13.6.4.1 An Example for Plana's Summation Formula. Comparison with Results obtained by other Methods in Mathematica

B. Schnizer

schnizer@itp.tu-graz.ac.at

```
$Version
```

7.0 for Mac OS X x86 (64-bit) (February 19, 2009)

ζ(3)

Symbolic Summation

```
ff[nn_] = 1 / nn^3
```

$\frac{1}{nn^3}$

```
Sum[ff[n], {n, Infinity}]
```

Zeta[3]

```
ne = N[%, 25]
```

1.202056903159594285399738

Numeric Summation

"Infinitely" many terms

```
ns = SetPrecision[NSum[ff[n], {n, Infinity}], 20]
```

1.2020569031403265381

```
ne - ns
```

$1.92677473 \times 10^{-11}$

100 terms

```
SetPrecision[N[Sum[ff[n], {n, 100}]], 20]
```

1.2020074006596777050

```
ScientificForm[ne - %, 3]
```

4.95×10^{-5}

1000 terms

```
SetPrecision[N[Sum[ff[n], {n, 1000}]], 20]
```

1.2020564036593444079

```
ScientificForm[ne - %, 3]
```

5.00×10^{-7}

Summation by Plana's formula

```
fz[zz_] = 1 / (zz + 1) ^ 3
```

$$\frac{1}{(1 + zz)^3}$$

```
t1 = fz[0] / 2
```

$$\frac{1}{2}$$

```
t2 = Integrate[fz[z], {z, 0, Infinity}]
```

$$\frac{1}{2}$$

```
it = I (fz[I y] - fz[-I y]) / (Exp[2 π y] - 1) // Simplify
```

$$-\frac{2\, y\, \left(-3 + y^2\right)}{\left(-1 + e^{2 \pi y}\right) \left(1 + y^2\right)^3}$$

```
t3 = SetPrecision[NIntegrate[it, {y, 0, Infinity}], 20]
```

0.20205690315959437542

```
SetPrecision[t1 + t2 + t3, 20]
```

1.2020569031595943754

```
ne - %
```

-9.00×10^{-17}

This shows that Planas formula as implemented above gives better results than the numeric summation.

ζ(3/2)

Symbolic Summation

```
ff[nn_] = 1 / nn^(3 / 2)
```

$$\frac{1}{nn^{3/2}}$$

```
Sum[ff[n], {n, Infinity}]
```

$$\text{Zeta}\left[\frac{3}{2}\right]$$

```
ne = N[%, 25]
```

2.612375348685488343348568

Numeric Summation

- **"Infinitely" many terms**

```
ns = SetPrecision[NSum[ff[n], {n, Infinity}], 20]
```

2.6123753485261729246

```
ne - ns
```

$1.593154188 \times 10^{-10}$

- **100 terms**

```
SetPrecision[N[Sum[ff[n], {n, 100}]], 20]
```

2.4128740987037162746

```
ScientificForm[ne - %, 3]
```

2.00×10^{-1}

- **1000 terms**

```
SetPrecision[N[Sum[ff[n], {n, 1000}]], 20]
```

2.5491456029175747489

```
ScientificForm[ne - %, 3]
```

6.32×10^{-2}

- ## Summation by Plana's formula

```
fz[zz_] = 1 / (zz + 1) ^ (3 / 2)
```

$$\frac{1}{(1 + zz)^{3/2}}$$

```
t1 = fz[0] / 2
```

$$\frac{1}{2}$$

```
t2 = Integrate[fz[z], {z, 0, Infinity}]
```

2

```
it = I (fz[I y] - fz[-I y]) / (Exp[2 π y] - 1) // Simplify
```

$$\frac{i \left(-\frac{1}{(1-i\, y)^{3/2}} + \frac{1}{(1+i\, y)^{3/2}}\right)}{-1 + e^{2 \pi y}}$$

```
t3 = SetPrecision[NIntegrate[it, {y, 0, Infinity}], 20] // Chop
```

0.11237534868548852695 6

```
SetPrecision[t1 + t2 + t3, 20]
```

2.6123753486854885270

```
ne - %
```

-1.836×10^{-16}

This shows that Planas formula as implemented above gives better results than the numeric summation.

ζ(11/10)

■ Symbolic Summation

```
ff[nn_] = 1 / nn^(11 / 10)
```

$\frac{1}{nn^{11/10}}$

```
Sum[ff[n], {n, Infinity}]
```

$\text{Zeta}\left[\frac{11}{10}\right]$

```
ne = N[%, 25]
```

10.58444846495080982638640

■ Numeric Summation

■ "Infinitely" many terms

```
ns = SetPrecision[NSum[ff[n], {n, Infinity}], 20]
```

10.584448464728067663

```
ne - ns
```

$2.22742164 \times 10^{-10}$

■ 100 terms

```
SetPrecision[N[Sum[ff[n], {n, 100}]], 20]
```

4.2780240231583706034

```
ScientificForm[ne - %, 3]
```

6.31

■ 1000 terms

```
SetPrecision[N[Sum[ff[n], {n, 1000}]], 20]
```

5.5728266763527418703

```
ScientificForm[ne - %, 3]
```

5.01

Summation by Plana's formula

```
fz[zz_] = 1 / (zz + 1) ^ (11 / 10)
```

$$\frac{1}{(1 + zz)^{11/10}}$$

```
t1 = fz[0] / 2
```

$$\frac{1}{2}$$

```
t2 = Integrate[fz[z], {z, 0, Infinity}]
```

10

```
it = I (fz[I y] - fz[-I y]) / (Exp[2 π y] - 1) // Simplify
```

$$\frac{i \left(-\frac{1}{(1-i\,y)^{11/10}} + \frac{1}{(1+i\,y)^{11/10}}\right)}{-1 + e^{2\pi y}}$$

```
t3 = SetPrecision[NIntegrate[it, {y, 0, Infinity}], 20] // Chop
```

0.084448464950810028795

```
SetPrecision[t1 + t2 + t3, 20]
```

10.584448464950810029

```
ne - %
```

-2.02×10^{-16}

This shows that Planas formula as implemented above gives better results than the numeric summation.