# Nonlinear Wave Equations,

a short introduction

based on lecture notes of F. Gesztesy and R. Weikard
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# 1. Examples of nonlinear wave equations

1. Korteweg-de Vries equation (KdV), the most famous example:

   $$KdV(q) = q_t - 6qq_x + q_{xxx} = 0$$

   Applications: model for small amplitude, long (water) waves, collision-free hydromagnetic waves, ion-acoustic waves in plasmas.

2. modified Korteweg-de Vries equation (mKdV):

   $$mKdV_{\pm}(\phi) = \phi_t \pm 6\phi^2\phi_x + \phi_{xxx} = 0$$

3. Nonlinear Schrödinger equation (NLS):

   $$NLS_{\pm}(u) = iu_t + u_{xx} \pm u|u|^2 = 0$$

   Applications: plasma waves, nonlinear optics.

4. Sine-Gordon equation, Hyperbolic-sine-Gordon equation (Sinh-Gordon):

   $$\begin{aligned} \phi_{tx} &= \pm \sin\phi \text{ or } u_{tt} - u_{xx} = \pm \sin u \\ \phi_{tx} &= \sinh\phi \text{ or } u_{tt} - u_{xx} = \sinh u \end{aligned}$$

   Applications: differential geometry, elementary particle physics.

5. Kadomtsev-Petviashvili equation (KP):

   $$V_t - 6VV_x + V_{xxx} \pm 3\partial_x^{-1}V_{yy} = 0 \text{ where } (\partial_x^{-1}f)(x,y,t) = \int_{-\infty}^{x} f(x',y,t)dx'$$

   This is a two-(space-)dimensional generalization of the KdV equation. It has also a modified version. The KP equation is mathematically very important.

# The linear one-dimensional wave equation

Let us first consider the linear one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

where $x$ is the space variable, $t$ is the time variable, $c$ is a constant (the velocity of the wave), and $u(x,t)$ denotes the amplitude.

This equation is solved by a simple transformation:

$$\xi = x + ct, \quad \eta = x - ct.$$

Define $\tilde{u}$ by

$$\tilde{u}(\xi,\eta) = u(x(\xi,\eta), t(\xi,\eta)).$$

Then

$$\tilde{u}_{\xi,\eta} = -\frac{1}{4c^2}(u_{tt} - c^2 u_{xx}) + \frac{1}{4c}(u_{tx} - u_{xt}) = 0.$$

If $u \in C^2$ satisfies the wave equation, then $\tilde{u}_{\xi,\eta} = 0$. Now integration yields

$$\begin{aligned} \tilde{u}_\xi &= g(\xi) \text{ with } g \in C^1, \\ \tilde{u} &= \int^\xi g(s)ds + F(\eta) = G(\xi) + F(\eta) \text{ with } F \in C^2 \end{aligned}$$

and hence

$$u(x,t) = G(x+ct) + F(x-ct).$$

$F$ describes a wave moving to the right with velocity $c > 0$ and $G$ describes a wave moving to the left with velocity $c > 0$. (Note that these waves retain their shape.)

Factorization:

$$(\partial_t \mp c\,\partial_x)(\partial_t \pm c\,\partial_x)v = 0,$$

$$v_t \pm c\,v_x = 0,$$

$$v(x,t) = H(x \mp c\,t).$$

# 2. Dispersion versus Nonlinearity

## Dispersion of a linear equation

Consider the linearized KdV equation $q_t + q_{xxx} = 0$ and try the ansatz (plane waves)

$$q(x,t) = e^{i(kx-\omega t)}. \tag{2.1}$$

One has $q_t = -i\omega q$ and $q_{xxx} = (ik)^3 q$. Therefore $q$ is a solution, iff the "dispersion relation"

$$\omega + k^3 = 0 \tag{2.2}$$

is fulfilled, yielding $q(x,t) = \exp(i(kx + k^3 t))$.

Next superpose solutions of this type

$$q(x,t) = \int_{-\infty}^{\infty} A(k) \exp(ikx + ik^3 t) dk$$

for appropriate $A(k) \in L^1(\mathbb{R})$. Under reasonable conditions on $A$ we can interchange differentiation with respect to $t$ and $x$ with integration with respect to $k$, we have

$$q_t + q_{xxx} = \int_{-\infty}^{\infty} A(k)(\partial_t + \partial_{xxx}) \exp(ikx + ik^3 t) dk = 0. \tag{2.3}$$

In the case of the linearized KdV equation we can determine $A(k)$ using the initial condition $q(x,0) = q_0(x)$. Since

$$q_0(x) = \int_{-\infty}^{\infty} A(k) e^{ikx} dk,$$

$A(k)$ is the Fourier transform of $q_0(x)$, i.e.,

$$A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q_0(x) e^{-ikx} dx.$$

For sufficiently nice $q_0$ the function $A$ will satisfy the above conditions, i.e., $(1+k^3)A(k)$ being in $L^1$.

In much the same way one can obtain the solution of many other initial value problems for partial differential equations with constant coefficients. In these cases one obtains global solutions, i.e. solutions which exists for all $t \geq 0$ (see John, Section 5.2.b).

Now we want to study dispersion.

**Lemma 1** *1. Let $\omega : \mathbb{R} \to \mathbb{R}$ be $C^\infty$ and let $A \in C_0^\infty(\mathbb{R})$. Let $G$ be an open set containing the compact set $\{\omega'(k) | k \in \operatorname{supp} A\}$. Define*

$$q(x,t) = \int_{-\infty}^{\infty} A(k) \exp(ikx - i\omega(k)t) dk. \tag{2.4}$$

*Then for any $m$, there is a constant $c$ depending on $m$, $A$ and $G$ such that $|q(x,t)| \leq c(1 + |x| + |t|)^{-m}$ for all $(x,t)$ with $\frac{x}{t} \notin G$.*

*2. In addition to the above let $\operatorname{supp} A \cap \{k | \omega''(k) = 0\} = \emptyset$. Then $|q(x,t)| \leq c|t|^{-1/2}$ for $|t| > 1$ and all $x$.*

For a proof see Reed and Simon III, Corollary to Theorem XI.15 (so called method of stationary phase.)

In our situation $\omega(k) = -k^3$ and $\omega''(k) = -6k$. Hence if $0 \notin \operatorname{supp} A$, then $\operatorname{supp} A \cap \{k | \omega''(k) = 0\} = \emptyset$ and the Lemma applies yielding $q(x,t) = O(|t|^{-1/2})$ uniformly in $x$ as $|t|$ tends to infinity. This describes the spreading of an initial wave packet, a phenomenon called dispersion.

# Nonexistence of Global Solutions in the Nonlinear Case

Consider the nonlinear equation

$$q_t + F(q) q_x = 0.$$

Formal solution:

$$q(x,t) = G(x - F(q)t).$$

The following proposition shows that this equation cannot have global solutions except for very special initial conditions. The following proposition can be found in Smoller (p.241-244).

**Proposition 2** *Assume $F \in C^1(\mathbb{R})$, $F' > 0$ and consider $q_t + F(q)q_x = 0$ for $t \geq 0$ and all $x$. Suppose that initially $q(x,0) = q_0(x)$, where $q_0 \in C^1(\mathbb{R})$. If $q_0' \geq 0$ then $q \in C^1(\mathbb{R}^2_+)$, ($\mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+$). But if $q_0'(x_0) < 0$ for some $x_0 \in \mathbb{R}$, then, in general, a $C^1$-solution can only exist for $t$ small enough.*

Note that the nonexistence of a global solution is a pure nonlinear effect. Also, if $F' < 0$, the condition in $q_0'$ has to be reversed in order to get a global solution. In any case one gets in trouble for either $t < 0$ or $t > 0$, even if $q_0'$ is of one sign. We conclude that the equation $q_t - 6qq_x = 0$ does not have global solutions.

# Nonlinearity plus dispersion

The linearized KdV equation admits global solutions but shows dispersion. The nonlinear equation $q_t - 6qq_x$, on the other hand, does not admit global solutions. Now let us consider the full KdV equation

$$q_t - 6qq_x + q_{xxx} = 0.$$

This equation has the special solution

$$q_1(x,t) = q_\infty - \frac{2\kappa^2}{(\cosh(\kappa x + (6q_\infty - 4\kappa^2)\kappa t + b))^2}$$

where $b, q_\infty \in \mathbb{R}$ and $\kappa \geq 0$.

$q_1$ is of the type of a traveling wave, i.e., $q_1(x,t) = f(x - ct)$ with $c = 4\kappa^2 - 6q_\infty$. Note that $q_\infty = \lim_{x\to\pm\infty} q_1 = \lim_{t\to\pm\infty} q_1$. $q_1$ is a global solution and non-dispersive, i.e., it does not spread out as $|t| \to \infty$. In fact as traveling wave it keeps its shape.

**Conclusion:** A proper combination of nonlinearity and dispersion might produce a nice surprise, namely a solitary wave that is globally defined and that is non-dispersive.

# 3. The Lax Approach for the KdV Equation

**Definition 3** *Let $A(t), t \in \mathbb{R}$, be a family of self-adjoint operators in some Hilbert space $H$ with common dense domain $D \subset H$. A two parameter family of operators $U(t,s)$, $t,s \in \mathbb{R}$, is called a unitary propagator for $A(t)$, if*

1. *$U(t,s)$ is unitary $\forall s,t \in \mathbb{R}$,*
2. *$U(t,t) = 1 \quad \forall t \in \mathbb{R}$,*
3. *$U(t,s)U(s,r) = U(t,r) \quad \forall r,s,t \in \mathbb{R}$,*
4. *$(t,s) \to U(t,s)$ is strongly continuous, i.e., $(t,s) \to U(t,s)\psi$ is continuous for all $\psi \in H$,*
5. *$U(t,s)\psi \in D$ for all $\psi \in D$ and for fixed $s \in \mathbb{R}$. The function $t \to U(t,s)\psi$ is differentiable for all $\psi \in D$ and*

$$\frac{\partial}{\partial t} U(t,s)\psi = -iA(t)U(t,s)\psi, \forall t,s \in \mathbb{R}, \forall \psi \in D.$$

Throughout this chapter the following hypothesis is used.

**Hypothesis 4** *Let $q \in C^\infty(\mathbb{R}^2)$ be real valued and let $q, q_x, q_t, q_{tx}$ be in $L^\infty(\mathbb{R}^2)$.*

Now we define a family of Schrödinger operators by

$$L(t) = -\partial_x^2 + q(\cdot, t)$$

in a Hilbert space $L^2(\mathbb{R})$.

Also let us define in $L^2(\mathbb{R})$

$$P(t) = -4\partial_x^3 + 6q(\cdot,t)\partial_x + 3q_x(\cdot,t).$$

**Proposition 5** *(i) Under Hypothesis 4 each $L(t)$ is a self-adjoint operator in $L^2(\mathbb{R})$ with domain $D(L(t)) = H^2(\mathbb{R})$.*
*(ii) Under Hypothesis 4 each $P(t)$ is skew-adjoint, (i.e., $\pm iP(t)$ is self-adjoint), in $L^2(\mathbb{R})$ with domain $D(P(t)) = H^3(\mathbb{R})$.*

**Lemma 6** *Suppose $q$ satisfies Hypothesis 4. Then $iP(t), t \in \mathbb{R}$ has a unitary propagator $U(t,s)$ in $L^2(\mathbb{R})$.*

If $P(t)$ were actually time-independent (stationary KdV solution) then the problem of Lemma 6 would be solved by Stone's theorem, i.e.,

$$U(t,s) = \exp(-i(t-s)iP) = \exp((t-s)P).$$

The significance of the Lax approach is given by the following theorem.

**Theorem 7** *Suppose q satisfies Hypothesis 4 and the KdV equation. Then there exists a family of unitary operators* $U(t,s)$, $t \in \mathbb{R}$, $U(t,t) = 1$ *in* $L^2(\mathbb{R})$ *which is a propagator for* $iP(t)$, *such that*

$$U(t,s)L(s)U(t,s)^{-1} = L(t) \quad \forall t \in \mathbb{R},$$

*i.e.,* $L(t)$ *is unitarily equivalent to* $L(s)$ *for all* $t$ *in* $\mathbb{R}$. *In particular all unitary invariants of* $L(t)$ *such as its spectrum, the nature of the spectrum (point spectrum, absolutely continuous spectrum etc.) and its multiplicity remain independent of* $t \in \mathbb{R}$.

$P$ and $L$ are called a Lax pair (Peter Lax). The connection with the KdV equation is given through

$$\frac{d}{dt}L(t) = \frac{d}{dt}\left(U(t,s)L(s)U(t,s)^{-1}\right)$$
$$= \underbrace{\frac{d}{dt}U(t,s)U(t,s)^{-1}}_{-i^2P(t)} L(t) + L(t) \underbrace{U(t,s)\frac{d}{dt}U(t,s)^{-1}}_{i^2P(t)}$$
$$= P(t)L(t) - L(t)P(t) = [P(t), L(t)].$$

Hence

$$\frac{d}{dt}L(t) - [P(t), L(t)] = 0$$

is equivalent to the KdV equation since $\frac{d}{dt}L(t) = q_t$ and $[P, L] = 6qq_x - q_{xxx}$.

The result of Theorem 7 is truly spectacular, since $q(x,t)$ will vary substantially in time in general, while the spectral properties of $L$ are independent of $t$.

# 4. Scattering and Inverse Scattering Theory for the one-dimensional Schrödinger Operator

## Direct Scattering

We are interested in solving the differential equation

$$-f'' + qf = k^2 f \tag{4.1}$$

for a real-valued potential $q$ and a complex parameter $k$.

For $p \geq 0$ let $L^1_p(\mathbb{R})$ denote classes of functions on $\mathbb{R}$:

$$L^1_p(\mathbb{R}) = \{q | q \; real - valued, \int_{-\infty}^{\infty} (1 + |x|^p) q(x) dx < \infty\}.$$

Note that $L^1_p(\mathbb{R}) \subset L^1_r(\mathbb{R})$, if $p \geq r$.

**Theorem 8** *1. Suppose $q \in L^1_1(\mathbb{R})$. For each $k$ with $\Im k \geq 0$ the integral equations*

$$f_\pm(k, x) = e^{\pm ikx} - \int_x^{\pm\infty} \frac{\sin(k(x - x'))}{k} q(x') f_\pm(k, x') dx' \tag{4.2}$$

*have unique solutions defined everywhere in $\mathbb{R}$, which solve the Schrödinger equation*

$$-f''_\pm + q(x) f_\pm = k^2 f_\pm.$$

*For each $x$ the functions $f_\pm(k, x), f'_\pm(k, x)$ are analytic in the upper half plane $\Im k > 0$ and continuous in $\Im k \geq 0$. They satisfy the following estimates:*

$$|f_\pm(k, x) - e^{\pm ikx}| \leq \frac{\text{const}}{|k|} \exp(\frac{\text{const}}{|k|}) e^{\mp(\Im k)x}, \quad k \neq 0 \tag{4.3}$$

$$|f_\pm(k, x)| \leq \text{const}(1 + \max\{x, 0\}) e^{\mp(\Im k)x} \tag{4.4}$$

$$|f'_\pm(k, x)| \leq \text{const}(\frac{1 + |k|}{|k|}) e^{\mp(\Im k)x}, \quad k \neq 0 \tag{4.5}$$

$$|f'_\pm(k, x)| \leq \text{const}(1 + |k| + |k||x|) e^{\mp(\Im k)x}. \tag{4.6}$$

*2. If $q \in L^1_2(\mathbb{R})$, then $\frac{\partial}{\partial k} f_\pm(k, x) = \dot{f}_\pm(k, x)$ exists for $\Im k \geq 0$ and is continuous there as a function of $k$. The following estimates hold*

$$|\frac{\partial}{\partial k}(e^{\mp ikx} f_\pm(k, x)| \leq \text{const}(1 + x^2) \tag{4.7}$$

$$|\frac{\partial}{\partial k} f_\pm(k, x)| \leq \text{const}(1 + x^2) e^{\mp(\Im k)x}, \tag{4.8}$$

$$|\frac{\partial^2}{\partial x \partial k}(e^{\mp ikx} f_\pm(k, x))| \leq \text{const}(1 + |x|) \tag{4.9}$$

$$|\frac{\partial^2}{\partial x \partial k} f_\pm(k, x)| \leq \text{const}(1 + |k|)(1 + x^2) e^{\mp(\Im k)x}. \tag{4.10}$$

In a similar way one can consider functions $g_\pm(k,x)$ defined by

$$g_\pm(k,x) = e^{\mp ikx} - \int_x^{\pm\infty} \frac{\sin(k(x-x'))}{k} q(x')g_\pm(k,x')dx'$$

for $\Im k \le 0$. These functions satisfy the relations

$$g_\pm(k,x) = f_\pm(k^*,x)^* \tag{4.11}$$

and for $k$ real

$$f_\pm(k) = g_\pm(-k) = g_\pm^*(k) = f_\pm^*(-k). \tag{4.12}$$

The functions $f_\pm$ and $g_\pm$ are called the Jost solutions of the Schrödinger equation.

If $q$ were continuous standard ode theory would imply that $W(g_\pm, f_\pm)$ were independent of $x$ because they are both solutions of the Schrödinger equation with the same value of $k^2$. Nevertheless this remains to be true even in our case, since $\frac{\partial}{\partial x}W(g_\pm, f_\pm)$ vanishes almost everywhere, which yields, upon integration from $x_1$ to $x_2$, that the Wronskians are indeed independent of $x$. Hence we may evaluate them by calculating the limit as $x \to \pm\infty$ and find

$$W(g_\pm, f_\pm) = \pm 2ik, \quad k \in \mathbb{R}. \tag{4.13}$$

For $\Im k \ge 0$ we define $W(k) = W(f_-(k,x), f_+(k,x))$, which is again independent of $x$. Note that $W$ is analytic for $\Im k > 0$ and continuous for $\Im k \ge 0$.

For real $k$ we have defined four solutions of a second order differential equation. These can not be linearly independent. In fact we have

$$f_\pm(k,x) = c_\mp(k) f_\mp(k,x) + d_\mp(k) g_\mp(k,x), \quad k \in \mathbb{R}, k \ne 0 \tag{4.14}$$

since, by (4.13) the pairs $(f_+, g_+)$ and $(f_-, g_-)$ are linearly independent.

The coefficients $c_\mp$ and $d_\mp$ may be expressed in terms of Wronskians:

$$c_\pm = \mp \frac{W(f_\mp, g_\pm)}{2ik} \tag{4.15}$$

$$d(k) = d_+(k) = d_-(k) = \frac{W(k)}{2ik}. \tag{4.16}$$

While $d_\pm$ is only defined for real $k \ne 0$ by (4.14), (4.16) may be used as definition for $\Im k \ge 0, k \ne 0$. Note that for real $k$ by (4.12)

$$d(k)^* = d(-k), \quad c_\pm(k)^* = c_\pm(-k) \text{ and } c_+(-k) = -c_-(k). \tag{4.17}$$

**Proposition 9** *Under the condition $q \in L^1_2(\mathbb{R})$ the following holds:*

1. *$W(k) \ne 0$ for $\Im k \ge 0$ unless $k$ is pure imaginary.*
2. *Suppose $W(k_0) = 0$ for some $k_0$ with $\frac{k_0}{i} > 0$. Then*

$$\dot W(k_0) = 2k_0 \int_{-\infty}^{\infty} f_-(k_0,x) f_+(k_0,x) dx \ne 0,$$

*i.e., all zeros of $W(k)$ are simple.*

3. $d(k) = 1 + O(|k|^{-1})$ *as* $|k| \to \infty$

4. $c_\pm(k) = O(|k|^{-1})$ *as* $k \to \pm\infty$

5. *The following alternative holds:*

   – *Either* $d(k)$ *is continuous at* $k = 0$ *with* $d(0) \neq 0$ *and* $c_\pm(k)$ *are continuous at* $k = 0$,

   – *or* $kd(k)$ *is continuous at* $k = 0$ *with* $\lim_{k\to 0} kd(k) = \alpha \neq 0$ *and* $kc_\pm(k)$ *are continuous at* $k = 0$ *with* $\lim_{k\to 0} kc_\pm(k) = \beta_\pm \neq 0$.

We are now turning to the Schrödinger operator associated to $q$.
Let $q \in L^1_2(\mathbb{R})$. Then define $H = -\frac{d^2}{dx^2} + q$ in
$D(H) = \{g \in L^2(\mathbb{R}) | g, g' \in AC_{\text{loc}}(\mathbb{R}), -g'' + qg \in L^2(\mathbb{R})\}$.

**Lemma 10** *The operator* $H$ *defined above is self-adjoint in* $L^2(\mathbb{R})$. *Its spectrum has the following properties*

1. $\sigma_{\text{ess}}(H) = \sigma_{\text{ac}}(H) = [0, \infty)$,

2. $\sigma_{\text{sing}}(H) = \emptyset$,

3. $\sigma_{\text{pp}}(H) \subset (-\infty, 0)$,

4. *all eigenvalues are simple.*

**Theorem 11** *The eigenvalues of* $H$ *are given through the zeros of the function* $d(k)$. *In particular there are only finitely many eigenvalues and* $\inf \sigma(H) > -\infty$.

$\sigma_{pp}(H) = \{-\kappa_j^2 | \kappa_j > 0, j = 1, ...N\}$ is the set of eigenvalues of $H$. To each eigenvalue $-\kappa_j^2$ there corresponds an eigenfunction

$$f_+(i\kappa_j, x) = \mu_j f_-(i\kappa_j, x) \tag{4.18}$$

for some non-zero $\mu_j$. We define norming constants $\gamma_{\pm,j}, j = 1, ...N$, by

$$\gamma_{\pm,j} = \|f_\pm(i\kappa_j, \cdot)\|_2^{-1}, \quad j = 1, ..., N. \tag{4.19}$$

We now turn to the continuous spectrum and introduce the scattering matrix.

**Definition 12** *For real* $k$ *let*

$$S(k) = \begin{pmatrix} T_-(k) & R_+(k) \\ R_-(k) & T_+(k) \end{pmatrix}$$

*where* $T_-(k) = T_+(k) = T(k) = 1/d(k)$ *denote the transmission coefficients (with respect to left and right incidence) and*

$$R_\pm(k) = c_\pm(k)/d(k)$$

*denote the reflection coefficients with respect to left and right incidence.*

Note that $T(k), R_{\pm}, (k)$ are well defined. This is true even if $k \to 0$ because of statement 5 in Proposition 9. The relations (4.16) imply

$$T(k)^* = T(-k), R_{\pm}(k)^* = R_{\pm}(-k) \tag{4.20}$$

**Theorem 13** *Assume $q \in L^1_2(\mathbb{R})$ and $k \in \mathbb{R}$. Then $S(k)$ is a continuous, unitary operator. In particular,*

$$|R_-(k)|^2 = |R_+(k)|^2 \tag{4.21}$$
$$|T(k)|^2 + |R_{\pm}(k)|^2 = 1. \tag{4.22}$$

In order to extract the physical meaning of the scattering matrix we define now the wave functions of $H$ by

$$\psi_{\pm}(k) = d(k)^{-1} f_{\pm}(k,x), \quad k \in \mathbb{R}, \quad x \in \mathbb{R}.$$

Of course $H\psi_{\pm} = k^2\psi_{\pm}$ in the distributional sense. The expansion of $f_{\pm}$ in terms of $f_-, g_-$ and $f_+, g_+$, respectively, and the asymptotic behavior of these translates into

$$\begin{aligned}
\psi_+(k,x) &= \begin{cases} d^{-1}(k)e^{ikx} & \text{as } x \to \infty \\ \frac{c_-(k)}{d(k)}e^{-ikx} + e^{ikx} & \text{as } x \to -\infty \end{cases} \\
&= \begin{cases} T(k)e^{ikx} & \text{as } x \to \infty \\ e^{ikx} + R_-(k)e^{-ikx} & \text{as } x \to -\infty. \end{cases}
\end{aligned}$$

Similarly

$$\psi_-(k,x) = \begin{cases} T(k)e^{-ikx} & \text{as } x \to -\infty \\ e^{-ikx} + R_+(k)e^{ikx} & \text{as } x \to \infty. \end{cases}$$

Now the interpretation is as follows: Consider a plane wave $e^{i(kx-\omega t)}$ of frequency $\omega$ and wave number $k$. For a given $\omega > 0$ the sign of $k$ gives the direction in which the wave travels, namely for $k > 0$ the wave is traveling to the right for $k < 0$ the wave is traveling to the left. Therefore at a fixed instant of time and positive $k$ $e^{ikx}$ and $e^{-ikx}$ are supposed to indicate waves travelling to the right and left respectively.

Now note that the condition $q \in L^1_2$ requires a certain decay of $q$ at $\pm\infty$. Therefore near $\pm\infty$ the solutions are close to plane waves. Then $\psi_+$ describes a wave incidenting

from $-\infty$, a part of it – $T(k)e^{+ikx}$ – being transmitted to $+\infty$ and another part of it – $R_-(k)e^{-ikx}$ – being reflected back to $-\infty$.
Similarly $\psi_-$ describes a wave incidenting from $+\infty$ a part of it being transmitted and a part of it being reflected. Our analysis has shown that the transmission coefficient is independent from whether the wave comes from $+\infty$ or $-\infty$. This is not true for the reflection coefficient but at least we have $|R_+| = |R_-|$. So the scattering matrix contains the asymptotic information describing the scattering process.

**Definition 14** *The sets*

$$S_\pm = \{R_\pm(k), k \in \mathbb{R}; \kappa_j, \gamma_{\pm,j}, j = 1, ..., N\}$$

*are called the scattering data $S_\pm$ for $H$.*

The direct scattering step consists in obtaining the scattering data for $q$, to determine the map

$$q \longmapsto S_\pm.$$

Note that all the information in $S_\pm$ actually sits in $f_\pm$ and $g_\pm$ and its $x$-derivatives, which can be obtained (theoretically) by solving the respective Volterra integral equations.

**Remarks:** Remember part 2 of Proposition 9 where

$$\dot{W}(i\kappa_j) = 2i\kappa_j \int_{-\infty}^{\infty} f_-(i\kappa_j, x) f_+(i\kappa_j, x) dx,$$

if $-\kappa_j^2$ is an eigenvalue of $H$. But then $W(i\kappa_j) = 0$ and $f_+(i\kappa_j, x) = \mu_j f_-(i\kappa_j, x)$. Hence

$$\begin{aligned} \left[\frac{d}{dk}\frac{1}{T(k)}\right]_{k=i\kappa_j} &= \left[\frac{d}{dk}\frac{W(k)}{2ik}\right]_{k=i\kappa_j} = -i\int_{-\infty}^{\infty} f_-(i\kappa_j, x) f_+(i\kappa_j, x) dx \\ &= -i\mu_j \|f_-\|_2^2 = \frac{-i}{\mu_j}\|f_+\|_2^2 = -i\mu_j\gamma_{-2}^{-2} = -i\mu_j^{-1}\gamma_{+,j}^{-2}. \end{aligned}$$

Therefore, from $T(k)$ and $\gamma_{+,j}$, one can calculate $\mu_j$ and also $\gamma_{-,j}$.
Investigating

$$h(k) = T(k)\prod_{i=1}^{N}\frac{k - i\kappa_j}{k + i\kappa_j}$$

one can show for $\Im k > 0$

$$T(k) = \prod_{j=1}^{N}\frac{k + i\kappa_j}{k - i\kappa_j}\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(1 - |R_+(\xi)|^2)}{\xi - k}d\xi\right\}, \tag{4.23}$$

and for real $k$

$$\begin{aligned} T(k) &= \lim_{\varepsilon\to 0} T(k + i\varepsilon) \\ &= \prod_{j=1}^{N}\frac{k + i\kappa_j}{k - i\kappa_j}\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(1 - |R_+(\xi)|^2)}{\xi - k} + \frac{1}{2}\ln(1 - |R_+(k)|^2)\right\} \end{aligned}$$

where the last integral is a principal value integral.

# Inverse Scattering

**Definition 15** *A function $h$ which is analytic in the upper half plane and satisfies*

$$\sup_{b>0} \int_{-\infty}^{\infty} |h(a+ib)|^2 da < \infty$$

*is called an Hardy function. Let $H^{2+}$ denote the space of all Hardy functions.*

**Lemma 16** *A function $h$ is of class $H^{2+}$ if and only if for all $\Im\omega > 0$ and for some $f \in L^2(\mathbb{R})$ vanishing on the negative real axis*

$$h(\omega) = \int_0^{\infty} f(x) e^{i\omega x} dx.$$

**Remark:** Levin representation of Jost solutions:

$$f_\pm(k,x) = e^{\pm ikx} \pm \int_x^{\pm\infty} dy K_\pm(x,y) e^{\pm iky}.$$

**Definition:**

$$K_\pm(x,y) = \frac{1}{2} B_\pm(x, \frac{y-x}{2}).$$

For $\Im k \geq 0$ and $x \in \mathbb{R}$ we introduce the functions

$$\begin{aligned} m_\pm(k,x) &= e^{\mp ikx} f_\pm(k,x), \\ n_\pm(k,x) &= T(k) e^{\pm ikx} f_\mp(k,x) = T(k) m_\mp(k,x), \\ N_\pm(k,x) &= n_\pm(k,x) - 1 - \sum_{j=1}^{N} \frac{A_{\pm,j}(x)}{k - i\kappa_j} \end{aligned}$$

where

$$A_{\pm,j}(x) = i\mu_j^{\pm 1} \gamma_{\pm j}^2 m_\mp(i\kappa_j, x) = i\gamma_{\pm,j}^2 e^{\mp 2\kappa_j x} m_\pm(i\kappa_j, x).$$

**Proposition 17** *The functions $(m_\pm(k,x) - 1)$ and $N_\pm(k,x)$ are Hardy functions for each $x \in \mathbb{R}$.*

**Corollary 18** *There exist functions $B_\pm(x,\cdot)$ and $\tilde{B}_\pm(x,\cdot) \in L^2(\mathbb{R})$, $x \in \mathbb{R}$ such that $B_\pm(x,y) = 0, \tilde{B}_\pm(x,y) = 0$ for $y < 0$ and all $x \in \mathbb{R}$ and*

$$\begin{aligned} m_\pm(k,x) &= 1 + \int_{-\infty}^{\infty} B_\pm(x,y) e^{iky} dy, \quad \Im k \geq 0, \quad x \in \mathbb{R} \\ n_\pm(k,x) &= 1 + \sum_{j=1}^{N} \frac{A_{\pm j}(x)}{k - i\kappa_j} + \int_{-\infty}^{\infty} \tilde{B}_\pm(x,y) e^{iky} dy, \quad \Im k \geq 0, \quad x \in \mathbb{R}. \end{aligned}$$

*Furthermore for each $x$ and real argument the function $(m_\pm(\cdot,x) - 1)$ is the Fourier transform of a certain function $B_\pm(x,\cdot) \in L^2(\mathbb{R})$ vanishing on the negative real axis. Similary $N_\pm(\cdot,x)$ is the Fourier transform of a certain function $\tilde{B}(x,\cdot) \in L^2(\mathbb{R})$ vanishing on the negative real axis.*

$B_-$ satisfies the integral equation

$$2B(x,y) = \int_{-\infty}^{x-y/2} q(x')dx' + \int_0^y \int_{-\infty}^{x+(z-y)/2} q(x')B(x',z)dx'dz \tag{4.24}$$

for $y > 0$ while $B_-(x,y) = 0$ would be a solution for $y < 0$ because of the missing nonhomogeneous term.

**Theorem 19** *Let $q \in L^1_2(\mathbb{R})$. Then the integral equation (4.24) has a (unique) solution $B(x,\cdot) : \mathbb{R}_+ \to \mathbb{R}$ satisfying*

1. $\|B(x,\cdot)\|_\infty \le \frac{1}{2}\eta(x)e^{\gamma(x)} < \infty$,
2. $\|B(x,\cdot)\|_1 \le \gamma(x)e^{\gamma(x)} < \infty$,
3. $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}B_-(x,y) + 2\frac{\partial}{\partial y}B_-(x,y)\right) = q(x)B(x,y)$,
4. $\frac{d}{dx}B(x,0) = \frac{1}{2}q(x)$ *almost everywhere,*
5. *the function* $e^{-ikx}\left(1 + \int_0^\infty B(x,y)e^{iky}dy\right)$ *is the Jost solution* $f_-(k,x)$.

The functions $\eta$ and $\gamma$ were defined by

$$\gamma(x) = \int_{-\infty}^x (x-t)|q(t)|dt \text{ and } \eta(x) = \int_{-\infty}^x |q(t)|dt. \tag{4.25}$$

Note that $B(x,\cdot) \in L^\infty(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ implies $B(x,\cdot) \in L^2(\mathbb{R}^+)$, so its Fourier transform exists in $L^2$. Since the Jost solution $f_-(k,x) = e^{-ikx}(1 + \int_0^\infty B(x,y)e^{iky}dy)$, we conclude that the function $B$, defined by the integral equation (4.24) is actually the $B_-$ that was defined in Corollary 18, thus justifying the formal reasoning that led to (4.24) in the first place. Furthermore property 4 of the theorem shows that one can obtainin $q$, if one knows $B_-$.

**Remark:** $q(x) = \mp\frac{d}{dx}K_\pm(x,x)$.

Using (4.12) we obtain from the fundamental relation

$$f_\pm(k,x) = c_\mp(k)f_\mp(k,x) + d(k)g_\mp(k,x)$$

the equation

$$n_\pm(k,x) = R_\pm(k)e^{\pm 2ikx}m_\pm(k,x) + m_\pm(-k,x) \tag{4.26}$$

or

$$\begin{aligned} N_\pm(k,x) = -\sum_{j=1}^N \frac{A_{\pm,j}(x)}{k - i\kappa_j} &+ (m_\pm(-k,x) - 1) + R_\pm(k)e^{\pm 2ikx} \\ &+ R_\pm(k)e^{\pm 2ikx}(m_\pm(k,x) - 1). \end{aligned}$$

Now we take inverse Fourier transforms on both sides and evaluate at $\mp 2y$. We get for $\pm y > 0$

$$0 = \omega_\pm(x+y) + B_\pm(x, \pm 2y) + 2\int_{-\infty}^{\infty} B_\pm(x, \pm 2z)\omega_\pm(x+y+z)dz. \tag{4.27}$$

Note that for a given set of scattering data both $\omega_+$ and $\omega_-$ are given functions, since the other set may be calculated from either set. Hence (4.27) is an integral equation, which can be used to obtain $B_\pm(x, y)$ from a given set of scattering data. It is called the Marchenko equation.

**Theorem 20** *The function $B_\pm(x, y)$ defined in Corollary 18 satisfies the Marchenko equation*

$$B_\pm(x, \pm 2y) + \omega_\pm(x+y) + 2\int_{-\infty}^{\infty} B_\pm(x, \pm 2z)\omega_\pm(x+y+z)dz = 0, \quad \pm y > 0$$

*where*

$$\omega_\pm(z) = \sum_{j=1}^{N} \gamma_{\pm j}^2 e^{\mp 2\kappa_j z} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R_\pm(k)e^{\pm 2ikz}dk.$$

**Remark:** Marchenko equation for $K_\pm(x, y)$:

$$K_\pm(x, y) + \Omega_\pm(x+y) \pm \int_{x}^{\pm\infty} K_\pm(x, z)\Omega_\pm(x+y+z)dz = 0, \quad x \gtrless y$$

where

$$\Omega_\pm(s) = \sum_{j=1}^{N} \gamma_{\pm j}^2 e^{\mp \kappa_j s} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R_\pm(k)e^{\pm iks}dk.$$

Now suppose the set of scattering data $S_-$ is given. This defines the function $\omega_-(z)$ using (4.27) and hence the nonhomogeneous term and the kernel of the Marchenko equation, which is a Fredholm integral equation. Under certain conditions on $S_-$ the Marchenko equation has a unique solution $B_-(x, y)$. Compute $2\frac{d}{dx}B_-(x, 0)$ to obtain $q(x)$. This solves the inverse problem. The next step to do is therefore to describe conditions under which the Marchenko equation has a unique solution.

We summarize the results on inverse scattering in the following theorem:

**Theorem 21** *Suppose the set $S_- = \{R_-(k), k \in \mathbb{R}; \kappa_1, ..., \kappa_N; \gamma_{-,1}, ..., \gamma_{-,N}\}$ satisfies the following conditions:*

1. *The numbers $\kappa_1, ..., \kappa_N$ and $\gamma_{-,1}, ..., \gamma_{-N}$ are positive. The numbers $\kappa_j$ are distinct.*
2. *The function $R_-(k)$ is continuous and satisfies*
   (a) $R_-(k) = R_-(-k)^*$

(b) $|R_-(k)| \leq 1, \quad |R_-(k)| = 1 \implies k = 0$

(c) If $|R_-(0)| = 1$, then $\lim_{k\to 0} \frac{1+R_-(k)}{k} = \rho \neq 0$

(d) $|R_-(k)| = 0(\frac{1}{|k|}) as |k| \to \infty$

(e) $R_-^\vee(\cdot)$, the inverse Fourier transform of $R_-$, is absolutely continuous and $\int_{-\infty}^{\infty}(1+|x|)|\frac{d}{dx}R_-^\vee(x)|dx < \infty$.

*Then $S_-$ determines a unique potential $q$. This potential is given through*

$$q(x) = 2\frac{d}{dx}B(x,0),$$

*where $B(x,y)$ is the unique solution of the Marchenko equation*

$$B(x,2y) + \omega(x-y) + 2\int_0^\infty B(x,2z)\omega(x-y-z)dz = 0, \quad y \geq 0$$

*and where $\omega(z) = \sum_{j=1}^N \gamma_{-,j}^2 e^{2\kappa_j z} + R_-^\vee(2z)$.*

# 5. The initial value problem for the KdV equation

The basic idea to solve the initial value problem for the KdV equation goes back to Gardner, Greene, Kruskal and Miura. It is given by the following diagram:

Here it should be remarked that this procedure resembles the strategy of solving a linear equation by using Fourier transforms and inverse Fourier transforms instead of scattering and inverse scattering methods. Consider for example the linearized KdV equation $q_t + q_{xxx} = 0$. Fourier transformation of the initial condition $q_0(x)$ yields

$$A(k) = \int_{-\infty}^{\infty} q_0(x) e^{ikx} dx.$$

Inserting $q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)\exp(-i(kx - \omega(k)t))dk$ into the equation yields $\omega(k) = -k^3$. Therefore

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)\exp(-ik^3 t)e^{-ikx} dk,$$

i.e. $q(x,t)$ is the inverse Fourier transform of $A(k)\exp(-ik^3 t)$.

The only missing step in the above diagram is now the time evolution of the scattering data. We start with some preparation.

**Proposition 22** *Suppose $q$ is a real valued solution of the KdV equation such that $q$ and its $x$-derivatives up to order three are continuous in $\mathbb{R}^2$. Furthermore assume that $\psi_0(k,x), k^2 \in \mathbb{R}, x \in \mathbb{R}$ satisfies the differential equation*

$$L(0)\psi_0 = -\psi_0'' + q(\cdot,0)\psi_0 = k^2\psi_0, \quad k^2 \in \mathbb{R}. \tag{5.1}$$

*Then there exists a unique real valued solution $\psi(k,x,t)$ of the initial value problem*

$$\begin{aligned} L(t)\psi &= -\psi_{xx} + q(\cdot,t)\psi = k^2\psi, \quad k^2 \in \mathbb{R}, & (5.2)\\ P(t)\psi &= -4\psi_{xxx} + 6q\psi_x + 3q_x\psi = \psi_t, & (5.3)\\ \psi(k,x,0) &= \psi_0(k,x), \quad k^2 \in \mathbb{R}. & (5.4) \end{aligned}$$

**Remark:** (5.3) is equivalent to

$$\psi_t = 2(q + 2k^2)\psi_x - q_x\psi. \tag{5.5}$$

In the following we require that $q$ satisfies the assumptions of chapter 3 as well as of chapter 4.

**Hypothesis 23** *Let $q \in C^\infty(\mathbb{R})$ be real valued and such that $q, q_x, q_t$ and $q_{xt} \in L^\infty(\mathbb{R}^2)$. Also require that $q(\cdot, t) \in L^1_2(\mathbb{R})$ and $q_t(\cdot, t) \in L^1(\mathbb{R})$.*

Remember that under this condition $L(t)$ is a self-adjoint operator with domain $D(L(t)) = H^2(\mathbb{R})$ for all $t$ (see Proposition 5).

**Proposition 24** *If $q$ satisfies Hypothesis 23 and the KdV equation then $q, q_x, q_{xx}$ tend to zero as $|x|$ tends to infinity.*

**Proposition 25** *If $q$ satisfies Hypothesis 23, then the Jost solutions $f_\pm$ of $L(t)f = k^2 f$ have the following properties*

$$\begin{aligned} e^{\mp ikx}\frac{\partial}{\partial x} f_\pm(k,x,t) &= \pm ik + O(1), \\ e^{\mp ikx}\frac{\partial}{\partial t} f_\pm(k,x,t) &= O(1) \end{aligned}$$

*as $x$ tends to $\pm\infty$.*

**Theorem 26** *If $q$ satisfies Hypothesis 23 and the KdV equation, then the scattering data $S_\pm(t)$ associated with $q(\cdot, t)$ evolve according to*

$$\begin{aligned} S_\pm(t) &= \{\exp(\pm 8ik^3 t)R_\pm(k, t=0); k \in \mathbb{R}; \\ &\quad \kappa_j(t=0), \exp(\pm 4\kappa_j^3 t)\gamma_{\pm,j}(t=0), j = 1, ..., N\} \end{aligned}$$

*Moreover the transmission coefficient $T$ is independent of time, i.e., $T(k,t) = T(k,0)$.*

**Remarks:** (i) In Theorem 7 it was proved that the operators $L(t)$ and $L(0)$ are unitarily equivalent. In particular their eigenvalues which are given through the numbers $\kappa_j, j = 1, ..., N$ do not depend on $t$.

(ii) Next define $\psi_{j,0}(x)$ to be the solution of $L(0)\psi = -\kappa_j^2\psi$ with $L^2$-norm one and $\psi_j(x,t)$ to be the solution of $L(t)\psi = 0$ and $\psi_t = P(t)\psi$ having initial values $\psi_{j,0}(x)$. $\psi_j$ exists uniquely according to Proposition 22. Therefore

$$\psi_j(x,t) = \gamma_{+,j}(t) f_+(i\kappa_j, x, t) \tag{5.6}$$

where $f_+$ is a Jost solution of $L(t)\psi = -\kappa_j^2\psi$ and $\gamma_{+,j}$ is the norming constant defined in (4.19). Differentiating (5.6) with respect to $t$ and using (5.5) once more one gets

$$\dot\gamma_{+,j} f_+ + \gamma_{+,j} f_{+,t} = 2(q - 2\kappa_j^2)\gamma_{+,j} f_{+,x} - q_x \gamma_{+,j} f_+.$$

Multiplying by $e^{\kappa_j x}$ and using Propositions 24 and 25 yields

$$\dot\gamma_{+,j} = 4\kappa_j^3\gamma_{+,j}, \text{ i.e., } \gamma_{+,j}(t) = \exp(4\kappa_j^3 t)\gamma_{+,j}(0).$$

(iii) Finally consider for real $k$ solutions $\psi_\pm$ of (5.2) and (5.3) with initial conditions $\psi_+(k,x,0) = f_+(k,x,0)$ and $\psi_-(k,x,0) = f_-(k,x,0)$. Compute, using (5.5)

$$\frac{d}{dt}W(\psi_-,\psi_+) \;=\; 0, \qquad \frac{d}{dt}W(\psi_-(k),\psi_+(-k)) = 0.$$

Since $f_+$ and $f_-$ are linearly independent for $k \neq 0$ we have

$$\psi_+(k,x,t) = a(k,t)f_+(k,x,t) + b(k,t)f_-(k,x,t)$$

and hence by differentiating with respect to $t$ and using (5.5)

$$\begin{aligned}
&\dot a e^{-ikx} f_+ + a e^{-ikx} f_{+,t} + \dot b e^{-2ikx} e^{ikx} f_- + b e^{-2ikx} e^{ikx} f_{-,t} \\
=\;& 2(q+2k^2)(a e^{-ikx} f_{+,x} + b e^{-2ikx} e^{ikx} f_{-,x} \\
&-q_x(a e^{-ikx} f_+ + b e^{-2ikx} e^{ikx} f_-)
\end{aligned}$$

As $x$ tends to infinity one gets using Propositions 24 and 25

$$\dot a - 4ik^3 a + e^{-2ikx}(\dot b + 4ik^3 b) + o(1) = 0.$$

This implies $\dot a = 4ik^3 a$ and $\dot b = -4ik^3 b$ with initial conditions $a(0) = 1$ and $b(0) = 0$ and therefore

$$\psi_+(k,x,t) = \exp(4ik^3 t) f_+(k,x,t)$$

and similarly

$$\psi_-(k,x,t) = \exp(-4ik^3 t) f_-(k,x,t).$$

Now

$$\frac{1}{T(k,t)} = d(k,t) = \frac{W(f_-(k,x,t), f_+(k,x,t))}{2ik} = \frac{W(\psi_-(k,x,t),\psi_+(k,x,t))}{2ik}$$

does not depent on $t$. Also

$$\begin{aligned}
R_\pm(k,t) \;&=\; \frac{c_\pm(k)}{d(k)} = \mp\frac{W(f_\mp(k,x,t), g_\pm(k,x,t))}{2ik\, d(k)} \\
&=\; \mp e^{\pm 8ik^3 t}\frac{W(\psi_\mp(k,x,t),\psi_\pm(k,x,t))}{2ik\, d(k)} = e^{\pm 8ik^3 t} R_\pm(k,0).
\end{aligned}$$

∎

The result of Theorem 26 is of course only a necessary condition for solutions of the KdV equation since the proof assumed already the existence of solutions. Therefore the follwing steps remain to be done.

- Check that $S_\pm(t)$ forms a set of scattering data, i.e., check that $S_\pm(t)$ satisfies the conditions of Theorem 21 for each fixed $t$.
- Show that starting from $q_0(x)$ the constructed $q(x,t)$ does indeed satisfy the KdV equation. (This can be done in principle while the necessary calculations are rather horrible.)

# 6. Solitons and soliton-like solutions

**Definition 27** *An N-soliton solution $q_N(x,t)$ of the KdV-equation is defined by either of the sets of scattering data*

$$S_\pm(t) = \{\kappa_j, \exp(\pm 4\kappa_j^3 t)\gamma_{\pm,j}(0), j = 1, ...N\}$$

*via the inverse scattering method, i.e., an N-soliton solution $q_N$ is a potential for the Schrödinger equation which is reflectionless and has $N$ bound states.*

**Lemma 28** *All reflectionless potentials $q$ satisfying $q \in L_2^1(\mathbb{R})$ are of the type*

$$q_N(x) = -2\frac{d^2}{dx^2}\ln\left(\det(1 + \Lambda_N(x))\right), \tag{6.1}$$

*where $\Lambda_N(x)$ is an $N \times N$ matrix with element*

$$\Lambda_{j,\ell} = \gamma_{+,j}\gamma_{+,\ell}(\kappa_j + \kappa_\ell)^{-1}\exp(-(\kappa_j + \kappa_\ell)x) \tag{6.2}$$

*in row $j$ and column $\ell$.*

**Remark:** Use $\Omega_\pm(s) = \sum_{j=1}^N c_{\pm,j}^2 e^{\mp\kappa_j s}$ and the Ansatz

$$K(x,y) = \sum_{j=1}^N \tilde{k}_{\pm,j}(x)e^{\mp\kappa_j y}$$

in the Marchenko equation.

Combining the lemma with the result of the previous chapter one obtains

**Theorem 29** *The N-soliton solutions of the KdV equation are of the form (6.1), where $\Lambda_N$ depends now also on $t$ and (6.2) is replaced by*

$$\Lambda_{j,l} = \gamma_{+,j}(0)\gamma_{+,l}(0)\exp(4(\kappa_j^3 + \kappa_\ell^3)t)(\kappa_j + \kappa_\ell)^{-1}\exp(-(\kappa_j + \kappa_\ell)x).$$

In order to study these solutions for $t \to \pm\infty$ we introduce

$$s(x,v) = -\frac{v}{2}\{\cosh(\frac{1}{2}xv^{1/2})\}^{-2} \text{ for } v > 0, \quad x \in \mathbb{R}.$$

Then

$$q_1(x,t) = s(x - 4\kappa_1^2 t - \delta_1, 4\kappa_1^2) \text{ with } \delta_1 = \frac{1}{2\kappa_1}\ln\left(\frac{\gamma_{+,1}(0)^2}{2\kappa_1}\right).$$

**Remark:** Note that $v = 4\kappa_1^2$ is the velocity of the solution and that $-\frac{v}{2}$ is the amplitude, i.e., taller solutions travel faster.

**Theorem 30** *Let $N \in \mathbb{N}$ and assume $\kappa_N > ... > \kappa_1$. Then*

$$\lim_{t\to\pm\infty}\left\{\sup_{x\in\mathbb{R}}\left|q_N(x,t)-\sum_{j=1}^{N}s(x-4\kappa_j^2t-\delta_j^{\pm},4\kappa_j^2)\right|\right\}=0,$$

*where*

$$\exp(2\kappa_j\delta_j^{\pm})=\frac{\gamma_{+,j}^2}{2\kappa_j}\begin{cases}\prod_{\ell=j+1}^{N} & (\kappa_j-\kappa_\ell)^2(\kappa_j+\kappa_\ell)^{-2}\\ \prod_{\ell=1}^{j-1} & (\kappa_j-\kappa_\ell)^2(\kappa_j+\kappa_\ell)^{-2}.\end{cases}$$

Theorem 30 shows that any $N$-soliton solution is asymptotically a superposition of $N$ solitary waves of different speed and amplitude as $t \to \pm\infty$. These are well separated from one another because of their different speed and the exponential decay of the $1/\cosh^2$-function. In a very distant past the tallest and fastest soliton is to the left of all others. But it catches up interacts - non linearly - with all other solitons and passes them finally. In a very distant future the tallest soliton will be to the right of all others. It does not change its speed, height or form once it has emerged from the interaction process but it has not come as far as it would have if no interaction had occured. Meeting all the other guys cost some time. On the other hand the slowest one is further ahead than it would be without interaction. It seems to have tried to keep the pace of the faster one for some time during the interaction.

For the general case, where $R_{\pm}(k) \neq 0$, the situation is more complicated. The following result is only stated.

**Theorem 31** *Let $q \in C^\infty(\mathbb{R}^2)$ satisfy the KdV equation and suppose $q(x,0) \in \mathcal{S}(\mathbb{R})$. Let $S_+(t)$ be the set of scattering data of $q$ and define the set $S_{+,N}(t)$ by putting the reflection coefficient equal to zero but keeping the other data from $S_+(t)$. This defines then an $N$-soliton solution of KdV which is called the $N$-soliton solution associated to $q$.*
*Now fix $\varepsilon > 0$. Then*

$$\lim_{t\to\pm\infty}\left(\sup_{\pm x>\pm\varepsilon t}|q(x,t)-q_N(x,t)|\right)=0.$$

*Ordering the eigenvalues of $q$ according to $\kappa_N > ... > \kappa_1$ one also has*

$$\lim_{t\to\pm\infty}\left(\sup_{\pm x>\pm\varepsilon t}|q(x,t)-\sum_{j=1}^{N}s(x-4\kappa_jt-\delta_j^{\pm},4\kappa_j^2)|\right)=0$$

*where the $\delta_j^{\pm}$ are given in Theorem 30.*

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