Sequence for $\sqrt{2}$

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Summary: The sequence of rational numbers given below converges towards $\sqrt{2}$, which is irrational, so not a rational number.
This sequence is found from the binomial series of $(1 + x)^{1/2}$ for x = 1.

■ Sum of the binomial series of $(1 + x)^{1/2}$ for x = 1

```
Sum[Binomial[1 / 2, k], {k, 0, Infinity}]
```

$\sqrt{2}$

■ partial sum of terms k = 0 up to k = n for x = 1

■ general term

```
ps[n_] = Sum[Binomial[1 / 2, k], {k, 0, n}]
```

$$\sqrt{2} - \frac{\sqrt{\pi}\ \text{Hypergeometric2F1}\left[1, \frac{1}{2} + n, 2 + n, -1\right]}{2\,\text{Gamma}\left[\frac{1}{2} - n\right]\,\text{Gamma}[2 + n]}$$

■ Lists of partial sums {n, s_n}

```
tsn = Table[{k, ps[k]}, {k, 0, 10}] // Simplify
```

$$\left\{\{0, 1\}, \left\{1, \frac{3}{2}\right\}, \left\{2, \frac{11}{8}\right\}, \left\{3, \frac{23}{16}\right\}, \left\{4, \frac{179}{128}\right\}, \left\{5, \frac{365}{256}\right\}, \left\{6, \frac{1439}{1024}\right\}, \left\{7, \frac{2911}{2048}\right\}, \left\{8, \frac{46147}{32768}\right\}, \left\{9, \frac{93009}{65536}\right\}, \left\{10, \sqrt{2} - \frac{4199\,\text{Hypergeometric2F1}\left[1, \frac{21}{2}, 12, -1\right]}{524288}\right\}\right\}$$

```
N[tsn]
```

```
{{0., 1.}, {1., 1.5}, {2., 1.375}, {3., 1.4375}, {4., 1.39844}, {5., 1.42578},
 {6., 1.40527}, {7., 1.42139}, {8., 1.40829}, {9., 1.4192}, {10., 1.40993}}
```

```
t50 = Table[{k, ps[k]}, {k, 0, 50}] // N;
```

```
sq2 = Sqrt[2] // N
```

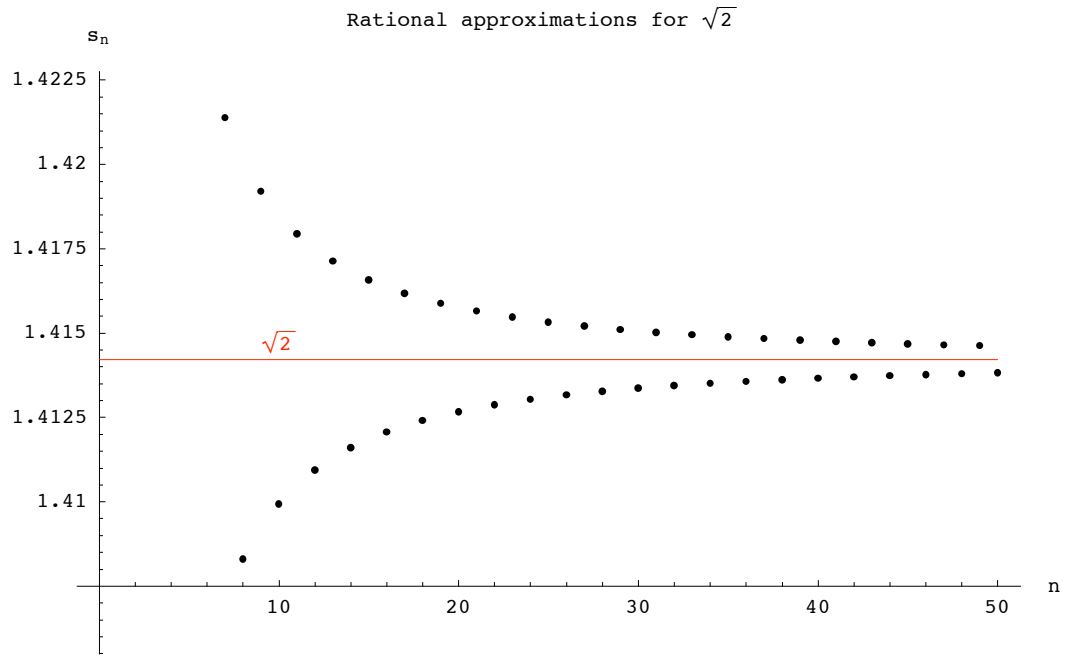
```
1.41421
```

```
In[8]:= N[93009/65536, 20]
```

```
Out[8]= 1.4192047119140625000
```

■ Plot of partial sums

```
ListPlot[t50,
  Epilog → {Hue[0], Text["√2", {10, sq2 + 0.0005}], Line[{{0, sq2}, {50, sq2}}]},
  AxesLabel → {"n", Subscript["s", "n"]},
  ImageSize → 500, AxesOrigin → {0, 1.4075},
  PlotLabel → "Rational approximations for √2 \n"];
```



■ Partial sums - $\sqrt{2}$

```
ScientificForm[Transpose[t50][[2]] - sq2, {6, 1}]
```

$$\{-4.1\times10^{-1},\ 8.6\times10^{-2},\ -3.9\times10^{-2},\ 2.3\times10^{-2},\ -1.6\times10^{-2},\ 1.2\times10^{-2},\ -8.9\times10^{-3},\ 7.2\times10^{-3},$$
$$-5.9\times10^{-3},\ 5.\times10^{-3},\ -4.3\times10^{-3},\ 3.7\times10^{-3},\ -3.3\times10^{-3},\ 2.9\times10^{-3},\ -2.6\times10^{-3},$$
$$2.4\times10^{-3},\ -2.2\times10^{-3},\ 2.\times10^{-3},\ -1.8\times10^{-3},\ 1.7\times10^{-3},\ -1.5\times10^{-3},\ 1.4\times10^{-3},$$
$$-1.3\times10^{-3},\ 1.3\times10^{-3},\ -1.2\times10^{-3},\ 1.1\times10^{-3},\ -1.\times10^{-3},\ 9.9\times10^{-4},\ -9.4\times10^{-4},$$
$$8.9\times10^{-4},\ -8.5\times10^{-4},\ 8.1\times10^{-4},\ -7.7\times10^{-4},\ 7.4\times10^{-4},\ -7.\times10^{-4},\ 6.7\times10^{-4},$$
$$-6.5\times10^{-4},\ 6.2\times10^{-4},\ -6.\times10^{-4},\ 5.7\times10^{-4},\ -5.5\times10^{-4},\ 5.3\times10^{-4},\ -5.1\times10^{-4},\ 5.\times10^{-4},$$
$$-4.8\times10^{-4},\ 4.6\times10^{-4},\ -4.5\times10^{-4},\ 4.3\times10^{-4},\ -4.2\times10^{-4},\ 4.1\times10^{-4},\ -4.\times10^{-4}\}$$