = 2, \end{cases} \tag{B.11}$$

$$\underline{\alpha}_1^{(\varepsilon)} = \begin{cases} -\underline{e}(g-n), & \varepsilon = 1, \\ \big(d_0^{(2)}\underline{e}(g-n) - \underline{e}(g)\big), & \varepsilon = 2, \end{cases} \tag{B.12}$$

$$\underline{\alpha}_3^{(\varepsilon)} = \begin{cases} \big(d_1^{(1)}\underline{e}(g) + c_1^{(1)}\underline{e}(g-n) - \underline{e}(g-1)\big), & \varepsilon = 1, \\ \big((2c_1^{(2)} - (d_0^{(2)})^3)\underline{e}(g-n) - \underline{e}(g-n-1) + (d_0^{(2)})^2\underline{e}(g)\big), & \varepsilon = 2, \end{cases} \tag{B.13}$$

etc.,

and

$$y(P) \underset{\zeta\to 0}{=} \big(c_0^{(\varepsilon)} + d_0^{(\varepsilon)}\zeta + c_1^{(\varepsilon)}\zeta^3 + d_1^{(\varepsilon)}\zeta^4 + O(\zeta^6)\big)\zeta^{-3n-2} \text{ as } P \to P_\infty, \tag{B.14}$$

with

$$(c_0^{(\varepsilon)}, d_0^{(\varepsilon)}) = \begin{cases} (0,1), & \varepsilon = 1, \\ (1, d_0^{(2)}), & \varepsilon = 2, \end{cases} \qquad d_0^{(2)} \in \mathbb{C}. \tag{B.15}$$

In particular, using (A.32), (B.10), and (B.11), one obtains

$$\frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_\infty,2} = \alpha^{(\varepsilon)}_{0,j} \quad \text{and} \quad \frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_\infty,3} = \frac{1}{2}\alpha^{(\varepsilon)}_{1,j}. \tag{B.16}$$

Finally, we turn our attention to special divisors.

From the theory of elementary symmetric polynomials one infers the following lemma.

Lemma B.2. *Pick $z \in \mathbb{C}$, and denote by $y_1(z)$, $y_2(z)$, and $y_3(z)$, the three solutions of (B.1). These solutions are distinct if and only if the discriminant $\Delta(z) \neq 0$. Moreover, introduce $Q_j = (z, y_j) \in \mathcal{K}_g$ for $j = 1,2,3$. Then*

(i) $\sum_{j=1}^{3} y_j(z) = 0$.

(ii) $\sum_{j<k}^{3} y_j(z)y_k(z) = S_m(z)$.

(iii) $\prod_{j=1}^{3} y_j(z) = T_m(z)$.

(iv) $\sum_{j=1}^{3} y_j(z)^2 = -2S_m(z)$.

(v) $\sum_{j=1}^{3} y_j(z)^3 = 3T_m(z)$.

(vi) $\sum_{j\neq k}^{3} y_j(z)^2 y_k(z) = -3T_m(z)$.

(vii) $\sum_{j<k}^{3} y_j(z)^2 y_k(z)^2 = S_m(z)^2$.

(viii) $\prod_{j=1}^{3} \big(3y_j(z)^2 + S_m(z)\big) = \Delta(z)$.

Lemma B.3. *Let $m_1, \dots, m_r \in \mathbb{N}$ with $\sum_{j=1}^{r} m_j = g$ and $Q_j = (z, y_j)$, $j = 1, 2, 3$ as in Lemma B.2. Suppose $P_1, \dots, P_r \in \mathcal{K}_g$. If*

$$\{Q_1, Q_2, Q_3\} \subseteq \{P_1, \dots, P_r\}, \tag{B.17}$$

then the divisor $\mathcal{D}_{m_1P_1+\cdots+m_rP_r} \in \sigma^g\mathcal{K}_g$ is special. In particular, if one of the points $P_j \in \{P_1, \dots, P_r\}$ is a triple point, then the divisor $\mathcal{D}_{m_1P_1+\cdots+m_rP_r} \in \sigma^g\mathcal{K}_g$ is special.

Proof. Using the identities in Lemma B.2, one readily computes

$$\sum_{j=1}^{3} \frac{1}{3y_j(z)^2 + S_m(z)} = 0, \quad \sum_{j=1}^{3} \frac{y_j(z)}{3y_j(z)^2 + S_m(z)} = 0. \tag{B.18}$$

Thus, choosing for simplicity the base point $P_0 = P_\infty$, a comparison of (A.56), (B.7), and (B.18) yields

$$\sum_{j=1}^{3} \underline{A}_{P_\infty}(Q_j) = 0 \,(\mathrm{mod}\; L_g). \tag{B.19}$$

Thus $\mathcal{D}_{m_1P_1+\cdots+m_rP_r} \in \sigma^g\mathcal{K}_g$ is special by Theorem A.21. □

Bibliography

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.

[2] N. I. Akhiezer, *Elements of the Theory of Elliptic Functions*, Amer. Math. Soc., Providence, RI, 1990.

[3] M. Adler, *On the trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-Devries type equations*, Invent. Math. **50**, 219–248 (1979).

[4] H. Airault, *Solutions of the Boussinesq equation*, Physica D **21**, 171–176 (1986).

[5] H. Airault, H. P. McKean, and J. Moser, *Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem*, Commun. Pure Appl. Math. **30**, 95–148 (1977).

[6] S. I. Al'ber, *Investigation of equations of Korteweg-de Vries type by the method of recurrence relations*, J. London Math. Soc. (2) **19**, 467–480 (1979). (Russian.)

[7] S. I. Al'ber, *On stationary problems for equations of Korteweg-de Vries type*, Commun. Pure Appl. Math. **34**, 259–272 (1981).

[8] R. Beals, P. Deift, and C. Tomei, *Direct and Inverse Scattering on the Line*, Amer. Math. Soc. Providence, R.I., 1988.

[9] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.

[10] J. L. Bona and R. L. Sachs, *Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation*, Commun. Math. Phys. **118**, 15–29 (1988).

[11] E. Brieskorn and H. Knörrer, *Plane Algebraic Curves*, Birkhäuser, Basel, 1981.

[12] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl, *Algebro-geometric quasi-periodic finite-gap solutions of the Toda and Kac-van-Moerbeke hierarchies*, Memoirs Amer. Math. Soc., Providence, R.I., 135/641, 1998.

[13] J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators*, Proc. London Math. Soc. (2) **21**, 420–440 (1923).

[14] J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators*, Proc. Roy. Soc. London **A118**, 557–583 (1928).

[15] J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators II. The identity* $P^n = Q^m$, Proc. Roy. Soc. London **A134**, 471–485 (1932).

[16] D. V. Chudnovsky, *Meromorphic solutions of nonlinear partial differential equations and many-particle completely integrable systems*, J. Math. Phys. **20**, 2416–2422 (1979).

[17] W. Craig, *An existence theory for water waves and the Boussinesq and Korteweg-deVries scaling limits*, Commun. Part. Diff. Eqs. **10**, 787–1003 (1985).

[18] P. Deift, C. Tomei, and E. Trubowitz, *Inverse scattering and the Boussinesq equation*, Commun. Pure Appl. Math. **35**, 567–628 (1982).

[19] L.A. Dickey, *Soliton Equations and Hamiltonian Systems*, World Scientific, Singapore, 1991.

[20] R. Dickson, F. Gesztesy, K. Unterkofler, *A new approach to the Boussinesq hierarchy*, Math. Nachr. **198** (1999), 51–108.

[21] R. Dickson, F. Gesztesy, K. Unterkofler, *Algebro-geometric solutions of the Boussinesq hierarchy*, Rev. Math. Phys. **11** (1999), 823–879.

[22] B. A. Dubrovin, *Periodic problems for the Korteweg-de Vries equation in the class of finite band potentials*, Funct. Anal. Appl. **9**, 215–223, (1975).

[23] B. A. Dubrovin, *Completely integrable Hamiltonian systems associated with matrix operators and Abelian varieties*, Funct. Anal. Appl. **11**, 265–277 (1977).

[24] B. A. Dubrovin, *Theta functions and nonlinear equations*, Russ. Math. Surv. **36:2**, 11–92 (1981).

[25] B. A. Dubrovin, *Matrix finite-zone operators*, Revs. Sci. Tech. **23**, 20–50 (1983).

[26] J. C. Eilbeck, V. Z. Enol'skii, *Elliptic Baker-Akhiezer functions and an application to an integrable dynamical system*, J. Math. Phys. **35** (1994), 1192–1201.

[27] J. C. Eilbeck, V. Z. Enol'skii, and D. V. Leykin, *On the Kleinian construction of Abelian functions of canonical algebraic curves*, Proceedings of the Conference SIDE III: Symmetries of Integrable Difference Equations, Saubadia, May 1998, CRM Proceedings and Lecture Notes (1999), 121–138.

[28] V. Z. Enol'skii, N. A. Kostov, *On the geometry of elliptic solitons*, Acta Appl. Math. **36** (1994), 57–86.

[29] Y.-F. Fang and M. G. Grillakis, *Existence and uniqueness for Boussinesq type equations on a circle,* Commun. Part. Diff. Eqs. **21**, 1253–1277 (1996).

[30] H. M. Farkas and I. Kra, *Riemann Surfaces*, 2nd ed., Springer, New York, 1992.

[31] A. R. Forsyth, *Theory of Differential Equations, Vol. 4*, Dover, New York, 1959.

[32] L. Gatto and S. Greco, *Algebraic curves and differential equations: an introduction*, The Curves Seminar at Queen's, Vol. VIII (ed. by A. V. Geramita), Queen's Papers Pure Appl. Math. **88**, Queen's Univ., Kingston, Ontario, Canada, 1991, B1–B69.

[33] I. M. Gel'fand and L. A. Dickey, *Fractional powers of operators and Hamiltonian systems*, Funct. Anal. Appl. **10**, 259–273 (1976).

[34] I. M. Gel'fand and L. A. Dickey, *Integrable nonlinear equations and the Liouville theorem*, Funct. Anal. Appl. **13**, 6–15 (1979).

[35] V. P. Gerdt, N. A. Kostov, *Computer algebra in the theory of ordinary differential equations of Halphen type.* Computers and mathematics, (ed. by E. Kaltofen, et al.) Proc. Conf., Cambridge, Mass., (1989), 279–288.

[36] F. Gesztesy and H. Holden, *A combined sine-Gordon and modified Korteweg-de Vries hierarchy and its algebro-geometric solutions*, preprint, 1997.

[37] F. Gesztesy and H. Holden, *Hierarchies of Soliton Equations and their Algebro-Geometric Solutions*, monograph in preparation.

[38] F. Gesztesy, D. Race, and R. Weikard, *On (modified) Boussinesq-type systems and factorizations of associated linear differential expressions*, J. London Math. Soc. (2) **47**, 321–340 (1993).

[39] F. Gesztesy and R. Ratneseelan, *An alternative approach to algebro-geometric solutions of the AKNS hierarchy*, Rev. Math. Phys. **10**, 345–391 (1998).

[40] F. Gesztesy, R. Ratnaseelan, and G. Teschl, *The KdV hierarchy and associated trace formulas*, in Proceedings of the *International Conference on Applications of Operator Theory*, I. Gohberg, P. Lancaster, and P. N. Shivakumar (eds.), Operator Theory: Advances and Applications, Vol. 87, Birkhäuser, Basel, 1996, pp. 125–163.

[41] F. Gesztesy and R. Weikard, *Spectral deformations and soliton equations*, in *Differential Equations with Applications to Mathematical Physics*, W. F. Ames, E. M. Harrell II, and J. V. Herod (eds.), Academic Press, Boston, 1993, pp. 101–139.

[42] F. Gesztesy and R. Weikard, *Lamé potentials and the stationary (m)KdV hierarchy*, Math. Nachr. **176**, 73–91 (1995).

[43] F. Gesztesy, R. Weikard, *A characterization of elliptic finite-gap potentials*, C. R. Acad. Sci. Paris **321** (1995), 837–841.

[44] F. Gesztesy and R. Weikard, *Treibich-Verdier potentials and the stationary (m)KdV hierarchy*, Math. Z. **219**, 451–476 (1995).

[45] F. Gesztesy, R. Weikard, *Picard potentials and Hill's equation on a torus*, Acta Math. **176** (1996), 73–107.

[46] F. Gesztesy, R. Weikard, *Toward a characterization of elliptic solutions of hierarchies of soliton equations*, Dorroh, J. Robert (ed.) et al., Applied analysis. Proceedings of a conference, Baton Rouge, LA, USA, April 19–21, 1996. Providence, RI, American Mathematical Society, Contemp. Math. **221** (1999), 133–161.

[47] F. Gesztesy, R. Weikard, *Elliptic algebro-geometric solutions of the KdV and AKNS hierarchies - an analytic approach*, Bull. Am. Math. Soc., **35** (1998), 271–317.

[48] B. Grébert, J. C. Guillot, and F. Klopp, *On the spectrum of odd order self adjoint ordinary differential operators on the real line with quasi-periodic coefficients,* preprint, 1997.

[49] S. Greco and E. Previato, *Spectral curves and ruled surfaces: projective models*, in The Curves Seminar at Queen's, Vol. VIII (ed. by A. V. Geramita), Queen's Papers Pure Appl. Math. **88**, Queen's Univ., Kingston, Ontario, Canada, 1991, F1–F33.

[50] R. C. Gunning, *Lectures on Riemann Surfaces: Jacobi Varieties*, Princeton University Press, Princeton, 1972.

[51] G. H. Halphen, *Sur une nouvelle classe d'équations différentielles linéaires intégrables*, C. R. Acad. Sc. Paris, **101** (1885), 1238–1240.

[52] G. H. Halphen, *Mémoire sur la réduction des équations différentielles linéaires aux formes intégrables*, Mem. prés. l'Acad. Sci. France **28** (1884), 1–300.

[53] C. Hermite, *Oeuvres*, tome 3, Gauthier–Villars, Paris, 1912.

[54] E. Kamke, *Differentialgleichungen, Lösungsmethoden und Lösungen, Vol. 1, Gewöhnliche Differentialgleichungen*, 5th ed., Akademische Verlagsgesellschaft, Leipzig, 1956.

[55] M. Krause, *Theorie der doppeltperiodischen Funktionen einer veränderlichen Grösse*, Vol. 1, 1895, Vol. 2, 1897, Teubner, Leipzig.

[56] E. L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.

[57] A. R. Its and V. B. Matveev, *Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg-de Vries equation*, Theoret. Math. Phys. **23**, 343–355 (1975).

[58] C. G. T. Jacobi, *Über eine neue Methode zur Integration der hyperelliptischen Differentialgleichungen und über die rationale Form ihrer vollständigen algebraischen Integralgleichungen*, J. Reine Angew. Math. **32**, 220–226 (1846).

[59] V. K. Kalantarov and O. A. Ladyzhenskaya, *The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types*, J. Sov. Math. **10**, 53–70 (1978).

[60] A. Krazer, *Lehrbuch der Thetafunktionen*, Chelsea Publ. Comp., New York, 1970.

[61] I. M. Krichever, *Algebraic-geometric construction of the Zaharov-Šabat equations and their periodic solutions*, Dokl. Akad. Nauk SSSR **227**, 394–397 (1976).

[62] I. M. Krichever, *Integration of nonlinear equations by the methods of algebraic geometry*, Funct. Anal. Appl. **11**, 12–26 (1977).

[63] I. M. Krichever, *Commutative rings of ordinary differential operators*, Funct. Anal. Appl. **12**, 175–185 (1978).

[64] G. A. Latham and E. Previato, *Darboux transformations for higher-rank Kadomtsev-Petviashvili and Krichever-Novikov equations*, Acta Applicandae Math. **39**, 405–433 (1995).

[65] Y. Liu, *Strong instability of solitary wave solutions of a generalized Boussinesq equation,* preprint.

[66] V. B. Matveev and A. O. Smirnov, *On the Riemann theta function of a trigonal curve and solutions of the Boussinesq and KP equations*, Lett. Math. Phys. **14**, 25–31 (1987).

[67] V. B. Matveev and A. O. Smirnov, *Simplest trigonal solutions of the Boussinesq and Kadomtsev-Petviashvili equations*, Sov. Phys. Dokl. **32**, 202–204 (1987).

[68] V. B. Matveev and A. O. Smirnov, *Symmetric reductions of the Riemann θ-function and some of their applications to the Schrödinger and Boussinesq equations*, Amer. Math. Soc. Transl. (2) **157**, 227–237 (1993).

[69] V. B. Matveev and M. I. Yavor, *Solutions presque périodiques et a N-solitons de l'équation hydrodynamique non linéaire de Kaup*, Ann. Inst. Henri Poincaré **A31**, 25–41 (1979).

[70] R. Miranda, *Algebraic Curves and Riemann Surfaces*, Graduate Studies in Mathematics, Vol. 5, Amer. Math. Soc., Providence, R.I., 1995.

[71] H. P. McKean, *Integrable Systems and Algebraic Curves*, in *Global Analysis*, M. Grmela and J. E. Marsden (eds.), Lecture Notes in Math., **755**, Springer, Berlin, 1979, pp. 83–200.

[72] H. P. McKean, *Boussinesq's equation on the circle*, Commun. Pure Appl. Math. **34**, 599–691 (1981).

[73] H. P. McKean, *Boussinesq's equation: How it blows up,* in *J. C. Maxwell, the Sesquiecentennial Symposium,* M. S. Berger (ed.), North-Holland, Amsterdam, 1984, pp. 91–105.

[74] H. P. McKean, *Variation on a theme of Jacobi*, Commun. Pure Appl. Math. **38**, 669–678 (1985).

[75] D. Mumford, *Tata Lectures on Theta II*, Birkhäuser, Boston, 1984.

[76] S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons*, Consultants Bureau, New York, 1984.

[77] R. Pego, *Origin of the KdV equation,* Notices Amer. Math. Soc. **45** (1998), 358.

[78] E. Picard, *Sur une généralisation des fonctions périodiques et sur certaines équations différentielles linéaires*, C. R. Acad. Sci. Paris, **89** (1879), 140–144.

[79] E. Picard, *Sur une classe d'équations différentielles linéaires*, C. R. Acad. Sci. Paris, **90** (1880), 128–131.

[80] E. Picard, *Sur les équations différentielles linéaires à coefficients doublement périodiques*, J. reine angew. Math., **90** (1881), 281–302.

[81] E. Previato, *The Calogero-Moser-Krichever system and elliptic Boussinesq solitons*, in *Hamiltonian Systems, Transformation Groups and Spectral Transform Methods*, J. Harnad and J. E. Marsden (eds.), CRM, Monréal, 1990, pp. 57–67.

[82] E. Previato, *Monodromy of Boussinesq elliptic operators*, Acta Applicandae Math. **36**, 49–55 (1994).

[83] E. Previato, *Seventy years of spectral curves*, in *Integrable Systems and Quantum Groups* (ed. by R. Donagi, B. Dubrovin, E. Frenkel, and E. Previato), Lecture Notes in Mathematics **1620**, Springer, Berlin, 1996, pp. 419–481.

[84] E. Previato and J.-L. Verdier, *Boussinesq elliptic solitons: the cyclic case*, in *Proceedings of the Indo-French Conference on Geometry*, Dehli, 1993, S. Ramanan and A. Beauville (eds.), Hindustan Book Agency, Delhi, 1993, pp. 173–185.

[85] R. L. Sachs, *On the integrable variant of the Boussinesq system: Painlevé property, rational solutions, a related many-body system, and equivalence with the AKNS hierarchy*, Physica D **30**, 1–27 (1988).

[86] G. Segal and G. Wilson, *Loop groups and equations of KdV type*, Publ. Math. IHES **61** (1985), 5–65.

[87] A. O. Smirnov, *Real finite-gap regular solutions of the Kaup-Boussinesq equation*, Theoret. Math. Phys. **66**, 19–31 (1986).

[88] A. O. Smirnov, *A matrix analogue of Appell's theorem and reductions of multidimensional Riemann theta-functions*, Math. USSR Sbornik **61**, 379–388 (1988).

[89] A. O. Smirnov, *On a class of elliptic solutions of the Boussinesq equations*, Theoret. Math. Phys. **109**, 1515–1522 (1996).

[90] G. Teschl, *Jacobi Operators and completely integrable nonlinear lattices*, Mathematical Surveys and Monographs, Vol. 72, Amer. Math. Soc., Providence, R.I., 1999.

[91] R. Weikard, *On rational and periodic solutions of stationary KdV equations,* Doc. Math. **4** (1999), 109–126.

[92] R. Weikard, *On Commuting Differential Operators*, (http://www.math.uab.edu/rudi/papers/cdo.dvi), preprint 1999.

[93] G. Wilson, *Algebraic curves and soliton equations*, in *Geometry Today*, E. Arbarello, C. Procesi, and E. Strickland (eds.), Birkhäuser, Boston, 1985, pp. 303–329.

[94] V. E. Zakharov, *On stochastization of one-dimensional chains of nonlinear oscillators*, Sov. Phys. JETP **38**, 108–110 (1974).

[95] V. E. Zakharov and S. V. Manakov, *Multidimensional nonlinear integrable systems and methods for constructing their solutions*, J. Sov. Math. **31**, 3307–3316 (1985).

[96] V. E. Zakharov and S. V. Manakov, *Construction of higher-dimensional nonlinear integrable systems and of their solutions*, Funct. Anal. Appl. **19**, 89–101 (1985).

[97] V. E. Zakharov and A. B. Shabat, *A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I*, Funct. Anal. Appl. **8**, 226–235 (1974).

[98] V. E. Zakharov and A. B. Shabat, *Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II*, Funct. Anal. Appl. **13**, 166–174 (1979).