Beispiele:

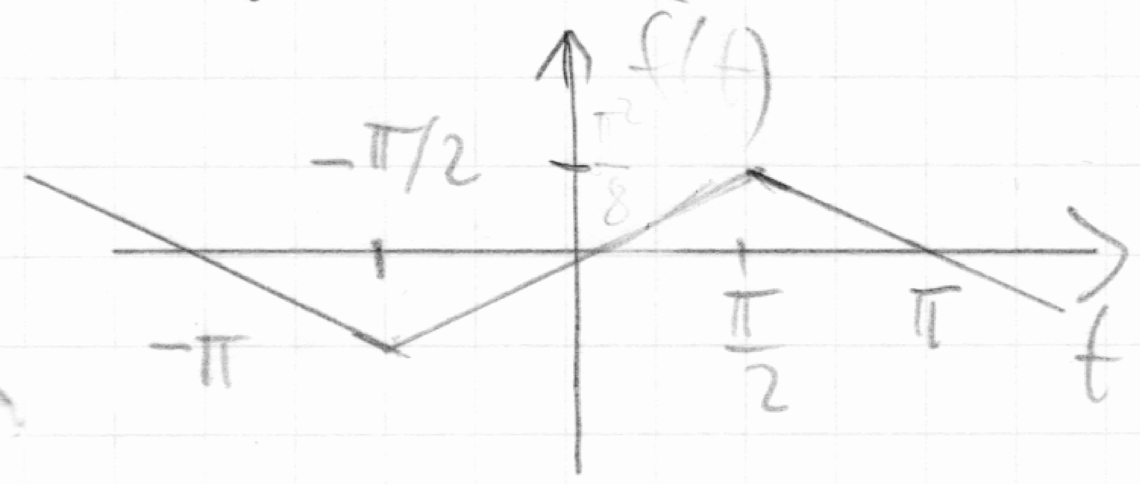
1. $h(z) = \dfrac{1}{z^2+a^2} = \dfrac{1}{(z+ia)(z-ia)}$, Pole an $z = \pm ia$, Residuum $= \pm\dfrac{1}{2ia}$.

$$\sum_{n}^{\infty} \frac{1}{n^2+a^2} = -\frac{\pi}{2ia}\cot(\pi i a) - \frac{\pi}{2ia}\cot(-\pi i a) =$$

$$= \frac{1}{a^2} + 2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a), \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \lim_{a\to 0}\frac{1}{2}\left[\frac{\pi}{a}\coth(\pi a) - \frac{1}{a^2}\right] = \frac{\pi^2}{3}$$

$$\left[\frac{\pi}{a}\left(\cancel{\frac{1}{\pi a}} + \frac{\pi a}{3} + O((\pi a)^3)\right) - \cancel{\frac{1}{a^2}}\right]$$

2. Summation einer Fourierreihe.



$$\bar f(t) = \frac{4}{\pi}\left(\sin t - \frac{\sin(3t)}{3^2} + \frac{\sin(5t)}{5^2} - + \dots\right) \quad (1)$$

$$= -\frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{\sin[(2n-1)t]}{(2n-1)^2}(-1)^n \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

$$-\frac{\pi \bar f(t)}{2} = \sum_{n=-\infty}^{\infty}(-1)^n\frac{\sin[(2n-1)t]}{(2n-1)^2} = \sum(-)^n h(n)$$

$h(z) = \dfrac{\sin[(2z-1)t]}{(2z-1)^2}$, Pol: $z = \frac{1}{2}$, Residuum$\left(h(z)\frac{\pi}{\sin(\pi z)}, z = \frac{1}{2}\right) = \frac{\pi t}{2}$.

$$\sum_{n=-\infty}^{\infty}(-1)^n h(n) = 2\sum_{n=1}^{\infty}(-1)^n h(n) = -2\sin t + 2\frac{\sin(3t)}{3^2} - 2\frac{\sin(5t)}{5^2} + - \dots = -\frac{\pi t}{2}$$

$$-\pi/2 \le t \le \frac{\pi}{2}$$

Untersuchung der Konvergenz des Integrals $\oint_{|z|=R_n\to\infty} f(z)\,dz \to 0$

$$\left|\frac{h(z)}{\sin(\pi z)}\right| \sim \left|\frac{\sin[(2z-1)t]}{(2z-1)^2\sin(\pi z)}\right| \sim \left|\frac{e^{it(2z-1)} - e^{-it(2z-1)}}{z^2\,(e^{i\pi z} - e^{-i\pi z})}\right| \qquad \alpha = e^{-2it(2x-1)}$$

$$\sim \left|\frac{e^{it(2x+2iy-1)} - e^{-it(2x+2iy-1)}}{|z|^2\,(e^{i\pi(x+iy)} - e^{-i\pi(x+iy)})}\right| \sim \left|\frac{e^{-2ty} - e^{2ty}\alpha}{|z|^2(e^{-\pi y} - \beta e^{\pi y})}\right| \qquad \beta = e^{-2i\pi x}, \quad |\alpha| = |\beta| = 1$$

$$y \ge 0: \sim \left|\frac{e^{-y(2t+\pi)} - \alpha e^{-y(\pi-2t)}}{|z|^2(\beta - e^{-2\pi y})}\right| \overset{!}{\underset{y\to+\infty}{\longrightarrow}} 0 \Rightarrow \begin{array}{ll} 2t+\pi \ge 0 & t \ge -\frac{\pi}{2} \\ \pi - 2t \ge 0 & t \le \frac{\pi}{2} \end{array}$$

$0 > y \to -\infty$ führt zu den selben Ungleichungen.

Um die Summe der FR (1) im Intervall $[\pi/2, 3\pi/2]$ zu bestimmen, muß man die Variable transformieren:

$s := \pi - t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2} \Rightarrow \frac{\pi}{2} \le s \le \frac{3\pi}{2}$; aber $\bar f(s) = \bar f(t)$!

$h(z) = \dfrac{\sin[(2z-1)(\pi-s)]}{(2z-1)^2}$, Pol an $z = \frac{1}{2}$, $\mathrm{Res}\left(h(z)\frac{\pi}{\sin(\pi z)}, z = \frac{1}{2}\right) = \pi(\pi - s)/2$.

also: $\bar f(s) = \frac{\pi}{4}(\pi - s)$, $\frac{\pi}{2} \le s \le \frac{3\pi}{2}$.