19.5.1 Point charge q with two dielectrica. Method of images

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```
                     z
                    /|\
                     |
   ∈1                |
                     . q, q1          Φ1
 ____________________|_d_________________ x
                     | -d
                     . q2             Φ2
   ∈2                |
```

A point charge **q** is at a distance **d** in front of an interface, z = 0, separating two dielectric half spaces. This problem is equivalent to a problem with dielectrica corresponding to the halfspace under consideration but with two additional image charges **q1** and **q2** as drawn above. This solution is computed below in the first part of this notebook. Thereeafter the same game is played with **q** located at **z = - d**.

Cylindrical coordinates $\rho = \sqrt{x^2 + y^2}$, ϕ, z .

Primary charge q in G1, z = d > 0

■ The potentials

```
R1 = Sqrt[ρ^2 + (z - d)^2]
R2 = Sqrt[ρ^2 + (z + d)^2]
```

$$\sqrt{(-d + z)^2 + \rho^2}$$

$$\sqrt{(d + z)^2 + \rho^2}$$

```
sΦ1 = 1 / (4 π ∈1) q / R1 + q2 / (4 π R2 ∈1)
```

$$\frac{q}{4 \pi \sqrt{(-d + z)^2 + \rho^2}\ \epsilon_1} + \frac{q_2}{4 \pi \sqrt{(d + z)^2 + \rho^2}\ \epsilon_1}$$

```
sΦ2 = 1 / (4 π ∈2) q1 / R1
```

$$\frac{q_1}{4 \pi \sqrt{(-d + z)^2 + \rho^2}\ \epsilon_2}$$

Field matching at the interface z = 0

```
bc1 = ( ε1 D[sΦ1 , z] /. z -> 0 ) == (ε2 D[sΦ2, z] /. z -> 0)
```

$$\left(\frac{d\,q}{4\,\pi\,\left(d^2+\rho^2\right)^{3/2}\,\epsilon_1}-\frac{d\,q_2}{4\,\pi\,\left(d^2+\rho^2\right)^{3/2}\,\epsilon_1}\right)\epsilon_1 == \frac{d\,q_1}{4\,\pi\,\left(d^2+\rho^2\right)^{3/2}}$$

```
bc2 = (sΦ1 /. z -> 0) == ( sΦ2 /. z -> 0)
```

$$\frac{q}{4\,\pi\,\sqrt{d^2+\rho^2}\,\epsilon_1}+\frac{q_2}{4\,\pi\,\sqrt{d^2+\rho^2}\,\epsilon_1} == \frac{q_1}{4\,\pi\,\sqrt{d^2+\rho^2}\,\epsilon_2}$$

```
so = Map[Factor, Solve[{bc1, bc2}, {q1 , q2}] // Flatten // Together, {2}]
```

$$\left\{q_1\to\frac{2\,q\,\epsilon_2}{\epsilon_1+\epsilon_2},\ q_2\to\frac{q\,(\epsilon_1-\epsilon_2)}{\epsilon_1+\epsilon_2}\right\}$$

```
qr2 = q2 / q /. so
```

$$\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}$$

```
Φ1 = sΦ1
```

$$\frac{q}{4\,\pi\,\sqrt{(-d+z)^2+\rho^2}\,\epsilon_1}+\frac{q_2}{4\,\pi\,\sqrt{(d+z)^2+\rho^2}\,\epsilon_1}$$

```
Φ2 = sΦ2
```

$$\frac{q_1}{4\,\pi\,\sqrt{(-d+z)^2+\rho^2}\,\epsilon_2}$$

The potential of the original and of the image charges

```
Φ1 = sΦ1 /. so
```

$$\frac{q}{4\,\pi\,\sqrt{(-d+z)^2+\rho^2}\,\epsilon_1}+\frac{q\,(\epsilon_1-\epsilon_2)}{4\,\pi\,\sqrt{(d+z)^2+\rho^2}\,\epsilon_1\,(\epsilon_1+\epsilon_2)}$$

```
Φ2 = sΦ2 /. so
```

$$\frac{q}{2\,\pi\,\sqrt{(-d+z)^2+\rho^2}\,(\epsilon_1+\epsilon_2)}$$

Checking continuity

```
Limit[Φ1, d → 0]
```

$$\frac{q}{\sqrt{z^2+\rho^2}\,(2\,\pi\,\epsilon_1+2\,\pi\,\epsilon_2)}$$

```
Limit[Φ2, d → 0]
```

$$\frac{q}{\sqrt{z^2+\rho^2}\,(2\,\pi\,\epsilon_1+2\,\pi\,\epsilon_2)}$$

```
% == %%
```

True

Parts Gi1 of the Green's function

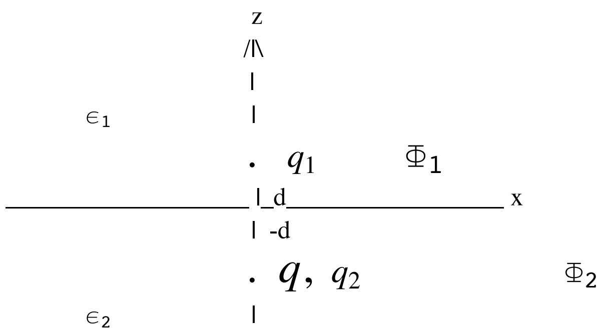
```
G11 = Φ₁ /. q → 1
```

$$\frac{1}{4\pi\sqrt{(-d+z)^2+\rho^2}\,\epsilon_1}+\frac{\epsilon_1-\epsilon_2}{4\pi\sqrt{(d+z)^2+\rho^2}\,\epsilon_1\,(\epsilon_1+\epsilon_2)}$$

```
G21 = Φ₂ /. q → 1
```

$$\frac{1}{2\pi\sqrt{(-d+z)^2+\rho^2}\,(\epsilon_1+\epsilon_2)}$$

Primary charge q in G2, z = - d < 0



The potentials

```
R₁ = Sqrt[ρ^2 + (z - d)^2]
R₂ = Sqrt[ρ^2 + (z + d)^2]
```

$$\sqrt{(-d+z)^2+\rho^2}$$

$$\sqrt{(d+z)^2+\rho^2}$$

```
sΦ₁ = 1 / (4 π ϵ₂) q₂ / R₂
```

$$\frac{q_2}{4\pi\sqrt{(d+z)^2+\rho^2}\,\epsilon_2}$$

```
sΦ₂ = q / (4 π ϵ₂) / R₂ + 1 / (4 π ϵ₁) q₁ / R₁
```

$$\frac{q_1}{4\pi\sqrt{(-d+z)^2+\rho^2}\,\epsilon_1}+\frac{q}{4\pi\sqrt{(d+z)^2+\rho^2}\,\epsilon_2}$$

■ Field matching at the interface z = 0

```
bc1 = ( ϵ1 D[sΦ1 , z] /. z -> 0 ) == (ϵ2 D[sΦ2, z] /. z -> 0)
```

$$-\frac{d\, q_2\, \epsilon_1}{4\pi\left(d^2+\rho^2\right)^{3/2}\epsilon_2} == \left(\frac{d\, q_1}{4\pi\left(d^2+\rho^2\right)^{3/2}\epsilon_1} - \frac{d\, q}{4\pi\left(d^2+\rho^2\right)^{3/2}\epsilon_2}\right)\epsilon_2$$

```
bc2 = (sΦ1 /. z -> 0) == ( sΦ2 /. z -> 0)
```

$$\frac{q_2}{4\pi\sqrt{d^2+\rho^2}\;\epsilon_2} == \frac{q_1}{4\pi\sqrt{d^2+\rho^2}\;\epsilon_1} + \frac{q}{4\pi\sqrt{d^2+\rho^2}\;\epsilon_2}$$

```
so = Map[Factor, Solve[{bc1, bc2}, {q1 , q2}] // Flatten // Together, {2}]
```

$$\left\{q_1 \to \frac{q\,\epsilon_1\,(-\epsilon_1+\epsilon_2)}{\epsilon_2\,(\epsilon_1+\epsilon_2)},\; q_2 \to \frac{2\,q\,\epsilon_2}{\epsilon_1+\epsilon_2}\right\}$$

```
qr2 = q2 / q /. so
```

$$\frac{2\,\epsilon_2}{\epsilon_1+\epsilon_2}$$

```
Φ1 = sΦ1
```

$$\frac{q_2}{4\pi\sqrt{(d+z)^2+\rho^2}\;\epsilon_2}$$

```
Φ2 = sΦ2
```

$$\frac{q_1}{4\pi\sqrt{(-d+z)^2+\rho^2}\;\epsilon_1} + \frac{q}{4\pi\sqrt{(d+z)^2+\rho^2}\;\epsilon_2}$$

■ The potential of the original and of the image charges

```
Φ1 = sΦ1 /. so
```

$$\frac{q}{2\pi\sqrt{(d+z)^2+\rho^2}\;(\epsilon_1+\epsilon_2)}$$

```
Φ2 = sΦ2 /. so
```

$$\frac{q}{4\pi\sqrt{(d+z)^2+\rho^2}\;\epsilon_2} + \frac{q\,(-\epsilon_1+\epsilon_2)}{4\pi\sqrt{(-d+z)^2+\rho^2}\;\epsilon_2\,(\epsilon_1+\epsilon_2)}$$

■ Checking continuity

```
Limit[Φ1, d → 0]
```

$$\frac{q}{\sqrt{z^2+\rho^2}\;(2\pi\epsilon_1+2\pi\epsilon_2)}$$

```
Limit[Φ2, d → 0]
```

$$\frac{q}{\sqrt{z^2+\rho^2}\;(2\pi\epsilon_1+2\pi\epsilon_2)}$$

```
% == %%
```

True

Parts G2i of the Green's function

```
G12 = Φ1 /. q → 1
```

$$\frac{1}{2\pi\sqrt{(d+z)^2+\rho^2}\;(\epsilon_1+\epsilon_2)}$$

```
G22 = Φ2 /. q → 1
```

$$\frac{1}{4\pi\sqrt{(d+z)^2+\rho^2}\;\epsilon_2}+\frac{-\epsilon_1+\epsilon_2}{4\pi\sqrt{(-d+z)^2+\rho^2}\;\epsilon_2\,(\epsilon_1+\epsilon_2)}$$

The resulting four pieces of the Green's function

```
G11
```

$$\frac{1}{4\pi\sqrt{(-d+z)^2+\rho^2}\;\epsilon_1}+\frac{\epsilon_1-\epsilon_2}{4\pi\sqrt{(d+z)^2+\rho^2}\;\epsilon_1\,(\epsilon_1+\epsilon_2)}$$

```
G12
```

$$\frac{1}{2\pi\sqrt{(d+z)^2+\rho^2}\;(\epsilon_1+\epsilon_2)}$$

```
G21
```

$$\frac{1}{2\pi\sqrt{(-d+z)^2+\rho^2}\;(\epsilon_1+\epsilon_2)}$$

```
G22
```

$$\frac{1}{4\pi\sqrt{(d+z)^2+\rho^2}\;\epsilon_2}+\frac{-\epsilon_1+\epsilon_2}{4\pi\sqrt{(-d+z)^2+\rho^2}\;\epsilon_2\,(\epsilon_1+\epsilon_2)}$$